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On admissibility and inadmissibility of estimators after selection under reflected gamma loss function

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Abstract

Let Π_1 and Π_2 denote two gamma populations with common known shape parameter $\alpha > 0$ and unknown scale parameters θ_1 and θ_2, respectively. Let X_1 and X_2 be two independent random variables from Π_1 and Π_2, and $X_{(1)} \leq X_{(2)}$ denote the ordered statistics of X_1 and X_2. Suppose the population corresponding to the largest $X_{(2)}$ or the smallest $X_{(1)}$ observation is selected. This paper concerns on the admissible estimation of the scale parameters θ_M and θ_J of the selected population under reflected gamma loss function. We provide sufficient conditions for the inadmissibility of invariant estimators of θ_M and θ_J. The admissibility and inadmissibility of estimators in the class of linear estimators of the form $cX_{(2)}$ and $dX_{(1)}$ are discussed. We apply our results on k-Records, censored data and extend to a subclass of exponential family.

Keywords: Admissibility; Estimation after selection; Inadmissibility; Invariant estimators; Gamma distribution; Reflected gamma loss function; k-Records data; Type-II censoring.

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1. Introduction

Estimation of the parameter(s) of the selected population is an important estimation problem and arises in various practical problems. For example, we wish to select the treatment with the highest cure rate and then estimate the actual probability of success with this treatment, see Tappin [30]. A car manufacturer, who has selected the most reliable model of engine for his cars, would like to know the reliability of the selected engine during actual use, see Kumar and Kar [12]. A textile designer chooses the best quality cloth from k available varieties for his usage. Naturally, he would be interested in estimating the durability of the best cloth that he has selected, see Gangopadhyay and Kumar [10].

The problem of estimation after selection is related to ranking and selection methodology. Readers may refer to Sarkadi [28], Dahiya [9], Sackrowitz and Cahn [26,27], Misra and Singh [17], Kumar and Kar [12], Balakrishnan et al. [5] and Kumar et al. [14].

During the past three decades, a lot of work has been done on estimation after selection from Gamma populations. Some of the main results are as follows: For positive integer value shape parameter, Vellaisamy and Sharma [35] derived the UMVUE of the scale parameter of the larger selected population and obtained estimators which are admissible (or inadmissible) within a subclass of equivariant estimators under the Squared Error Loss (SEL) function. Some of their results were extended to real valued shape parameter by Vellaisamy and Sharma [36]. Later, Vellaisamy [33] obtained estimators which dominates natural estimators under the SEL function. Vellaisamy [34] showed that the UMVUE of the selected scale parameters are inadmissible under the SEL function. Misra et al. [18,19] extended the admissibility and inadmissibility results of Vellaisamy and Sharma [35] to the case of known and arbitrary shape parameter. Motamed-Shariati and Nematollahi [20] derived the minimax estimator of the scale parameter of the larger selected population under the Scale Invariant Squared Error Loss (SISEL) function. Nematollahi and Motamed-Shariati [22] dealt with estimating the scale parameter of the selected gamma population under the entropy loss function and extended their results to a subclass of exponential family.

Let X_1 and X_2 be two independent random variables from populations Π_1 and Π_2, respectively, where Π_i has probability density function (pdf)

$$(1.1)\qquad f(x|\theta_i,\alpha) = \frac{1}{\theta_i^{\alpha}\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\frac{x}{\theta_i}},\ \ x>0,\ \ \alpha>0,\ \ \theta_i>0,\ \ i=1,2,$$

where α is the common known shape parameter and θ_1, θ_2 are unknown scale parameters. Let $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$. For selecting the population corresponding to the larger (or smaller) θ_i's, we use the natural selection rule and select the population corresponding to the $X_{(2)}$ (or $X_{(1)}$). Therefore the scale parameter associated with the larger and smaller selected population are given by

$$\theta_M = \begin{cases} \theta_1 & X_1 \geq X_2 \\ \theta_2 & X_1 < X_2 \end{cases} \quad \textit{and} \quad \theta_J = \begin{cases} \theta_2 & X_1 \geq X_2 \\ \theta_1 & X_1 < X_2. \end{cases}$$

In literature the estimation of the selected gamma scale parameters θ_M and θ_J considered under SEL, SISEL and entropy loss functions, which are either symmetric or asymmetric and unbounded. In some estimation problems, the use of unbounded loss function may be inappropriate. For example in estimating the mean life θ of a component, the amount of loss for estimating θ by an estimator is essentially bounded.

For estimation of the parameter of the selected population under a bounded loss function, see Naghizadeh Qomi et al. [21]. They estimate the mean of the selected

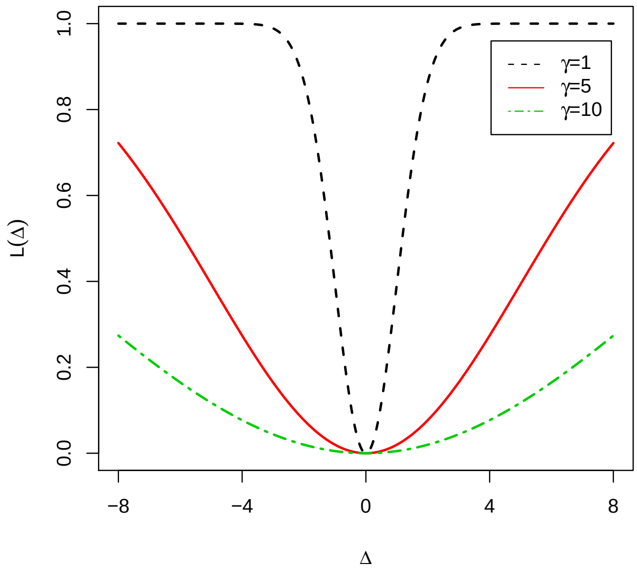

Figure 1. Plot of the RNL function for $k = 1$ and certain values of γ

normal population under Reflected Normal Loss (RNL) function given by

$$L(\Delta) = k\left\{1 - e^{-\frac{\Delta^2}{2\gamma^2}}\right\},$$

where $\Delta = \delta - \theta$, $k > 0$ is the maximum loss and $\gamma > 0$ is a shape parameter. The RNL function is a symmetric and bounded function of Δ (see Figure 1).

Although the RNL function is bounded, but it is symmetric and give the same penalty for underestimation and overestimation. Also, it is appropriate for location parameter θ. In some estimation problem, underestimation may be more serious than overestimation or vise versa. For example, in estimating the average life of the components of an aircraft, overestimation is usually more serious than underestimation. In such cases, for estimation the average life, which is a multiple of a scale parameter, an asymmetric bounded scale invariant loss function is appropriate to use.

In this paper, we discuss the estimation of the scale parameter of the selected gamma population under Reflected Gamma Loss (RGL) function. The RGL function is a simple transformation of the gamma density and introduced by Spiring and Yeung [29] in response to the criticisms of the SISEL function and is defined by

$$L(\theta, \delta) = k\left\{1 - \left(\frac{\delta}{\theta}\right)^{\gamma} e^{-\gamma(\frac{\delta}{\theta}-1)}\right\} = k\left\{1 - e^{-\gamma\left(\frac{\delta}{\theta}-\ln\frac{\delta}{\theta}-1\right)}\right\} \tag{1.2}$$

where $k > 0$ is the maximum loss and $\gamma > 0$ is a shape parameter. The RGL function is a bounded and asymmetric function of δ but not convex in δ and is essentially a gamma density flipped upside down, whence its name (see Figure 2). This loss is scale invariant, which is appropriate for estimating scale parameter θ, and it penalizes heavily underestimation. Towhidi and Behboodian [31,32] used this loss in some problem of scale

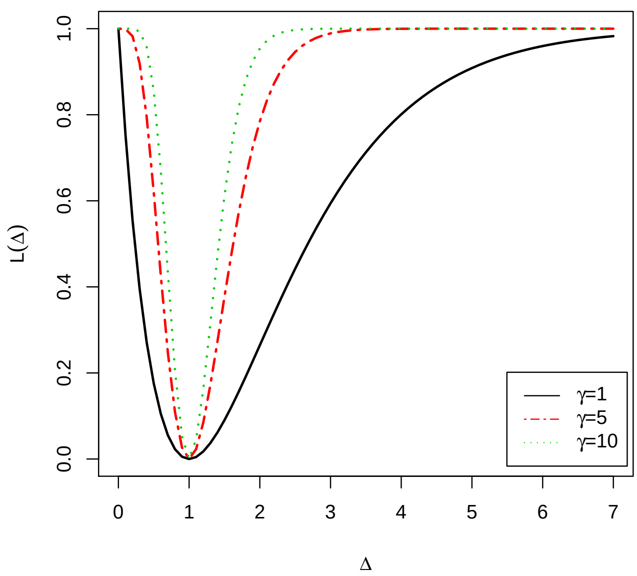

Figure 2. Plot of the RGL function for $\Delta = \frac{\delta}{\theta}$, $k = 1$ and certain values of γ

parameter estimation. Clearly the value of $k > 0$ does not have any influence on our results, therefore without loss of generality, we shall take $k = 1$ in the rest of the paper.

Since the RGL function is bounded, so by a result of Basu [6], Uniformly Minimum Risk Unbiased estimator of any unknown parameter does not exist under the RGL function. We are interested in estimation of the random parameters θ_M and θ_J of the selected gamma population under the RGL function and we concentrate our attention on admissible and inadmissible estimators of θ_M and θ_J. To this end, in section 2, we employ the technique of Brewster and Zidek [7] for finding dominating estimators for some intended scale and permutation invariant estimators. In section 3, we discuss the admissibility of invariant estimators of the form $cX_{(2)}$ and $dX_{(1)}$ for estimating θ_M and θ_J, respectively. In section 4, applications on k-records, censored data and extension of the problem to a subclass of exponential family are considered.

2. Sufficient Conditions for Inadmissibility

Let X_1 and X_2 be two independent random variables, where $X_i,\ i = 1, 2$ has pdf (1.1). In estimation of unknown random parameters θ_M and θ_J under the RGL function, the problem is invariant under the scale and permutation groups of transformations. Therefore, it is natural to consider only those estimators which are permutation and scale invariant, i.e, estimators satisfying $\delta(X_1, X_2) = \delta(X_2, X_1)$ and $\delta(cX_1, cX_2) = c\delta(X_1, X_2),\ \forall c > 0$. For this purpose, consider the following classes of invariant estimators

$$D_U = \{\delta_\psi : \delta_\psi(X_1, X_2) = X_{(2)}\psi(Y)\}, \tag{2.1}$$

and

$$D_L = \{\delta_\varphi : \delta_\varphi(X_1, X_2) = X_{(1)}\varphi(T)\}, \tag{2.2}$$

for θ_M and θ_J respectively, where $Y = \frac{X_{(1)}}{X_{(2)}}$, $T = \frac{1}{Y}$ and ψ and φ are some real valued functions defined on $(0, 1]$ and $[1, \infty)$, respectively. In this section, we will employ the technique of Brewster and Zidek [7] to derive dominating estimators to show the inadmissibility of some scale and permutation invariant estimators of θ_M and θ_J, under the RGL function. As a consequence, we show that several of the proposed estimators are inadmissible and present improved estimators for those.

The following lemma will be useful in deriving the improved estimators on estimating θ_M and θ_J.

2.1. Lemma *Let $Y = \frac{X_{(1)}}{X_{(2)}}$, $T = \frac{1}{Y}$, $\mu = \frac{\max(\theta_1,\theta_2)}{\min(\theta_1,\theta_2)}$ and ψ and φ are real valued functions defined on $(0, 1]$ and $[1, \infty)$, respectively. Define the functions $\eta_{y,\psi}(\mu)$ and $\xi_{t,\varphi}(\mu)$ as*

$$\eta_{y,\psi}(\mu) = (2\alpha + \gamma)\, \frac{\left[\frac{\mu}{(1+\gamma\psi(y))\mu+y}\right]^{2\alpha+\gamma+1} + \frac{1}{\mu^{\gamma+1}}\left[\frac{\mu}{1+\gamma\psi(y)+\mu y}\right]^{2\alpha+\gamma+1}}{\left[\frac{\mu}{(1+\gamma\psi(y))\mu+y}\right]^{2\alpha+\gamma} + \frac{1}{\mu^{\gamma}}\left[\frac{\mu}{1+\gamma\psi(y)+\mu y}\right]^{2\alpha+\gamma}}, \quad 0 < y \le 1,\ \mu \ge 1,$$

and

$$\xi_{t,\varphi}(\mu) = (2\alpha + \gamma)\, \frac{\left[\frac{\mu}{(1+\gamma\varphi(t))\mu+t}\right]^{2\alpha+\gamma+1} + \frac{1}{\mu^{\gamma+1}}\left[\frac{\mu}{1+\gamma\varphi(t)+\mu t}\right]^{2\alpha+\gamma+1}}{\left[\frac{\mu}{(1+\gamma\varphi(t))\mu+t}\right]^{2\alpha+\gamma} + \frac{1}{\mu^{\gamma}}\left[\frac{\mu}{1+\gamma\varphi(t)+\mu t}\right]^{2\alpha+\gamma}}, \quad t \ge 1,\ \mu \ge 1.$$

(i) *For $y \in (0, 1]$, the conditional pdf of $S = \frac{X_{(2)}}{\theta_M}$ given $Y = y$ is*

$$f_{S|Y=y}(s) = \frac{y^{\alpha-1}s^{2\alpha-1}}{\Gamma^2(\alpha)f_Y(y)}\left[\mu^{-\alpha}e^{-(\frac{y}{\mu}+1)s} + \mu^{\alpha}e^{-(1+\mu y)s}\right], \quad s > 0,$$

where $f_Y(y)$ denotes the pdf of Y.

(ii) *For $t \in [1, \infty)$, the conditional pdf of $U = \frac{X_{(1)}}{\theta_J}$ given $T = t$ is*

$$f_{U|T=t}(u) = \frac{t^{\alpha-1}u^{2\alpha-1}}{\Gamma^2(\alpha)f_T(t)}\left[\mu^{-\alpha}e^{-(\frac{t}{\mu}+1)u} + \mu^{\alpha}e^{-(1+\mu t)u}\right], \quad u > 0,$$

where $f_T(t)$ denotes the pdf of T.

(iii) *For $y \in (0, 1]$*

$$\sup_{\mu\ge1} \eta_{y,\psi}(\mu) = \frac{2\alpha + \gamma}{1 + \gamma\psi(y)} = \frac{1}{\psi^\star(y)}. \tag{2.3}$$

(iv) *For $t \in [1, \infty]$*

$$\sup_{\mu\ge1} \xi_{t,\varphi}(\mu) = \frac{2\alpha + \gamma}{1 + \gamma\varphi(t)} = \frac{1}{\varphi^\star(t)}. \tag{2.4}$$

Proof. **(i),(ii)** For a proof, see Lemma 16(i) and 16(ii) of Misra et al. [18].

(iii) Note that

$$\lim_{\mu\uparrow\infty} \eta_{y,\psi}(\mu) = \frac{2\alpha + \gamma}{1 + \gamma\psi(y)}.$$

So, we need to show that $\eta_{y,\psi}(\mu) \leq \frac{2\alpha+\gamma}{1+\gamma\psi(y)}$. But this inequality is equivalent to:

$$
\begin{aligned}
&[1+\gamma\psi(y)]\eta_{y,\psi}(\mu) \leq (2\alpha+\gamma)\\
\Leftrightarrow \quad & [1+\gamma\psi(y)]\mu^{\gamma+1}\big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma+1)}\\
&\qquad +[1+\gamma\psi(y)]\big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma+1)}\\
&\quad \leq \mu^{\gamma}\big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma)}+\big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma)}\\
\Leftrightarrow \quad & \big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma+1)}\Big\{\mu^{\gamma+1}[1+\gamma\psi(y)]-\mu^{\gamma}\big\{[1+\gamma\psi(y)]\mu+y\big\}\Big\}\\
+ \quad & \big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma+1)}\Big\{1+\gamma\psi(y)-[1+\gamma\psi(y)+y\mu]\Big\} \leq 0\\
\Leftrightarrow \quad & -y\mu^{\gamma}\big\{[1+\gamma\psi(y)]\mu+y\big\}^{-(2\alpha+\gamma+1)}-y\mu\big\{1+\gamma\psi(y)+y\mu\big\}^{-(2\alpha+\gamma+1)} \leq 0
\end{aligned}
$$

which is always true for $\mu \geq 1$ and $y \in (0,1]$. So, the result follows.

(iv) Similar to the proof of (**iii**).

The following theorem provides a sufficient condition for invariant estimators $\delta_\psi(X_1, X_2) \in D_U$ to be inadmissible under the RGL function.

2.2. Theorem *Let $\delta_\psi(X_1,X_2) \in D_U$ be an invariant estimator of θ_M, $\psi_{11}(y)$ be any function defined on $(0,1]$ such that $P_\theta\big(\psi(Y) < \psi_{11}(Y) \leq \psi^\star(Y)\big) > 0$, $\forall\theta = (\theta_1,\theta_2) \in (0,\infty)\times(0,\infty) = \Re^2_+$. Then under the RGL function, the invariant estimator δ_ψ is inadmissible for estimating θ_M, and is dominated by $\delta_{\psi_1}(X_1,X_2) = X_{(2)}\psi_1(Y)$, where*

$$
\psi_1(Y) \quad = \quad \begin{cases} \psi_{11}(Y) & \psi(Y) < \psi_{11}(Y) \leq \psi^\star(Y) \\ \psi(Y) & o.w. \end{cases}
$$

Proof. For $\mu \geq 1$, the risk difference of δ_ψ and δ_{ψ_1} is

$$
\begin{aligned}
\Delta(\mu) &= R(\theta_M,\delta_\psi) - R(\theta_M,\delta_{\psi_1})\\
&= E_\theta\left[e^{-\gamma\left(\frac{X_{(2)}\psi_1(Y)}{\theta_M} - \ln\frac{X_{(2)}\psi_1(Y)}{\theta_M} - 1\right)}\right] - E_\theta\left[e^{-\gamma\left(\frac{X_{(2)}\psi(Y)}{\theta_M} - \ln\frac{X_{(2)}\psi(Y)}{\theta_M} - 1\right)}\right]\\
&= e^\gamma E_\theta\left[e^{-\gamma\left(S\psi_1(Y)-\ln S\psi_1(Y)\right)} - e^{-\gamma\left(S\psi(Y)-\ln S\psi(Y)\right)}\right]\\
&= e^\gamma E_\theta[D_\theta(Y)],
\end{aligned}
$$

where

$$
D_\theta(y) = E_\theta\left[e^{-\gamma\left(S\psi_1(y)-\ln S\psi_1(y)\right)} - e^{-\gamma\left(S\psi(y)-\ln S\psi(y)\right)}|Y=y\right], \qquad y \in (0,1].
$$

Using the fact that $e^a - e^b \geq (a-b)e^b$, $\forall a,b \in \Re$, we have

$$
\begin{aligned}
D_\theta(y) \quad &\geq \quad \gamma E_\theta\Big\{\big[(S\psi(y)-\ln S\psi(y)) - (S\psi_1(y)-\ln S\psi_1(y))\big]\\
&\qquad\qquad \times e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\Big\}\\
&= \quad \gamma(\psi(y)-\psi_1(y))\; E_\theta\big\{e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\big\}
\end{aligned}
$$

$$\times\left[\frac{\ln\psi_1(y)-\ln\psi(y)}{\psi(y)-\psi_1(y)}+\frac{E_\theta\{Se^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\}}{E_\theta\{e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\}}\right]. \tag{2.5}$$

Let $K(y,\mu)=E_\theta\{e^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\}$, then from Lemma 2.1(i), we have

$$\begin{aligned} K(y,\mu) &= [\psi(y)]^\gamma\int_0^\infty s^\gamma e^{-\gamma\psi(y)s}f_{S|Y=y}(s)ds \\ &= \frac{[\psi(y)]^\gamma y^{\alpha-1}\Gamma(2\alpha+\gamma)\mu^\alpha}{\Gamma^2(\alpha)f_Y(y)} \\ &\quad\times\left[\frac{\mu^\gamma}{[(1+\gamma\psi(y))\mu+y]^{2\alpha+\gamma}}+\frac{1}{[1+\gamma\psi(y)+y\mu]^{2\alpha+\gamma}}\right] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &E_\theta\{Se^{-\gamma(S\psi(y)-\ln S\psi(y))}|Y=y\} \\ &\quad= [\psi(y)]^\gamma\int_0^\infty s^{\gamma+1} e^{-\gamma\psi(y)s}f_{S|Y=y}(s)ds \\ &\quad= \frac{[\psi(y)]^\gamma y^{\alpha-1}\Gamma(2\alpha+\gamma+1)\mu^\alpha}{\Gamma^2(\alpha)f_Y(y)} \\ &\quad\quad\times\left[\frac{\mu^{\gamma+1}}{[(1+\gamma\psi(y))\mu+y]^{2\alpha+\gamma+1}}+\frac{1}{[1+\gamma\psi(y)+y\mu]^{2\alpha+\gamma+1}}\right]. \end{aligned} \tag{2.7}$$

Now, substituting (2.6) and (2.7) in (2.5), we have

$$D_\theta(y)\geq\gamma K(y,\mu)(\psi(y)-\psi_1(y))\left[\frac{\ln\frac{\psi_1(y)}{\psi(y)}}{\psi(y)-\psi_1(y)}+\eta_{y,\psi}(\mu)\right],$$

where $\eta_{y,\psi}(\mu)$ is defined in Lemma 2.1. Clearly, if the condition $\psi(y)<\psi_{11}(y)\leq\psi^\star(y)$ does not hold, then $D_\theta(y)=0,\ \forall\theta\in\Re_+^2$ and $\forall y\in(0,1]$. For $\psi(y)<\psi_{11}(y)\leq\psi^\star(y)$, using (2.3) and the inequality $\ln a>1-\frac{1}{a},\ \forall a>0$, we have

$$D_\theta(y)\geq\gamma K(y,\mu)(\psi(y)-\psi_{11}(y))\left[\frac{\ln\frac{\psi_{11}(y)}{\psi(y)}}{\psi(y)-\psi_{11}(y)}+\frac{1}{\psi^\star(y)}\right]>0,\quad\forall\theta\in\Re_+^2 \textit{ and } \forall y\in(0,1].$$

Since $P_\theta(\psi(Y)<\psi_{11}(Y)\leq\psi^\star(Y))>0,\ \forall\theta\in\Re_+^2$, it follows that $\Delta(\mu)>0,\ \forall\theta\in\Re_+^2$.

2.3. Remark *In Theorem 2.2 we have the condition $\psi(Y)<\psi_{11}(Y)\leq\psi^\star(Y)$ with positive probability. So, $P_\theta(\psi(Y)<\psi^\star(Y))=P_\theta(\psi(Y)<\frac{1}{2\alpha})>0$. Therefore, a necessary condition on the function $\psi(Y)$ for Theorem 2.2 to actually offer an improved estimator (i.e., $\psi_1(Y)$ is different than $\psi(Y)$ with positive probability) is that $P_\theta(\psi(Y)<\frac{1}{2\alpha})>0$.*

The following Corollary is an immediate consequence of the Theorem 2.2.

2.4. Corollary *Let $\delta_\psi(X_1,X_2)\in D_U$ be an invariant estimator of θ_M such that $P_\theta(\psi(Y)<\frac{1}{2\alpha})>0$. If for some function $\psi^{\star\star}(Y)$, $P_\theta\big(\psi^{\star\star}(Y)\leq\psi(Y)<\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma}\big)>0,\ \forall\theta\in\Re_+^2$, then under the RGL function, the invariant estimator δ_ψ is inadmissible for estimating θ_M, and is dominated by $\delta_{\psi_1^\star}(X_1,X_2)=X_{(2)}\psi_1^\star(Y)$, where*

$$\psi_1^\star(Y)=\begin{cases}\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma} & \psi^{\star\star}(Y)\leq\psi(Y)<\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma}\\ \psi(Y) & o.w.\end{cases}$$

Proof. Use Theorem 2.2 with $\psi_{11}(Y)=\frac{1+\gamma\psi^{\star\star}(Y)}{2\alpha+\gamma}\leq\psi^\star(Y)$.

2.5. Corollary *Let $\delta_\psi(X_1, X_2) \in D_U$ be an invariant estimator of θ_M such that $P_\theta(\psi(Y) < \frac{1}{2\alpha}) > 0$. Define*

$$\psi_1(Y) = \begin{cases} \frac{1+\gamma\psi(Y)}{2\alpha+\gamma} & \psi(Y) < \frac{1}{2\alpha} \\ \psi(Y) & o.w. \end{cases}$$

Then the estimator $\delta_{\psi_1}(X_1, X_2) = X_{(2)}\psi_1(Y)$ dominates $\delta_\psi(X_1, X_2)$.

Proof. Apply Corollary 2.4 with $\psi^{\star\star}(Y) = \psi(Y)$.

2.6. Remark *Consider the following mixed estimators of θ_M*

$$\begin{aligned} \delta_{p,\psi}(X_1, X_2) &= pX_{(2)} + (1-p)X_{(1)} \\ &= X_{(2)}[p + (1-p)Y] \end{aligned}$$

where $p \geq 0$. Following the Corollary 2.4 and taking $\psi^{\star\star}(y) = \frac{y}{2\alpha}$ for $\alpha > \frac{1}{2}$, this estimator is inadmissible and is dominated by

$$\delta^\star_{p,\psi}(X_1, X_2) = \begin{cases} \frac{2\alpha X_{(2)} + \gamma X_{(1)}}{2\alpha(2\alpha+\gamma)} & \frac{Y}{2\alpha} < p + (1-p)Y < \frac{2\alpha+\gamma Y}{2\alpha(2\alpha+\gamma)} \\ \delta_{p,\psi}(X_1, X_2) & o.w. \end{cases}$$

Also, using Corollary 2.5 one can get another improved estimator of $\delta_{p,\psi}(X_1, X_2)$, which is given by

$$\delta_{p,\psi_1}(X_1, X_2) = \begin{cases} \frac{X_{(2)} + \gamma[pX_{(2)} + (1-p)X_{(1)}]}{2\alpha+\gamma} & p + (1-p)Y < \frac{1}{2\alpha} \\ \delta_{p,\psi}(X_1, X_2) & o.w. \end{cases}$$

The following Theorem gives a sufficient condition for inadmissibility of invariant estimators δ_φ in D_L under the RGL function.

2.7. Theorem *Let $\delta_\varphi(X_1, X_2) \in D_L$ be an invariant estimator of θ_J, $\varphi_{11}(t)$ be any function defined on $[1, \infty)$ such that $P_\theta\big(\varphi(T) < \varphi_{11}(T) \leq \varphi^\star(T)\big) > 0$, $\forall \theta \in \Re^2_+$. Then under the RGL function, the invariant estimator δ_φ is inadmissible for estimating θ_J, and is dominated by $\delta_{\varphi_1}(X_1, X_2) = X_{(1)}\varphi_1(T)$, where*

$$\varphi_1(T) = \begin{cases} \varphi_{11}(T) & \varphi(T) < \varphi_{11}(T) \leq \varphi^\star(T) \\ \varphi(T) & o.w. \end{cases}$$

Proof. The proof is similar to the proof of Theorem 2.2 by replacing $Y, \psi, \psi^\star, \psi_1$ and ψ_{11} by $T, \varphi, \varphi^\star, \varphi_1$ and φ_{11}, respectively.

2.8. Remark *In Theorem 2.7 we have the condition $\varphi(T) < \varphi_{11}(T) \leq \varphi^\star(T)$ with positive probability. So, $P_\theta(\varphi(T) < \varphi^\star(T)) = P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$. Therefore, a necessary condition on the function $\varphi(T)$ for Theorem 2.7 to actually offer an improved estimator (i.e., $\varphi_1(T)$ is different than $\varphi(T)$ with positive probability) is that $P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$.*

The following Corollary is an immediate consequence of the Theorem 2.7.

2.9. Corollary *Let $\delta_\varphi(X_1, X_2) \in D_L$ be an invariant estimator of θ_J such that $P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$. If for some function $\varphi^{\star\star}(T)$, $P_\theta\big(\varphi^{\star\star}(t) \leq \varphi(T) < \frac{1+\gamma\varphi^{\star\star}(t)}{2\alpha+\gamma}\big) >$*

0, $\forall \theta \in \Re_+^2$, *then under the RGL function, the invariant estimator* δ_φ *is inadmissible for estimating* θ_J, *and is dominated by* $\delta_{\varphi_1^\star}(X_1, X_2) = X_{(1)}\varphi_1^\star(T)$, *where*

$$\varphi_1^\star(T) \quad = \quad \begin{cases} \frac{1+\gamma\varphi^{\star\star}(T)}{2\alpha+\gamma} & \varphi^{\star\star}(T) \leq \varphi(T) < \frac{1+\gamma\varphi^{\star\star}(T)}{2\alpha+\gamma} \\ \\ \varphi(T) & o.w. \end{cases}$$

Proof. Use Theorem 2.7 with $\varphi_{11}(T) = \frac{1+\gamma\varphi^{\star\star}(T)}{2\alpha+\gamma} \leq \varphi^\star(T)$.

2.10. Corollary *Let* $\delta_\varphi(X_1, X_2) \in D_L$ *be an invariant estimator of* θ_J *such that* $P_\theta(\varphi(T) < \frac{1}{2\alpha}) > 0$. *Define*

$$\varphi_1(T) \quad = \quad \begin{cases} \frac{1+\gamma\varphi(T)}{2\alpha+\gamma} & \varphi(T) < \frac{1}{2\alpha} \\ \\ \varphi(T) & o.w. \end{cases}$$

Then the estimator $\delta_{\varphi_1}(X_1, X_2) = X_{(2)}\varphi_1(T)$ *dominates* $\delta_\varphi(X_1, X_2)$.

Proof. Apply Corollary 2.9 with $\varphi^{\star\star}(T) = \varphi(T)$.

2.11. Remark *Consider the following mixed estimators of* θ_J

$$\begin{aligned} \delta_{p,\varphi}(X_1, X_2) &= pX_{(1)} + (1-p)X_{(2)} \\ &= X_{(1)}[1 + (1-p)(T-1)] \end{aligned}$$

where $p \geq 0$. *Following the Corollary 2.9 and taking* $\varphi^{\star\star}(t) = 1$ *for* $\alpha < \frac{1}{2}$, *this estimator is inadmissible and is dominated by*

$$\delta_{p,\varphi}^\star(X_1, X_2) \quad = \quad \begin{cases} \frac{1+\gamma}{2\alpha+\gamma}X_{(1)} & 1 \leq p + (1-p)T < \frac{1+\gamma}{2\alpha+\gamma} \\ \\ \delta_{p,\varphi}(X_1, X_2) & o.w. \end{cases}$$

Also, using Corollary 2.10 we get another improved estimator of $\delta_{p,\varphi}(X_1, X_2)$, *which is given by*

$$\delta_{p,\varphi_1}(X_1, X_2) \quad = \quad \begin{cases} \frac{X_{(1)}+\gamma[pX_{(1)}+(1-p)X_{(2)}]}{2\alpha+\gamma} & p + (1-p)T < \frac{1}{2\alpha} \\ \\ \delta_{p,\varphi}(X_1, X_2) & o.w. \end{cases}$$

3. Discussion on Admissible Estimators

An important problem in estimation of θ_M and θ_J in the family of scale distributions, is to determine admissible estimators of the form $cX_{(2)}$ and $dX_{(1)}$ in the class of scale invariant estimators of the form

$$D_{1U} = \{\delta_{1c} : \delta_{1c}(X_1, X_2) = cX_{(2)},\ c > 0\} \tag{3.1}$$

and

$$D_{1L} = \{\delta_{2d} : \delta_{2d}(X_1, X_2) = dX_{(1)},\ d > 0\}, \tag{3.2}$$

respectively. In this section, we discuss the admissibility of δ_{1c} and δ_{2d} within the subclass D_{1U} and D_{1L}, respectively under the RGL function.

The following lemma will be useful in characterization of admissible estimators of θ_M and θ_J within the subclasses D_{1U} and D_{1L}, respectively, under the RGL function.

Table 1. Values of $c^\star(1,\gamma)$ for $\alpha = 1,2,4$ and certain values of γ

α	γ					
	0.25	0.5	0.75	1	5	10
1	0.6693	0.6714	0.6732	0.6747	0.6843	0.6874
2	0.3643	0.3650	0.3655	0.3660	0.3702	0.3721
4	0.1965	0.1966	0.1968	0.1969	0.1983	0.1992

3.1. Lemma *Let X_1 and X_2 be two independent random variables such that X_i, $i = 1,2$ has pdf (1.1) and $X_{(1)} \le X_{(2)}$ be the ordered statistics of X_1 and X_2.*
(i) *If $S = \frac{X_{(2)}}{\theta_M}$, then the pdf of S is*

$$f_S(s) = \left[F_\alpha(\mu s) + F_\alpha\left(\frac{s}{\mu}\right)\right] f_\alpha(s), \quad s > 0, \tag{3.3}$$

where $\mu = \frac{\max(\theta_1,\theta_2)}{\min(\theta_1,\theta_2)} \ge 1$, F_α and f_α denote the cumulative distribution function (cdf) and the pdf of Gamma$(\alpha,1)$-distribution, respectively.
(ii) *If $U = \frac{X_{(1)}}{\theta_J}$, then the pdf of U is given by*

$$f_U(u) = \left[2 - F_\alpha(\mu u) - F_\alpha\left(\frac{u}{\mu}\right)\right] f_\alpha(u), \quad u > 0. \tag{3.4}$$

Proof. For a proof, see Lemma 7(i) and 7(ii) of Misra et al. [18].

3.1. Admissibility of δ_{1c}. For deriving admissible estimators of θ_M in the class of invariant estimators (3.1), we find values of c that minimizes the risk function $\delta_{1c} = cX_{(2)}$ which is

$$\begin{aligned} R(\theta_M, \delta_{1c}) &= 1 - E\left[e^{-\gamma(\frac{cX_{(2)}}{\theta_M} - \ln\frac{cX_{(2)}}{\theta_M} - 1)}\right] \\ &= 1 - E\left[e^{-\gamma(cS - \ln cS - 1)}\right], \end{aligned} \tag{3.5}$$

where $S = \frac{X_{(2)}}{\theta_M}$. The risk function (3.5) is not necessarily convex, but has a unique minimum w.r.t. c. Figure 3 shows the graph of the risk function as a function of c for certain values of μ, $\gamma = 1$ and $\alpha = 1,2,4$. It seems that the minimum point c, which depends on the values of μ and γ, of the risk function is near to α^{-1} when μ gets larger and larger. Therefore $R(\theta_M, cX_{(2)})$ will be minimized at the point c given by $\frac{\partial R(\theta_M,\delta_{1c})}{\partial c} = 0$ which reduces to

$$E\left[\left(S - \frac{1}{c(\mu,\gamma)}\right) e^{-\gamma(c(\mu,\gamma)S - \ln(c(\mu,\gamma)S) - 1)}\right] = 0 \tag{3.6}$$

The behavior of $c(\mu,\gamma)$ can not be determined analytically. The graph of $c(\mu,\gamma)$ as a function of $\mu \ge 1$ for $\alpha = 1,2,4$ and certain values of γ are shown in Figure 4. It is seen from Figure 4 (and also from numerical solution of equation (3.6)) that for fixed γ, $c(\mu,\gamma)$ increases as μ increases and $c(\mu,\gamma) \to \alpha^{-1}$ as $\mu \to \infty$. So

$$\inf_{\mu\ge 1} c(\mu,\gamma) = c(1,\gamma) = c^\star \quad \textit{and} \quad \sup_{\mu\ge 1} c(\mu,\gamma) = \lim_{\mu\to\infty} c(\mu,\gamma) = \alpha^{-1}.$$

where $c^\star = c^\star(1,\gamma)$ is the root of equation (3.6) with $\mu = 1$. Table 1 shows the root $c^\star = c^\star(1,\gamma)$ for certain values of $\gamma > 0$. Therefore for each $c \in [c^\star, \alpha^{-1}]$ there is a μ for which $R(\theta_M, \delta_{1c})$ is minimum, which implies that for $c \in [c^\star, \alpha^{-1}]$, δ_{1c} is admissible in the class of estimators (3.1). So, we have the following conjecture.

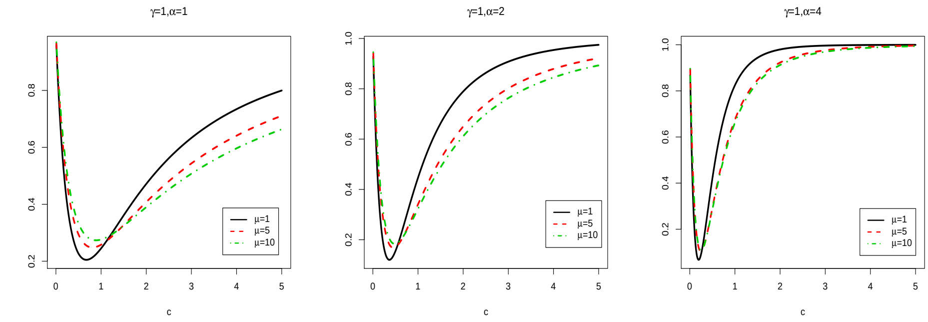

Figure 3. Plots of risk function for $\gamma = 1,\ \alpha = 1, 2, 4$ and certain values of μ

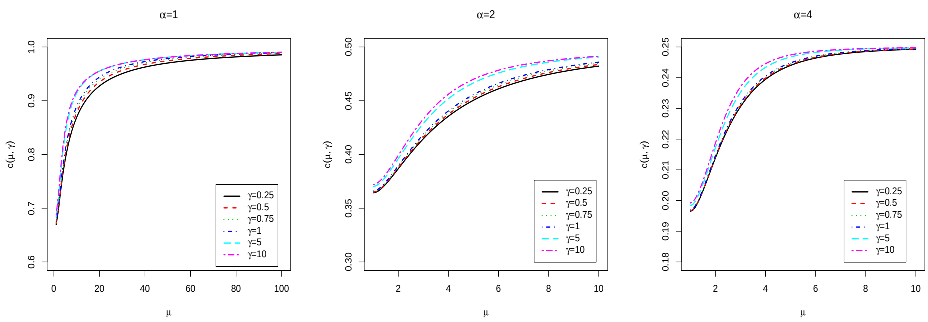

Figure 4. Graph of $c(\mu, \gamma)$ for $\alpha = 1, 2, 4$ and certain values of γ

3.2. Conjecture *Let $c^\star$ be the root of equation (3.6) with $\mu = 1$. Then, under the RGL function, the estimators $\delta_{1c}(X_1, X_2) = cX_{(2)}$ are admissible within the subclass D_{1U} of invariant estimators of θ_M, if and only if $c \in [c^\star, \alpha^{-1}]$.*

3.3. Remark *From Corollary 2.5 the estimator $\delta_{1c}(X_1, X_2) = cX_{(2)}$ for $c < \frac{1}{2\alpha}$ is inadmissible and is dominated by $\delta_1(X_1, X_2) = \frac{1+\gamma c}{2\alpha+\gamma} X_{(2)}$. Note that from Table 1, $c^\star > \frac{1}{2\alpha}$ for certain values of γ, which satisfy the condition of Conjecture 3.2.*

3.4. Remark *Based on the Conjecture 3.2, the natural and generalized Bayes estimator $\frac{X_{(2)}}{\alpha}$ of θ_M, which is the analog of the maximum likelihood and best scale invariant estimators of θ_2, is admissible within the subclass D_{1U} of invariant estimators of θ_M.*

3.2. Admissibility of δ_{2c}. Similarly, the risk function of $\delta_{2d} = dX_{(1)}$ as an estimator of θ_J has a unique minimum w.r.t. d, and can be yield from $\frac{\partial R(\theta_J, dX_{(1)})}{\partial d} = 0$ which is

$$E\left[\left(U - \frac{1}{d(\mu,\gamma)}\right) e^{-\gamma(d(\mu,\gamma)U - \ln(d(\mu,\gamma)U) - 1)}\right] = 0. \tag{3.7}$$

For $\mu = 1$, the root $d^\star = d^\star(1, \gamma)$ of this equation are summarized in Table 2 for the values $\alpha = 1, 2, 4$ and for certain values of γ. Note that we are able to prove analytically that the root $d^\star(1, \gamma)$ for $\alpha = 1$ and arbitrary $\gamma > 0$ is always equal to 2 (see the Appendix).

Table 2. Values of $d^{\star}(1,\gamma)$ for $\alpha = 1, 2, 4$ and certain values of γ

	γ					
α	0.25	0.5	0.75	1	5	10
1	2	2	2	2	2	2
2	0.7982	0.7967	0.7954	0.7944	0.7870	0.7845
4	0.3437	0.3433	0.3430	0.3427	0.3400	0.3386

The graph of $d(\mu, \gamma)$ as a function of $\mu \geq 1$ (and also numerical solution of equation (3.7)) shows that for fixed $\gamma > 0$, $d(\mu, \gamma)$ decreases as μ increases and $d(\mu, \gamma) \to \alpha^{-1}$ as $\mu \to \infty$. So

$$\inf_{\mu \geq 1} d(\mu, \gamma) = \lim_{\mu \to \infty} d(\mu, \gamma) = \alpha^{-1} \quad \textit{and} \quad \sup_{\mu \geq 1} d(\mu, \gamma) = d(1, \gamma) = d^{\star}$$

Therefore, we conjecture that the estimators $\delta_{2d}(X_1, X_2) = dX_{(1)}$ are admissible within the subclass D_{1L} of invariant estimators of θ_J, under the RGL function, if and only if $d \in [\alpha^{-1}, d^{\star}]$.

3.5. Remark *Let $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2,$ be a pair of independent random samples from $\Pi_i,\ i = 1, 2,$ and Π_i has p.d.f. (1.1). Then $T_i(\boldsymbol{X}_i) = \sum_{i=1}^{n} X_{ij},\ i = 1, 2,$ is complete sufficient statistic for θ_i and has gamma distribution with parameters $(n\alpha, \theta_i)$, respectively, where $\boldsymbol{X}_i = (X_{i1}, \ldots, X_{in})$. Therefore, the results of Sections 2-3 hold true upon replacing α by $n\alpha$ and X_i by $T_i(\boldsymbol{X}_i),\ i = 1, 2,$ in this case.*

4. Applications and Extensions

In this section, we apply the results of Sections 2 and 3 to k-records and Type-II censored data and extend these results to a subclass of exponential family.

4.1. Estimation After Selection Based on k-Record Data. In statistical inference, a rich literature has developed on record data since Chandler [8] formulated the theory of records. Let $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1, 2,$ be a pair of independent random samples from negative exponential populations Π_1, Π_2 with Π_i having the associated pdf

$$f(x|\theta_i) = \frac{1}{\theta_i}\, e^{-\frac{x}{\theta_i}}, \quad \theta_i > 0, \quad i = 1, 2, \tag{4.1}$$

where θ_1, θ_2 are unknown scale parameters. Let $R^i_{m(k)}$ be upper k-records of i-th sample, $i = 1, 2$. It is easy to verify that the mth k-Records, $R^i_{m(k)}$, has a Gamma$\big(m, \frac{\theta_i}{k}\big)$-distribution and $kR^i_{m(k)}$ has a Gamma$\big(m, \theta_i\big)$-distribution, see Arnold et al. [4], Nevzorov [23], Ahmadi et al. [1] and Ahmadi et al. [2,3] and references therein. Let $R^{(1)}_{m(k)} \leq R^{(2)}_{m(k)}$ represent the ordered statistics of $R^1_{m(k)}$ and $R^2_{m(k)}$. Suppose the population corresponding to largest $R^{(2)}_{m(k)}$ (or the smallest $R^{(1)}_{m(k)}$) observation is selected. The problems that we are interested here are the estimation of the following random parameters:

$$\theta^m_M = \begin{cases} \theta_1 & R^1_{m(k)} \geq R^2_{m(k)} \\ \theta_2 & R^1_{m(k)} < R^2_{m(k)} \end{cases} \quad \textit{and} \quad \theta^m_J = \begin{cases} \theta_2 & R^1_{m(k)} \geq R^2_{m(k)} \\ \theta_1 & R^1_{m(k)} < R^2_{m(k)}. \end{cases}$$

Since $kR^i_{m(k)}$ has Gamma$\big(m, \theta_i\big)$-distribution, therefore the results of Sections 2-3, except Remark 2.11, hold for this case upon replacing α by m and X_i by $kR^i_{m(k)},\ i = 1, 2.$

4.2. Estimation after Selection using Type-II Censored Data. The most common censoring scheme which is of importance in the field of reliability and life-testing, is Type-II censoring. In this scheme, after starting the life-testing experiment with n items, the experiment continues until a pre-specified number of failures , say $r(\leq n)$ occur. For more details about this scheme, see Lawless [16].

Let $X_{i1}, X_{i2}, \ldots, X_{in},\ i = 1,2$, be a pair of independent random samples from negative exponential populations with pdf (4.1). It is easy to show that in this scheme $T_i = \sum_{j=1}^{r} X_{i(j)} + (n-r)X_{i(r)},\ i = 1,2$, has a Gamma$(r, \theta_i)$-distribution, see Lehmann and Romano [15]. Now, Suppose $T_{(1)} = \min(T_1, T_2)$ and $T_{(2)} = \max(T_1, T_2)$ and the population corresponding to the largest $T_{(2)}$ (or smallest $T_{(1)}$) is selected. We are interested in estimation of the random parameters

$$\theta_M = \begin{cases} \theta_1 & T_1 \geq T_2 \\ \theta_2 & T_1 < T_2 \end{cases} \quad and \quad \theta_J = \begin{cases} \theta_2 & T_1 \geq T_2 \\ \theta_1 & T_1 < T_2. \end{cases}$$

Since $T_i,\ i = 1,2$, has Gamma$\big(r, \theta_i\big)$-distribution, therefore the results of Sections 2-3, except Remark 2.11, hold true upon replacing α by r and X_i by $T_i,\ i = 1,2$, in this case.

4.3. Extension to a Subclass of Exponential Family. Let $\mathbf{X}_i = (X_{i1}, X_{i2}, \cdots, X_{in})$, $i = 1,2$, be a random sample of size n from the ith population $\Pi_i,\ i = 1,2$, with the joint scale probability density function

$$f(\mathbf{x}_i, \tau_i) = \frac{1}{\tau_i^n} f\big(\frac{\mathbf{x}_i}{\tau_i}\big), \quad i = 1,2,$$

where $\mathbf{x}_i = (x_{i1}, \cdots, x_{in})$. In some cases the above model reduces to

$$f(\mathbf{x}_i, \theta_i) = C(\mathbf{x}_i, n)\theta_i^{-\gamma} e^{-T_i(\mathbf{x}_i)/\theta_i}, \qquad i = 1,2, \tag{4.2}$$

where $C(\mathbf{x}_i, n)$ is a function of $\mathbf{x}_i$ and n, $\theta_i = \tau_i^r$ for some $r > 0, \gamma$ is a function of n and $T_i(\mathbf{X}_i)$ is a complete sufficient statistic for θ_i with $Gamma(\gamma, \theta_i)$-distribution. For example $Exponential(\beta_i)$, $Gamma(\nu, \beta_i)$, $Inverse\,Gaussian(\infty, \lambda_i)$, $Normal(0, \sigma_i^2)$, $Weibull(\eta_i, \beta)$, $Rayleigh(\beta_i)$, $Generalized\ Gamma(\alpha, \lambda_i, p_i)$, $Generalized\ laplace(\lambda_i, k)$ belong to the family of distributions (4.2), see Parsian and Nematollahi [24] and references therein.

Since $T_i = T_i(\mathbf{X}_i), i = 1,2$, has a $Gamma(\gamma, \theta_i)$-distribution, therefore we can extend the results of Sections 2-3 to the subclass of exponential family (4.2) by replacing α and X_i by γ and $T_i(\mathbf{X}_i)$, respectively.

The results of Section 2-3 can also be extended to the family of transformed chi-square distributions which is introduced by Rahman and Gupta [25] and includes Pareto and beta distributions. For details see Jafari Jozani et al. [11].

5. Appendix

In this section, we show analytically that the root $d^\star(1, \gamma)$ for $\alpha = 1$ and arbitrary $\gamma > 0$ is always equal to 2. To see this, note that the pdf of U given in (3.4), for $\mu = \alpha = 1$, reduces to

$$f_U(u) = 2e^{-2u}, \quad u > 0.$$

Therefore $R(\theta_J, dX_{(1)})$ will be minimized at the point d given by $\frac{\partial R(\theta_J, \delta_{2d})}{\partial d} = 0$ which reduces to (3.7) and for $\mu = 1$ can be written as

$$\int_0^\infty \left(u - \frac{1}{d(1,\gamma)}\right) e^{-\gamma(d(1,\gamma)u - \ln(d(1,\gamma)u) - 1)} f_U(u)d(u) = 0$$

and with simple computations is equivalent to

$$2(ed(1,\gamma))^{\gamma}\left(\frac{1}{\gamma d(1,\gamma)+2}\right)^{\gamma+1}\left\{\frac{\Gamma(\gamma+2)}{\gamma d(1,\gamma)+2}-\frac{\Gamma(\gamma+1)}{d(1,\gamma)}\right\}=0.$$

The root of the above equation is simply equal to $d^{\star}(1,\gamma)=2$.

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