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EFFICIENT FAMILY OF EXPONENTIAL ESTIMATORS FOR THE POPULATION MEAN

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Abstract

This paper deals with the estimation of the population mean with improved family of exponential estimators for the variable under study using some known population parameters of the auxiliary variable. The expressions for the bias and mean square error (MSE) of the estimators of the proposed family have been derived to the first degree of approximation. A comparison has been made with the exponential family of estimators of Singh et al. [10]. An improvement has been shown over the family of estimators of Singh et al. [10] through an empirical study.

Keywords: Exponential estimator, auxiliary variable, bias, mean square error, efficiency.

2000 AMS Classification: 62D05

1. Introduction

The use of auxiliary information increases the precision of the estimates of the parameters under consideration. When the variable under study, y, is highly correlated with the auxiliary variable, x, the ratio and product estimators are used for the improved estimation of parameters of the variable under study. To obtain the most efficient estimator, many authors have proposed ratio and product type estimators using some known parameters of the auxiliary variable. The first estimator utilizing the auxiliary information is well known ratio estimator due to Cochran [2] as

$$(1.1) \qquad t_R = \bar{y}(\frac{\bar{X}}{\bar{x}}).$$

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To the first order of approximation, the bias and mean square error, respectively, are

$$B(t_R) = f\,\bar{Y}C_x^2[1 - C],$$

$$(1.2)\qquad MSE(t_R) = f\ \bar{Y}^2[C_y^2 + C_x^2(1 - 2C)],$$

where $f = \frac{N-n}{Nn}$. Bahl and Tuteja [1] was the first to suggest an exponential ratio type estimator as

$$(1.3)\qquad t_1 = \bar{y}\exp(\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}})$$

with the bias and mean square error, respectively, up to the first order of approximation as

$$B(t_1) = f\ C_x^2\left(\frac{\bar{Y}}{8}\right)(3 - 4C),$$

$$(1.4)\qquad MSE(t_1) = f\ \bar{Y}^2[C_y^2 + \frac{C_x^2}{4}(1 - 4C)]$$

Upadhyaya et al. [13] proposed a modified exponential ratio type estimator as

$$(1.5)\qquad t_{Re}^{(a)} = \bar{y}\exp(\frac{\bar{X} - \bar{x}}{\bar{X} + (a-1)\bar{x}})$$

where a is some suitable constant.

To the first order of approximation, the bias and mean square error, respectively, are

$$B(t_{Re}^{(a)}) = f\,\bar{Y}(\frac{C_x^2}{2a^2})[2a(1 - C) - 1],$$

$$(1.6)\qquad MSE(t_{Re}^{(a)}) = f\,\bar{Y}^2[C_y^2 + \frac{C_x^2}{a^2}(1 - 2aC)]$$

where $C = \rho\frac{C_y}{C_x}$.

MSE in (1.4) is minimum for $a = \frac{1}{C}$ and the minimum mean square error is equal to the MSE of the usual linear regression estimator.

There are several authors who used the known parameters of the auxiliary variable to find more precise estimate of the population parameters of the variable under study, including Sen [7], Sisodia and Dwivedi [11], Upadhyaya and Singh [12], Singh and Kakran [8]. Singh and Tailor [9], Kadilar and Cingi [3, 4] and Khoshnevisan et al. [5] have suggested modified ratio estimators utilizing different known values of population parameters of the auxiliary variable.

In this paper, we have proposed an improved exponential family of ratio type estimators for the population mean of the variable under study using some known parameters of the auxiliary variable.

Let the population consists of N units and a sample of size n is drawn with the simple random sampling without replacement (SRSWOR). Let Y_i and X_i be the values of the study and auxiliary variables for the i^{th} unit ($i = 1, 2, \ldots, N$) of the population, respectively. Further, let $\bar{y}$ and $\bar{x}$ be the sample means of the study and auxiliary variables, respectively.

To obtain the bias and mean square error (MSE) of the estimators, let $\bar{y}= \bar{Y}(1 + e_0)$ and $\bar{x}= \bar{X}(1 + e_1)$ such that $E(e_i)= 0$, $i= 0, 1$ and $E(e_0^2)= f\,C_y^2$, $E(e_1^2)= f\,C_x^2$, and $E(e_0e_1)= f\,C_{yx}= f\rho\,C_yC_x$, where $C_y^2 = \frac{S_y^2}{\bar{Y}^2}$, and $C_x^2= \frac{S_x^2}{\bar{X}^2}$.

2. The Suggested Exponential Family of Estimators

Singh et al. [10] defined an exponential family of estimators for the population mean in the simple random sampling as

$$(2.1)\qquad t = \bar{y}\exp\left[\frac{(a\bar{X}+b)-(a\bar{x}+b)}{(a\bar{X}+b)+(a\bar{x}+b)}\right],$$

where $a(\neq 0)$ and b are either real numbers or the functions of the known parameters of the auxiliary variable, x, such as coefficient of variation (C_x), coefficient of kurtosis ($\beta_2(x)$) and correlation coefficient (ρ).

The bias and MSE of this family of estimators, to the first degree of approximation are, respectively, as follows:

$$(2.2)\qquad B(t) = f\,\bar{Y}(2\theta^2 C_x^2 - \theta\rho C_y C_x),$$

$$(2.3)\qquad MSE(t) = f\,\bar{Y}^2(C_y^2 + \theta^2 C_x^2 - 2\theta\rho C_y C_x),$$

where $\theta = \frac{a\bar{X}}{2(a\bar{X}+b)}$.

The ratio estimators, given in Table 1, are the members of the t-family of estimators in (2.1) and the mean square error for these estimators is

$$(2.4)\qquad MSE(t_i) = f\,\bar{Y}^2(C_y^2 + \theta_i^2 C_x^2 - 2\theta_i\rho C_y C_x)\ ,\ i = 2,\,3,\,...,\,10$$

where $\theta_2 = \frac{\bar{X}}{2(\bar{X}+\beta_2(x))}$, $\theta_3 = \frac{\bar{X}}{2(\bar{X}+C_x)}$, $\theta_4 = \frac{\bar{X}}{2(\bar{X}+\rho)}$, $\theta_5 = \frac{\beta_2(x)\bar{X}}{2(\beta_2(x)\bar{X}+C_x)}$, $\theta_6 = \frac{C_x\bar{X}}{2(C_x\bar{X}+\beta_2(x))}$, $\theta_7 = \frac{C_x\bar{X}}{2(C_x\bar{X}+\rho)}$, $\theta_8 = \frac{\rho\bar{X}}{2(\rho\bar{X}+C_x)}$, $\theta_9 = \frac{\beta_2(x)\bar{X}}{2(\beta_2(x)\bar{X}+\rho)}$, $\theta_{10} = \frac{\rho\bar{X}}{2(\rho\bar{X}+\beta_2(x))}$.

Many more estimators can be generated from the estimator, t, in (2.1) just by putting different values of a and b.

Motivated by Koyuncu and Kadilar [6], we propose a new family of exponential estimators as

$$(2.5)\qquad \xi = k\,\bar{y}\exp\left[\frac{(a\bar{X}+b)-(a\bar{x}+b)}{(a\bar{X}+b)+(a\bar{x}+b)}\right],$$

where k is suitably chosen constant to be determined later and $a(\neq 0)$ and b are either real numbers or the functions of the known parameters of the auxiliary variable, such as the coefficient of variation (C_x), the coefficient of kurtosis ($\beta_2(x)$), and the correlation coefficient (ρ).

Now, expressing the estimator ξ in terms of $e_i(i = 0,\,1)$, we can write (2.5) as

$$(2.6)\qquad \xi = k\,\bar{Y}(1+e_0)\exp\left\{-\theta\,e_1(1+\theta\,e_1)^{-1}\right\}.$$

Expanding (2.6) on right hand side to the first degree of approximation and subtracting $\bar{Y}$ from both sides, we get

$$(2.7)\qquad \xi - \bar{Y} = k\,\bar{Y}(1 + e_0 - \theta\,e_1 + 2\theta^2 e_1^2 - \theta\,e_0 e_1) - \bar{Y}.$$

Taking the expectation on both sides of (2.7), we get the bias of the estimator ξ as

$$(2.8)\qquad B(\xi) = k\,\bar{Y}f(2\theta^2 C_x^2 - \theta\rho C_y C_x) + \bar{Y}(k-1).$$

Squaring both sides of (2.7), we have

$$(2.9)\qquad (\xi - \bar{Y})^2 = k^2\,\bar{Y}^2(1 + e_0 - \theta\,e_1 + 2\theta^2 e_1^2 - \theta\,e_0 e_1)^2 + \bar{Y}^2 - 2k\bar{Y}^2(1 + e_0 - \theta\,e_1 + 2\theta^2 e_1^2 - \theta\,e_0 e_1).$$

Taking expectation of both sides, we get the MSE of the estimator, ξ, to the first degree of approximation as

$$(2.10)\qquad MSE(\xi) = \bar{Y}^2\left\{k^2 f\,C_y^2 + (5k^2-4k)f\theta^2 C_x^2 - 2(2k^2-k)f\theta\rho C_y C_x + (k-1)^2\right\}.$$

The minimum $MSE(\xi)$ is obtained for the optimal value of k which is $k_{opt} = \frac{A}{B}$, where $A = f(2\theta^2 C_x^2 - \theta\rho C_y C_x) + 1$ and $B = f(C_y^2 + 5\theta^2 C_x^2 - 4\theta\rho C_y C_x) + 1$. Therefore, the minimum MSE of the estimator, ξ, is

$$MSE_{\min}(\xi) = \bar{Y}^2 \left[1 - \frac{A^2}{B}\right]. \tag{2.11}$$

For the ratio estimators, given in Table 2, the expression for the MSE is obtained by

$$MSE(\xi_i) = \bar{Y}^2\{k_{opt}^2 f\, C_y^2 + (5k_{opt}^2 - 4k_{opt}) f \theta_i^2 C_x^2 \tag{2.12}$$
$$-2(2k_{opt}^2 - k_{opt}) f \theta_i \rho C_y C_x + (k_{opt} - 1)^2\}, \quad i = 1,\ 2,\ 3,\ ...,\ 10$$

Many more estimators can be generated from the estimator, ξ, in (2.5) just by putting different values of a and b.

TABLE 1. Some Members of the t-family of Estimators

Estimators	a	b
$t_0 = \overline{y}$ (The Sample Mean)	0	0
$t_1 = \overline{y} \exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right]$ (Bahl and Tuteja [1])	1	1
$t_2 = \overline{y} \exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2\beta_2(x)}\right]$	1	$\beta_2(x)$
$t_3 = \overline{y} \exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2C_x}\right]$	1	C_x
$t_4 = \overline{y} \exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2\rho}\right]$	1	ρ
$t_5 = \overline{y} \exp\left[\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}-\overline{x}\right)+2C_x}\right]$	$\beta_2(x)$	C_x
$t_6 = \overline{y} \exp\left[\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}-\overline{x}\right)+2\beta_2(x)}\right]$	C_x	$\beta_2(x)$
$t_7 = \overline{y} \exp\left[\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}-\overline{x}\right)+2\rho}\right]$	C_x	ρ
$t_8 = \overline{y} \exp\left[\frac{\rho\left(\overline{X}-\overline{x}\right)}{\rho\left(\overline{X}-\overline{x}\right)+2C_x}\right]$	ρ	C_x
$t_9 = \overline{y} \exp\left[\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}-\overline{x}\right)+2\rho}\right]$	$\beta_2(x)$	ρ
$t_{10} = \overline{y} \exp\left[\frac{\rho\left(\overline{X}-\overline{x}\right)}{\rho\left(\overline{X}-\overline{x}\right)+2\beta_2(x)}\right]$	ρ	$\beta_2(x)$

3. Efficiency Comparisons

The t-family of estimators is more efficient than the Bahl and Tuteja [1] estimator, t_1, if

$$MSE(t_i) < MSE(t_1) \quad i = 2, \ldots, 10, \quad \text{i.e., } (\theta_i^2 - \frac{1}{4})C_x^2 + (2\theta_i - 1)\rho C_y C_x > 0. \tag{3.1}$$

When the condition (3.1) is satisfied, we may say that t-family is more efficient than the estimator t_1.

The proposed ξ- family of estimators is more efficient than the estimator t_1 if

$$MSE_{\min}(\xi_i) < MSE(t_1) \quad i = 1,\ 2, \ldots,\ 10, \quad \text{i.e., } (1 - \frac{A^2}{B}) < f(C_y^2 + \frac{C_x^2}{4} - \rho C_y C_x). \tag{3.2}$$

When the condition (3.2) is satisfied, we may infer that ξ-family is more efficient than the estimator t_1.

TABLE 2. Some Members of the ξ- family of Estimators

Estimators	a	b
$\xi_1 = k\,\overline{y}\exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}}\right]$	1	1
$\xi_2 = k\,\overline{y}\exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2\beta_2(x)}\right]$	1	$\beta_2\,(x)$
$\xi_3 = k\,\overline{y}\exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2C_x}\right]$	1	C_x
$\xi_4 = k\,\overline{y}\exp\left[\frac{\overline{X}-\overline{x}}{\overline{X}+\overline{x}+2\rho}\right]$	1	ρ
$\xi_5 = k\,\overline{y}\exp\left[\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}-\overline{x}\right)+2C_x}\right]$	$\beta_2\,(x)$	C_x
$\xi_6 = k\,\overline{y}\exp\left[\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}-\overline{x}\right)+2\beta_2(x)}\right]$	C_x	$\beta_2\,(x)$
$\xi_7 = k\,\overline{y}\exp\left[\frac{C_x\left(\overline{X}-\overline{x}\right)}{C_x\left(\overline{X}-\overline{x}\right)+2\rho}\right]$	C_x	ρ
$\xi_8 = k\,\overline{y}\exp\left[\frac{\rho\left(\overline{X}-\overline{x}\right)}{\rho\left(\overline{X}-\overline{x}\right)+2C_x}\right]$	ρ	C_x
$\xi_9 = k\,\overline{y}\exp\left[\frac{\beta_2(x)\left(\overline{X}-\overline{x}\right)}{\beta_2(x)\left(\overline{X}-\overline{x}\right)+2\rho}\right]$	$\beta_2\,(x)$	ρ
$\xi_{10} = k\,\overline{y}\exp\left[\frac{\rho\left(\overline{X}-\overline{x}\right)}{\rho\left(\overline{X}-\overline{x}\right)+2\beta_2(x)}\right]$	ρ	$\beta_2\,(x)$

The proposed family ξ of estimators is more efficient than t -family of estimators if

$$MSE_{\min}(\xi_i) < MSE(t_i) \quad i = 1,\ldots,10, \text{ i.e., } (1-\frac{A^2}{B}) < f(C_y^2+\theta_i^2C_x^2-2\theta_i\rho C_yC_x). \tag{3.3}$$

The proposed family ξ of estimators is more efficient than Upadhyaya et al. [13] estimator if

$$(1-\frac{A^2}{B}) < f\,[C_y^2 + \frac{C_x^2}{a^2}(1-2aC)]$$

for suitable positive value of a.

4. The Empirical Study

We have used the data in Koyuncu and Kadilar [6], given in Table 3, to compare the efficiencies between the t-family and the proposed ξ- family of estimators for the population mean under the simple random sampling.

TABLE 3. Data Statistics

$N = 923$	$n = 180$	$\overline{Y} = 436.4345$
$\overline{X} = 11440.4984$	$C_y = 1.7183$	$C_x = 1.8645$
$\rho = 0.9543$	$\beta_1\,(x) = 3.9365$	$\beta_2\,(x) = 18.7208$

The MSE values of the t and ξ estimators have been obtained using (2.4) and (2.11), respectively, and these values are presented in Table 4.

Table 4. Mean Square Error of the t and ξ Families

Estimators	MSE	Estimators	MSE
t_1	655.1199	ξ_1	651.4819
t_2	656.9777	ξ_2	653.3295
t_3	655.3104	ξ_3	651.6914
t_4	655.2152	ξ_4	651.5962
t_5	655.1390	ξ_5	651.5009
t_6	656.1295	ξ_6	652.4724
t_7	655.1771	ξ_7	651.5199
t_8	655.3295	ξ_8	651.6724
t_9	655.1390	ξ_9	651.5009
t_{10}	657.0628	ξ_{10}	653.4247

5. Conclusion

From Table 4, we observe that $MSE_{\min}(\xi_i) < MSE(t_i)\ for\ i = 1, 2, 3, ..., 10$ and therefore ξ_1 as well as every member of ξ- family is more efficient than the Bahl and Tuteja [1] estimator. Every member of ξ- family is more efficient than the corresponding as well as every member of t-family given by Singh et al. [10]. Thus, from the results of the empirical study, we may conclude that the proposed ξ- family of exponential estimators of the population mean is more efficient than t-family of exponential estimators. Hence, the use of suggested estimators should be preferred in practice.

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