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Some properties of ordered 0-minimal $(0, 2)$-bi-Γ-ideals in po-Γ-semigroups

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Abstract

In this paper, we introduce ordered (generalized) (m, n)-Γ-ideals in po-Γ-semigroups. Then we characterize the po-Γ-semigroup through ordered (generalized) $(0, 2)$-Γ-ideals, ordered (generalized) $(1, 2)$-Γ-ideals and ordered (generalized) 0-minimal $(0, 2)$-Γ-ideals. Also, we investigate the notion of ordered (generalized) $(0, 2)$-bi-Γ-ideals, ordered 0-$(0, 2)$ bisimple po-Γ-semigroups and ordered 0-minimal (generalized) $(0, 2)$-bi-Γ-ideals in po-Γ-semigroups. It is proved that a po-Γ-semigroup S with a zero 0 is 0-$(0, 2)$-bisimple if and only if it is left 0-simple.

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1. Introduction and Preliminaries

The notion of the Γ-semigroup was introduced by M. K. Sen in [9] as a generalization of semigroups and ternary semigroups and the concept of the po-Γ-semigroup was given by Y. I. Kwon and S. K. Lee in [5]. Thereafter different aspects of ideal-theoretic results have been extensively studied in semigroups and po-Γ-semigroups in [1-4, 6].

The concept of the (m, n)-ideal in semigroups was given by S. Lajos in [8] as a generalization of one-sided ideals of semigroups. Thereafter, the notion of the generalized bi-ideal [(or generalized (1,1)-ideal] was introduced in semigroups by S. Lajos in [7] as a generalization of bi-ideals of semigroups. In this paper, we define and use the notion of ordered (generalized) (m, n)-Γ-ideals in po-Γ-semigroups to examine some important classical results and properties in po-Γ-semigroups. As an application of the results of

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this paper, the corresponding results of Γ-semigroups (without order) and semigroups (without order) can also be obtained.

Let S and Γ be two nonempty sets. Then a system $(S, \Gamma, \cdot)$ is called a Γ-semigroup, where $\cdot$ is a ternary operation $S \times \Gamma \times S \to S$ such that $(x \cdot \alpha \cdot y) \cdot \beta \cdot z = x \cdot \alpha \cdot (y \cdot \beta \cdot z)$, for all $x, y, z \in S$ and all $\alpha, \beta \in \Gamma$. Let A be a nonempty subset of $(S, \Gamma, \cdot)$. Then A is called a sub-Γ-semigroup of $(S, \Gamma, \cdot)$ if $a \cdot \gamma \cdot b \in A$, for all $a, b \in A$ and $\gamma \in \Gamma$. Furthermore, a Γ-semigroup S is said to be commutative if $a \cdot \gamma \cdot b = b \cdot \gamma \cdot a$, for all $a, b \in S$ and $\gamma \in \Gamma$.

A po-Γ-semigroup is an ordered set $(S, \leq)$ at the same time a Γ-semigroup $(S, \Gamma, \cdot)$ such that $a \leq b \Rightarrow a \cdot \alpha \cdot x \leq b \cdot \alpha \cdot x$ and $x \cdot \beta \cdot a \leq x \cdot \beta \cdot b$, for all $a, b, x \in S$ and $\alpha, \beta \in \Gamma$.

Notation 1: For subsets A, B of a po-Γ-semigroup S, the product set $A \cdot B$ of the pair (A, B) relative to S is defined as $A \cdot \Gamma \cdot B = \{a \cdot \gamma \cdot b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \cdot A$ relative to S is defined as $A^2 = A \cdot A = A \cdot \Gamma \cdot A$.

Notation 2: For $M \subseteq S$, $(M] = \{s \in S \mid s \leq m, \text{ for some } m \in M\}$. Also, we write $(s]$ instead of $(\{s\}]$ for $s \in S$.

Notation 3: Let $B \subseteq S$. Then for a non-negative integer m, the power of $B^m = B\Gamma B\Gamma B\Gamma B \cdots$, where B occurs m times. Note that the power is suppressed when $m = 0$. So $B^0\Gamma S = S = S\Gamma B^0$.

In what follows we denote the po-Γ-semigroup $(S, \Gamma, \cdot, \leq)$ by S unless otherwise specified. Throughout the paper, for the sake of brevity, we denote $a \cdot \gamma \cdot b$ by $a\gamma b$.

1.1. Example. Let S be the set of all $m \times n$ matrices with entries from a field, where m, n are positive integers. Let $P(S)$ be the power set of S. Then it is easy to see that $P(S)$ is not a semigroup under multiplication of matrices because for A, $B \in P(S)$, the product AB is not defined. Let Γ be the set of $n \times m$ matrices with entries from the same field. Then for A, B, $C \in P(S)$ and P, $Q \in \Gamma$, we have $APB \in P(S)$, $AQB \in P(S)$ and since the matrix multiplication is associative, we get that S is a Γ-semigroup. Furthermore, define $A \leq B$ if and only if $A \subseteq B$ for all $A, B \in P(S)$, then $P(S)$ is a po-Γ-semigroup.

Suppose A and B are two nonempty subsets of S. Then we have the following (see [3]).

(1) $(A]\Gamma(B] \subseteq (A\Gamma B]$;

(2)$A \subseteq B \Rightarrow (A] \subseteq (B]$;

(3) $((A]] = (A]$.

Suppose S is a po-Γ-semigroup and I is a nonempty subset of S. Then I is called an ordered right (resp. left) Γ-ideal of S if

(i) $I\Gamma S \subseteq I (rep.\ S\Gamma I \subseteq I)$,

(ii) $a \in I, b \leq a$ for $b \in S \Rightarrow b \in I$.

Equivalent definition:

(i) $I\Gamma S \subseteq I$ (resp. $S\Gamma I \subseteq I$).

(ii) $(I] = I$.

An ordered Γ-ideal I of S is both a right and a left ordered Γ-ideal of a po-Γ-semigroup S. A right, left or (two-sided) ordered Γ-ideal I of S is called proper if $I \neq S$.

Let S be a semigroup and A be a nonempty subset of S then A is called a generalized (m,n)-ideal of S if $A^mSA^n \subseteq A$, where m, n are arbitrary non-negative integers. Note that if A is a subsemigroup of S, then A is called an (m,n)-ideal of S. We now introduce the following definition.

1.2. Definition. Suppose B is a sub-Γ-semigroup (resp. nonempty subset) of a po-Γ-semigroup S. Then B is called an (resp. generalized) (m,n)-Γ-ideal of S if (i) $B^m\Gamma S\Gamma B^n \subseteq B$ and (ii) for $b \in B$, $s \in S$, $s \leq b \Rightarrow s \in B$.

Note that in the above Definition 1.2, if we set $m = n = 1$, then B is called a (generalized) bi-Γ-ideal of S. Moreover, if $m = 0$ and $n = 2$, then we obtain an ordered (generalized) $(0,2)$-Γ-ideal of S. In a similar fashion, we can obtain an ordered (generalized) $(1,2)$-Γ-ideal and an ordered (generalized) $(2,1)$-Γ-ideal of S.

If B is a nonempty subset of S, then to see that $(B^2 \cup B\Gamma S\Gamma B^2]$ is an ordered (generalized) bi-Γ-ideal of S, we present the verification of it as follows:

$$\begin{aligned}
((B^2 \cup B\Gamma S\Gamma B^2]] &= (B^2 \cup B\Gamma S\Gamma B^2] \text{ and} (\mathrm{B}^2 \cup \mathrm{B\Gamma S\Gamma B}^2]\Gamma\mathrm{S}\Gamma(\mathrm{B}^2 \cup \mathrm{B\Gamma S\Gamma B}^2] \\
&= (B^2 \cup B\Gamma S\Gamma B^2]\Gamma(S]\Gamma(B^2 \cup B\Gamma S\Gamma B^2] \\
&\subseteq (B^2\Gamma S\Gamma B^2 \cup B^2\Gamma S\Gamma B\Gamma S\Gamma B^2 \cup B\Gamma S\Gamma B^2\Gamma S\Gamma B^2 \cup B\Gamma S\Gamma B^2\Gamma S\Gamma B\Gamma S\Gamma B^2] \\
&\subseteq (B\Gamma S\Gamma B^2] \\
&\subseteq (B^2 \cup B\Gamma S\Gamma B^2].
\end{aligned}$$

2. Main Results

We now develop ideal theory for po-Γ-semigroups. We begin our study with proving the following Lemma.

2.1. Lemma. The following assertions are equivalent for a subset B of a po-Γ-semigroup S.

(i) B is an ordered (generalized) $(0,2)$-Γ-ideal of S;

(ii) B is an ordered left Γ-ideal of some ordered left Γ-ideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose B is an ordered (generalized) $(0,2)$-Γ-ideal of a po-Γ-semigroup S. Then we obtain $(B \cup S\Gamma B]\Gamma B = (B^2 \cup S\Gamma B^2] \subseteq (B] = B$ and $((B]] = (B]$ and so B is an ordered left Γ-ideal of the ordered left Γ-ideal $(B \cup S\Gamma B]$ of S.
$(ii) \Rightarrow (i)$. Suppose L is an ordered left Γ-ideal of S and B is an ordered left Γ-ideal of L. Then, $S\Gamma B^2 \subseteq S\Gamma L\Gamma B \subseteq L\Gamma B \subseteq B$. Suppose $b \in B$ and $s \in S$ are such that $s \leq b$. As $b \in L$, we get $s \in L$ and so $s \in B$. Consequently, B is an ordered (generalized) $(0,2)$-Γ-ideal of S.

2.2. Theorem. Suppose B is a subset of a po-Γ-semigroup S. Then the following results are equivalent:

(i) B is an ordered (generalized) $(1,2)$-Γ-ideal of S;

(ii) B is an ordered left Γ-ideal of some ordered (generalized) bi-Γ-ideal of S;

(iii) B is an ordered (generalized) bi-Γ-ideal of some left ordered Γ-ideal of S;

(iv) B is an ordered (generalized) $(0,2)$-Γ-ideal of some ordered right Γ-ideal of S;

(v) B is an ordered right-Γ-ideal of some ordered (generalized) $(0,2)$-Γ-ideal of S.

Proof. *(i)* $\Rightarrow$ *(ii)*. Suppose B is an ordered (generalized) $(1,2)$-Γ-ideal of S. This means B is a sub-Γ-semigroup (nonempty subset) of S and $B\Gamma S\Gamma B^2 \subseteq B$. So $(B^2 \cup B\Gamma S\Gamma B^2]\Gamma B = (B^2 \cup B\Gamma S\Gamma B^2]\Gamma(B] \subseteq (B^3 \cup B\Gamma S\Gamma B^3] \subseteq (B^2 \cup B\Gamma S\Gamma B^2] \subseteq (B] = B$. Obviously, if $b \in B$, $s \in (S^2 \cup B\Gamma S\Gamma B^2]$ so that $s \leq b$ then $s \in B$. Hence, B is an ordered left Γ-ideal of the ordered (generalized) bi- Γ-ideal $(B^2 \cup B\Gamma S\Gamma B^2]$ of S.
(ii) $\Rightarrow$ *(iii)*. Suppose B is an ordered left Γ-ideal of some ordered (generalized) bi-Γ-ideal A of S. Recall that $(B \cup S\Gamma B]$ is an ordered left Γ-ideal of S. According to our hypothesis, $B\Gamma(B \cup S\Gamma B]\Gamma B \subseteq (B]\Gamma(B \cup S\Gamma B]\Gamma(B] \subseteq (B^3 \cup B\Gamma S\Gamma B^2] \subseteq (B \cup A\Gamma S\Gamma A\Gamma B] \subseteq (B \cup A\Gamma B] \subseteq (B] = B$. Suppose $b \in B$, $s \in (B \cup S\Gamma B]$ such that $s \leq b$. As $b \in B$, $b \in A$. So $s \in A$ and therefore, $s \in B$. Hence, B is an ordered (generalized) bi-Γ-ideal of the left ordered Γ-ideal $(B \cup S\Gamma B]$ of S.
(iii) $\Rightarrow$ *(iv)*. Suppose B is an ordered (generalized) bi-Γ-ideal of some left ordered Γ-ideal L of S. This implies that $B \subseteq L$, $B\Gamma L^1\Gamma B \subseteq B$ and $S\Gamma L \subseteq L$. Therefore $(B \cup B\Gamma S]\Gamma B^2 \subseteq (B \cup B\Gamma S]\Gamma(B^2] \subseteq (B^3 \cup B\Gamma S\Gamma B^2] \subseteq (B \cup B\Gamma S\Gamma L\Gamma B] \subseteq (B \cup B\Gamma L\Gamma B] \subseteq (B] = B$. Furthermore, suppose that $b \in B$, $s \in (B \cup B\Gamma S]$ such that $s \leq b$, so $b \in L$. Then $s \in L$, therefore $s \in B$. Hence, B is an ordered (generalized) $(0,2)$-Γ-ideal of the ordered right Γ-ideal $(B \cup B\Gamma S]$ of S.
(iv) $\Rightarrow$ *(v)*. Suppose B is an ordered (generalized) $(0,2)$-Γ-ideal of some ordered right Γ-ideal R of S. This implies that $B \subseteq R$, $R\Gamma B^2 \subseteq B$ and $R\Gamma S \subseteq R$. Then $B\Gamma(B \cup S\Gamma B^2] \subseteq (B]\Gamma(B \cup S\Gamma B^2] \subseteq (B^2 \cup B\Gamma S\Gamma B^2] \subseteq (B \cup R\Gamma S\Gamma B^2] \subseteq (B \cup R\Gamma B^2] \subseteq (B] = B$. Let $b \in B$, $s \in (B \cup S\Gamma B^2]$ such that $s \leq b$. Then $b \in R$, so $s \in R$, thus $s \in B$. Hence, B is an ordered right Γ-ideal of the (generalized) $(0,2)$-Γ-ideal $(B \cup S\Gamma B^2]$ of S.
(v) $\Rightarrow$ *(i)*. Suppose B is an ordered right Γ-ideal of an ordered (generalized) $(0,2)$-Γ-ideal R of S. This further implies that $B \subseteq R$, $B\Gamma R \subseteq B$ and $S\Gamma R^2 \subseteq R$. Then $B\Gamma S\Gamma B^2 \subseteq B\Gamma S\Gamma R^2 \subseteq B\Gamma R \subseteq B$. Suppose $b \in B$, $s \in S$ such that $s \leq b$. As $b \in R$, so $s \in B$. Hence B is an ordered (generalized) $(1,2)$-Γ-ideal of S. Hence, B is an ordered (generalized) bi-Γ-ideal of S.

2.3. Lemma. A sub-Γ-semigroup (nonempty subset) A of a po-Γ-semigroup S such that $A = (A]$ is an ordered (generalized) $(1,2)$-Γ-ideal of S if and only if there exists an ordered (generalized) $(0,2)$-Γ-ideal L of S and an ordered right Γ-ideal R of S so that $R\Gamma L^2 \subseteq A \subseteq R \cap L$.

Proof. Suppose A is an ordered (generalized)$(1,2)$-Γ-ideal of S. We know that $(A \cup S\Gamma A^2]$ and $(A \cup A\Gamma S]$ are an ordered (generalized) $(0,2)$-Γ-ideal and an ordered right Γ-ideal of S, respectively. Furthermore, assume $L = (A \cup S\Gamma A^2]$ and $R = (A \cup A\Gamma S]$. Then $R\Gamma L^2 \subseteq (A^3 \cup A^2\Gamma S\Gamma A^2 \cup A\Gamma S\Gamma A^2 \cup A\Gamma S\Gamma A\Gamma S\Gamma A^2] \subseteq (A^3 \cup A\Gamma S\Gamma A^2] \subseteq (A] = A$. Hence, $R \subseteq R \cap L$.

Conversely, suppose R is an ordered right Γ-ideal of S and L is an ordered (generalized) $(0,2)$-Γ-ideal of S so that $R\Gamma L^2 \subseteq A \subseteq R\cap L$. Then $A\Gamma S\Gamma A^2 \subseteq (R\cap L)\Gamma S\Gamma(R\cap L)\Gamma(R\cap L) \subseteq R\Gamma S\Gamma L^2 \subseteq R\Gamma L^2 \subseteq A$. Hence, A is an ordered (generalized) $(1,2)$-Γ-ideal of S.

2.4. Definition. An ordered (generalized) $(0,2)$-bi-Γ-ideal B of S is called 0-minimal if (i) $B \neq \{0\}$ and (ii) $\{0\}$ is the only ordered (generalized) $(0,2)$-bi-Γ-ideal of S properly contained in B.

2.5. Lemma. Suppose L is an ordered 0-minimal left Γ-ideal of a po-Γ-semigroup S with 0 and I is a sub-Γ-semigroup (nonempty subset) of L such that $I = (I]$. Then I is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in L if and only if $(I\Gamma I] = \{0\}$ or $I = L$.

Proof. Suppose I is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in L. As $(S\Gamma I^2]$ is an ordered left Γ-ideal of S and $(S\Gamma I^2] \subseteq I \subseteq L$, we obtain $(S\Gamma I^2] = \{0\}$ or $(S\Gamma I^2] = \{L\}$. If $(S\Gamma I^2] = L$, then $L = (S\Gamma I^2] \subseteq (I]$. So $I = L$. Suppose $(S\Gamma I^2] = \{0\}$. As $S\Gamma(I^2] \subseteq (S\Gamma I^2] = \{0\} \subseteq (I^2]$, then $(I^2]$ is an ordered left Γ-ideal of S contained in L. By the minimality of L, we obtain $(I^2] = \{0\}$ or $(I^2] = L$. If $(I^2] = L$, then $I = L$. Therefore, $I^2 = \{0\}$ or $I = L$.
The converse part is straightforward.

2.6. Lemma. Suppose M is an ordered 0-minimal (generalized) $(0,2)$-Γ-ideal of a po-Γ-semigroup S with a zero 0. Then $(M^2] = \{0\}$ or M is an ordered 0-minimal left Γ-ideal of S.

Proof. As $M^2 \subseteq M$ and $S\Gamma(M^2]^2 = S\Gamma(M^2]\Gamma(M^2] \subseteq (S\Gamma M^2]\Gamma(M^2] \subseteq (M]\Gamma(M^2] \subseteq (M^2]$. Then we obtain $(M^2]$ is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in M. Therefore $(M^2] = \{0\}$ or $(M^2] = M$. Suppose $(M^2] = M$. As $S\Gamma M = S\Gamma(M^2] \subseteq (S\Gamma M^2] \subseteq (M] = M$, it follows that M is an ordered left Γ-ideal of S. Suppose B is an ordered left Γ-ideal of S contained in M. Therefore, $S\Gamma B^2 \subseteq B^2 \subseteq B \subseteq M$. Hence, B is an ordered (generalized) $(0,2)$-Γ-ideal of S contained in M and so $B = \{0\}$ or $B = M$.

2.7. Corollary. Suppose S is a po-Γ-semigroup without a zero 0. Then M is an ordered minimal (generalized) $(0,2)$-Γ-ideal of S if and only if M is an ordered minimal left Γ-ideal of S.

Proof. It follows by Lemma 2.5 and Lemma 2.6.

2.8. Lemma. Suppose S is a po-Γ-semigroup without a zero 0. Further suppose that M is a nonempty subset of S. Then the following results are equivalent:

(i) M is an ordered (generalized) minimal $(2,1)$-Γ-ideal of S;

(ii) M is an ordered (generalized) minimal bi-Γ-ideal of S.

Proof. Suppose S is a po-Γ-semigroup without zero and M is an ordered minimal (generalized) $(2,1)$-Γ-ideal of S. Then $(M^2\Gamma S\Gamma M] \subseteq M$ and so $(M^2\Gamma S\Gamma M]$ is an ordered (generalized) $(2,1)$-Γ-ideal of S. Therefore, we obtain $(M^2\Gamma S\Gamma M] = M$.
As $M\Gamma S\Gamma M = (M^2\Gamma S\Gamma M]\Gamma S\Gamma M \subseteq (M^2\Gamma S\Gamma M\Gamma S\Gamma M] \subseteq (M^2\Gamma S\Gamma M] = M$, we have that M is an ordered (generalized) bi-Γ-ideal of S. Let there exist an ordered (generalized) bi-Γ-ideal B of S contained in M. Then $B^2\Gamma S\Gamma B \subseteq B^2 \subseteq B \subseteq M$, therefore, B is an ordered (generalized) $(2,1)$-Γ-ideal of S contained in M. Applying the minimality of

M, we obtain $B = M$.
Conversely, suppose M is an ordered minimal (generalized) bi-Γ-ideal of S. Then M is an ordered (generalized) $(2,1)$-Γ-ideal of S. Suppose T is an ordered (generalized) $(2,1)$-Γ-ideal of S contained in M. As $(T^2\Gamma S\Gamma T]\Gamma S\Gamma(T^2\Gamma S\Gamma T] \subseteq (T^2\Gamma(S\Gamma T\Gamma S\Gamma T^2\Gamma S)\Gamma T] \subseteq (T^2\Gamma S\Gamma T]$, we obtain $(T^2\Gamma S\Gamma T]$ is an ordered (generalized) bi-Γ-ideal of S. This shows that $(T^2\Gamma S\Gamma T] = M$. As $M = (T^2\Gamma S\Gamma T] \subseteq (T] = T$, $M = T$. Hence, M is an ordered minimal (generalized) $(2,1)$-Γ-ideal of S.

2.9. Definition. A sub-Γ-semigroup (nonempty subset) B of a po-Γ-semigroup S is called an ordered (generalized) $(0,2)$-bi-Γ-ideal of S if B is an ordered (generalized) bi-Γ-ideal of S and also an ordered (generalized) $(0,2)$-Γ-ideal of S.

2.10. Lemma. Suppose B is a subset of a po-Γ-semigroup S. Then the following conditions are equivalent :

(i) B is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S;

(ii) B is an ordered Γ-ideal of some ordered left Γ-ideal of S.

Proof. $(i) \Rightarrow (ii)$. Suppose B is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S. This implies that $B\Gamma S\Gamma B \subseteq B$ and $S\Gamma B^2 \subseteq B$. Then $S\Gamma(B^2 \cup S\Gamma B^2] \subseteq (S\Gamma B^2 \cup S^2\Gamma B^2] \subseteq (S\Gamma B^2] \subseteq (B^2 \cup S\Gamma B^2]$ Therefore, $(B^2 \cup S\Gamma B^2]$ is an ordered left Γ-ideal of S. As $B\Gamma(B^2 \cup S\Gamma B^2] \subseteq (B^3 \cup B\Gamma S\Gamma B^2] \subseteq (B] = B$, $(B^2 \cup S\Gamma B^2]\Gamma B \subseteq (B^3 \cup S\Gamma B^3] \subseteq (B] = B$. Hence B is an ordered Γ-ideal of the left Γ-ideal $(B^2 \cup S\Gamma B^2]$ of S.
$(ii) \Rightarrow (i)$. Suppose B is an ordered Γ-ideal of some ordered left Γ-ideal L of S. By Lemma 2.1, B is an ordered (generalized) $(0,2)$-Γ-ideal of S and hence B is an ordered (generalized) bi-Γ-ideal of S.

2.11. Theorem. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S with a zero 0. Then exactly one of the followings cases arises:

(i) $B = \{0, b\}$, $(b\Gamma S^1\Gamma b] = \{0\}$;

(ii) $B = (\{0, b\}]$, $b^2 = 0$, $(b\Gamma S\Gamma b] = B$;

(iii) $(S\Gamma b^2] = B$ for all $b \in B \setminus \{0\}$.

Proof. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S. Furthermore, suppose $b \in B \setminus \{0\}$. Then $(S\Gamma b^2] \subseteq B$ and $(S\Gamma b\Gamma b]$ is an ordered left Γ-ideal of S, therefore $(S\Gamma b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S. Hence $(S\Gamma b^2] = \{0\}$ or $(S\Gamma b^2] = B$.
Let $(S\Gamma b^2] = \{0\}$. As $b^2 \in B$, we obtain either $b^2 = b$ or $b^2 = 0$ or $b^2 \in B \setminus \{0, b\}$. If $b^2 = b$, then $b = 0$. This is a contradiction. Let $b^2 \in B \setminus \{0, b\}$. Then $S^1\Gamma(\{0, b^2\}]^2 \subseteq (\{0, S\Gamma b^2\}] = (\{0\}] \cup (S\Gamma b^2] = \{0\} \subseteq (\{0\} \cup b^2]$, $(\{0\} \cup b^2]\Gamma S\Gamma(\{0\} \cup b^2] \subseteq (b^2\Gamma S\Gamma b^2] \subseteq (S\Gamma b^2] = \{0\} \subseteq \{0, b^2\}$. So $(\{0\} \cup b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S contained in B and we notice that $(\{0\} \cup b^2] \neq \{0\}$, $(\{0\} \cup b^2] \neq B$. This is too not possible since B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of S. So $b^2 = \{0\}$ and hence by Lemma 2.10, $B = (\{0, b\}]$. Now since we have $(b\Gamma S\Gamma b]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S contained in B, we get $(b\Gamma S\Gamma b] = \{0\}$ or $(b\Gamma S\Gamma b] = B$. So, $(S\Gamma b^2] = \{0\}$ and it follows that either $B = \{0, b\}$ and $(b\Gamma S^1\Gamma b] = \{0\}$ or $B = \{0, b\}$,

$b^2 = \{0\}$ and $(b\Gamma S\Gamma b] = B$. If $(S\Gamma b^2] \neq \{0\}$, then $(S\Gamma b^2] = B$.

2.12. Corollary. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S with a zero 0 so that $(B^2] \neq \{0\}$. Then $B = (S\Gamma b^2]$, for every $b \in B \setminus \{0\}$.

2.13. Definition. A po-Γ-semigroup S with a zero 0 is called 0-$(0,2)$- bisimple if (i) $(S^2] \neq \{0\}$ and $\{0\}$ is the only ordered proper (generalized) $(0,2)$-bi-Γ-ideal of S.

2.14. Corollary. A po-Γ-semigroup S with a zero 0 is 0-$(0,2)$-bisimple if and only if $(S\Gamma s^2] = S$, for every $s \in S \setminus \{0\}$.

Proof. If S is 0-$(0,2)$-bisimple, then $(S\Gamma S] \neq \{0\}$ and S is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal. By Corollary 2.12, we have $S = (S\Gamma s^2]$, for every $s \in S \setminus \{0\}$.
Conversely, suppose $S = (S\Gamma s^2]$, for every element $s \in S \setminus \{0\}$ and further suppose that B is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S such that $B \neq \{0\}$. Suppose $b \in B \setminus \{0\}$. Then $S = (S\Gamma b^2] \subseteq (S\Gamma B^2] \subseteq (B] = B$, therefore, $S = B$. As $S = (S\Gamma b^2] \subseteq (S\Gamma S] = (S^2]$, we obtain $\{0\} \neq S = (S\Gamma S] = (S^2]$. Hence S is 0-$(0,2)$-bisimple.

2.15. Theorem. A po-Γ-semigroup S with a zero 0 is 0-$(0,2)$-bisimple if and only if S is left 0-simple.

Proof. We recall that every ordered left Γ-ideal B of a po-Γ-semigroup S is an ordered 0-$(0,2)$-bi-Γ-ideal of S. So $B = \{0\}$ or $B = S$. Therefore, if S is 0-$(0,2)$-bisimple then S is left 0-simple.
Conversely, if S is left 0-simple then $(S\Gamma s] = S$, for every $s \in S \setminus \{0\}$ from which it follows that $S = (S\Gamma s] = ((S\Gamma s]\Gamma s] \subseteq ((S\Gamma s^2]] = (S\Gamma s^2]$. Therefore, using Corollary 2.14, S is 0-$(0,2)$-bisimple.

2.16. Theorem. Suppose B is an ordered 0-minimal (generalized) $(0,2)$-bi-Γ-ideal of a po-Γ-semigroup S. Then either $(B\Gamma B] = \{0\}$ or B is left 0-simple.

Proof. Suppose $(B\Gamma B] \neq \{0\}$. Then, by Corollary 2.12, we obtain $(S\Gamma b^2] = B$, for every $b \in B \setminus \{0\}$. As $b^2 \in B \setminus \{0\}$, for every $b \in B \setminus \{0\}$, we obtain $b^4 = (b^2)^2 \in B \setminus \{0\}$. Suppose $b \in B \setminus \{0\}$. As $(B\Gamma b^2]\Gamma S^1\Gamma(B\Gamma b^2] \subseteq (B\Gamma B\Gamma b^2] \subseteq (B\Gamma b^2]$ and $S\Gamma(B\Gamma b^2]^2 \subseteq (S\Gamma B\Gamma b^2\Gamma B\Gamma b^2] \subseteq (S\Gamma B^2\Gamma b^2] \subseteq (B\Gamma b^2]$, we get that $(B\Gamma b^2]$ is an ordered (generalized) $(0,2)$-bi-Γ-ideal of S contained in B. Therefore, $(B\Gamma b^2] = \{0\}$ or $(B\Gamma b^2] = B$. As $b^4 \in B\Gamma b^2 \subseteq (B\Gamma b^2]$ and $b^4 \in B \setminus \{0\}$, we obtain $(B\Gamma b^2] = B$. By Corollary 2.14 and Theorem 2.15, it follows that B is left 0-simple.

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