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STRONGLY EXTENDING MODULES

Burcu Ungor [a] [*] and Sait Halicioglu [a] [†]

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Abstract

In this paper, we recall the concept of strong largeness to define strongly extending modules which are particular extending modules, and investigate some properties of strongly extending modules. We supply some examples showing that extending modules need not be strongly extending. Under some conditions we prove that extending modules are strongly extending.

Keywords: Large submodules, extending modules, strongly large submodules, strongly extending modules.

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1. Introduction

Throughout all rings have identities and all modules are unital right modules. Let R be a ring and M an R-module. For submodules A and B of M, $A \leq B$ denotes A is a submodule of B and $S = \mathrm{End}_R(M)$ denotes the ring of right R-module endomorphisms of M. Then M is a left S-module, right R-module and (S, R)-bimodule. In this work, for any rings S and R and any (S, R)-bimodule M, $r_R(.)$ and $l_M(.)$ denote the right annihilator of a subset of M in R and the left annihilator of a subset of R in M, respectively. Similarly, $l_S(.)$ and $r_M(.)$ are the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. For $m \in M$ and $N \leq M$, the right ideal $\{r \in R \mid mr \in N\}$ of R is denoted by $m^{-1}N$. When $N = 0$ the right ideal $m^{-1}N$ and $r_R(m)$ coincide. It is clear that N is a large submodule of M if and only if $m(m^{-1}N) \neq 0$ for each nonzero $m \in M$.

In this paper, our aim is to introduce and study strongly extending modules by using the concept of strong largeness. We see that some known essential objects are in fact strongly large and clearly every nonzero ideal in a commutative domain enjoys this property. We make use strong large submodules to define strongly extending modules. At first we give some elementary properties of strongly large submodules and introduce strongly

[a]Department of Mathematics, Ankara University, Ankara, Turkey
[*]Email: bungor@science.ankara.edu.tr
[†]Email: halici@ankara.edu.tr
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extending modules strengthening extending modules. We produce some examples showing that extending modules need not be strongly extending. Under some conditions we prove that extending modules are strongly extending.

In what follows, we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/n\mathbb{Z}$ integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n, respectively. And $\mathbb{Z}_{p^\infty}$ is the Prüfer p-group for any prime p. For unexplained concepts and notations, we refer the reader to [1].

2. Strong Largeness

In this section we establish the notation and state some results on strongly large submodules which are required later.

2.1. Definition. A submodule N of a module M is called *strongly large in M* in case of, for each $m \in M$ and each right ideal I of R, if $mI \neq 0$ then $m(m^{-1}N)I \neq 0$.

2.2. Definition. Let N be a submodule of a module M. Then N is called an *SL-closed submodule* of M if it has no proper strongly large extensions in M.

Let M be a module with a submodule N. The submodule N is called *dense* in M if for any $y \in M$ and $0 \neq x \in M$ there exists $r \in R$ such that $xr \neq 0$ and $yr \in N$ (see namely [6]). Note that every closed submodule of any module is SL-closed. But the converse is not true in general.

2.3. Example. Let F be any field. Consider the ring $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and the right R-module $M = R$. Then $N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is SL-closed and large in M.

Assume that R is a commutative ring. Then every dense submodule of any R-module is strongly large. In this case for a nonsingular module M, a submodule N is essential in M implies it is dense in M, and also a submodule N is SL-closed in M if and only if it is closed in M.

2.4. Proposition. *Let M be a module with a submodule N. Then the following are equivalent.*

(1) *N is a strongly large submodule of M.*
(2) *For each $m \in M$ and $s \in R$ with $ms \neq 0$, we have $m(m^{-1}N)s \neq 0$.*
(3) *For each $m \in M$ and $s \in R$ with $ms \neq 0$, there exists $r \in R$ such that $mr \in N$ and $mrs \neq 0$.*

2.5. Lemma. *Let N and L be submodules of a module M with $N \leq L$. If N is strongly large in L, then*

(1) *$N \cap K = 0$ implies $L \cap K = 0$ for any submodule K of M.*
(2) *$NI = 0$ implies $LI = 0$ for any right ideal I of R.*

2.6. Lemma. *Let M be a module and N, K submodules of M with $N \leq K$. Then N is strongly large in M if and only if N is strongly large in K and K is strongly large in M.*

2.7. Lemma. *Let M be a module. If N_1 is strongly large in $K_1 \leq M$ and N_2 is strongly large in $K_2 \leq M$, then $N_1 \cap N_2$ is strongly large in $K_1 \cap K_2$.*

It is obvious that any direct summands of a module are SL-closed. Now we can say that there are abundant examples of SL-closed submodules. For instance, consider the following proposition.

2.8. Proposition. *Every cyclic submodule $N = (a,b)\mathbb{Z}$ of the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$ where $\gcd(a,b) = 1$ is SL-closed in $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. Let L be a strongly large extension of N in $\mathbb{Z} \oplus \mathbb{Z}$ and $0 \neq (x,y) \in L$. Since N is large in L, there exists $t \in \mathbb{Z}$ with $0 \neq (x,y)t \in N$. So $xt = az, yt = bz$ for some $z \in \mathbb{Z}$. Hence $xb = ya$. Then we have $(x,y)b = (xb, yb) = (ya, yb) = (a,b)y$. On the other hand, $aa' + bb' = 1$ for some $a', b' \in \mathbb{Z}$ due to $\gcd(a,b) = 1$. Then $(x,y) = (x,y)aa' + (x,y)bb' = (xa, ya)a' + (a,b)yb' = (xa, xb)a' + (a,b)yb' = (a,b)xa' + (a,b)yb'$. Thus $(x,y) \in N$, and so $N = L$. Therefore N is an SL-closed submodule of $\mathbb{Z} \oplus \mathbb{Z}$. □

2.9. Proposition. *Let N be a cyclic SL-closed submodule of the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. Then there exists $(a,b) \in N$ such that $\gcd(a,b) = 1$ and $N = (a,b)\mathbb{Z}$.*

Proof. Let $N = (a,b)\mathbb{Z}$ be an SL-closed submodule of $\mathbb{Z} \oplus \mathbb{Z}$. If $\gcd(a,b) = 1$, there is nothing to show. Now let $\gcd(a,b) = d$ and $d \neq 1$. There exist $a', b' \in \mathbb{Z}$ such that $a = da'$, $b = db'$ and $\gcd(a', b') = 1$. It is clear that $N \subseteq (a', b')\mathbb{Z}$. Let $(a', b')u \in (a', b')\mathbb{Z}$ and $s \in \mathbb{Z}$ with $(a', b')us \neq 0$. For $d \in \mathbb{Z}$, $(a', b')ud = (a,b)u \in N$ and $(a', b')uds \neq 0$. So N is strongly large in $(a', b')\mathbb{Z}$. Since N is an SL-closed submodule of $\mathbb{Z} \oplus \mathbb{Z}$, we have $N = (a', b')\mathbb{Z}$. This completes the proof. □

2.10. Corollary. *Let N be a cyclic submodule of the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. Then N is SL-closed in $\mathbb{Z} \oplus \mathbb{Z}$ if and only if $N = (a,b)\mathbb{Z}$ for some $(a,b) \in \mathbb{Z} \oplus \mathbb{Z}$ where $\gcd(a,b) = 1$.*

Proof. Clear from Proposition 2.8 and Proposition 2.9. □

We have the following proposition due to Zorn's Lemma.

2.11. Proposition. *Every submodule N of a module M is contained in an SL-closed submodule of M in which N is strongly large.*

3. Strongly Extending Modules

A module M is called *strongly extending* if for every submodule N of M, there exists a decomposition $M = K \oplus L$ such that N is a strongly large submodule of K. Since each strongly large submodule is large, every strongly extending module is extending. But the converse is not true in general, as it will be shown later with some examples. There are some cases largeness implies strongly largeness.

3.1. Proposition. *Let M be a module. Then every strongly large submodule of M is large. The converse holds if M is a strongly extending module.*

Proof. Clear. □

3.2. Theorem. *Let M be a module. The following conditions are equivalent.*

(1) *M is a strongly extending module.*
(2) *For every submodule N of M, there exists a decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L$ is a strongly large submodule of M.*
(3) *Every SL-closed submodule of M is a direct summand. (In fact, for any submodule N of M, N is SL-closed in M if and only if N is a direct summand.)*

Proof. (1) $\Rightarrow$ (2) Let M be a strongly extending module and $N \leq M$. Since M is extending, there exists a decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L$ is large in M. From Proposition 3.1, M is a strongly large extension of $N \oplus L$.
(2) $\Rightarrow$ (1) Let N be a submodule of M. By (2), there exists a decomposition $M = K \oplus L$ such that $N \leq K$ and $N \oplus L$ is strongly large in M. Let $k \in K$ and $s \in R$ with $ks \neq 0$. There exists $r \in R$ such that $kr \in N \oplus L$ and $krs \neq 0$ because of strongly largeness of

$N \oplus L$ in M. Since $N \leq K$ and $K \cap L = 0$, we have $kr \in N$. Hence N is strongly large in K. Therefore M is a strongly extending module.
(1) $\Rightarrow$ (3) Let K be an SL-closed submodule of M. Then there exists a direct summand M_1 of M such that K is strongly large in M_1. Since K has no proper strongly large extensions in M, $K = M_1$.
(3) $\Rightarrow$ (1) Let N be a submodule of M. From Proposition 2.11, there exists an SL-closed submodule L of M in which N is strongly large. So L is a direct summand of M. Thus M is a strongly extending module. □

We present some examples for motivation.

3.3. Examples. (1) Every semisimple module is strongly extending.
(2) Let R be a commutative domain. Then R is strongly extending as a right R-module.
(3) Let M be a module in which every nonzero submodule is strongly large. Then M is strongly extending.

Of course, not every module is strongly extending. Note the following facts.

3.4. Example. Let $R = \mathbb{Z}[x]$ and consider $M = R \oplus R$ as an R-module. It is shown in [2] that M is not an extending module, therefore it is not strongly extending. In fact the cyclic submodule $N = (x, 2)R$ is SL-closed in M but not a direct summand.

3.5. Example. Consider the ring $R = \begin{bmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$ of upper triangular matrices over the ring $\mathbb{Z}_4$ and let M denote R as an R-module. The submodule $N = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} R$ of M is a complement of $K = \left\{ \begin{bmatrix} 0 & x \\ 0 & x \end{bmatrix} : x \in \mathbb{Z}_4 \right\}$ in M and consequently N is SL-closed in M. On the other hand, N should include a nonzero idempotent in order to be a direct summand of M. But this does not hold, and so M is not a strongly extending module.

As we mentioned before, every strongly extending module is extending, but there exists an extending module which is not strongly extending.

3.6. Example. Let $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ where F is any field. Consider R-module $M = R$. It is easy to check that M is an extending module. Now consider R-submodules $N_1 = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, $N_2 = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $N_3 = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ of M. For the right ideal $I = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$ of R, $N_1 I = 0$ but $N_2 I \neq 0$. Thus N_1 is not strongly large in N_2 by Lemma 2.5. On the other hand, $N_1 \cap N_4 = 0$ for the nonzero submodule $N_4 = \left\{ \begin{bmatrix} 0 & x \\ 0 & x \end{bmatrix} : x \in F \right\}$ of N_3. Hence N_1 is not a large submodule of N_3. And so N_1 is not strongly large in N_3. Since N_1 has no proper strongly large extensions in M, N is an SL-closed submodule of M. However, N_1 is not a direct summand of M. Therefore M is not a strongly extending module.

3.7. Example. Let M denote $\mathbb{Z}$-module $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z})$. It is well known that M is extending. Let $N = (1 + 2\mathbb{Z}, 0 + 4\mathbb{Z})\mathbb{Z}$ and $K = (1 + 2\mathbb{Z}, 2 + 4\mathbb{Z})\mathbb{Z}$. Then $(N \oplus K)I = 0$, however $MI \neq 0$ for $I = 2\mathbb{Z}$. From Lemma 2.5, $N \oplus K$ is not strongly large in M. So $N \oplus K$ is an SL-closed submodule of M. On the other hand, $N \oplus K$ is not a direct summand. Hence M is not a strongly extending module.

3.8. Example. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$. Then M is extending by [4]. Let $N = (0, 1/p + \mathbb{Z})\,\mathbb{Z} \leq M$. Then N is SL-closed in M and not a direct summand since it is large in $0 \oplus \mathbb{Z}_{p^\infty}$. Hence M is not strongly extending.

We point out that strongly extending modules do not coincide with extending modules over a principal ideal domain. On the other hand, in Example 3.4 the module M is finitely generated torsion-free over a domain which is not principal ideal domain. In Example 3.7 the module M is finitely generated torsion over a principal ideal domain. Now the next results provide another source of examples for strongly extending modules.

3.9. Proposition. *Every finitely generated torsion-free module over a principal ideal domain is strongly extending.*

Proof. Let M be a finitely generated torsion-free module over a principal ideal domain R and N a submodule of M. Assume firstly M/N is torsion-free. Then M/N is isomorphic to a finite direct sum of copies of R and so it is a projective R-module. Hence N is a direct summand of M. Suppose that M/N is not torsion-free. There exists a submodule K of M containing N such that M/K is torsion-free and K/N is torsion. Hence M/K is a torsion-free and finitely generated R-module. So M/K is projective and so K is a direct summand of M. In order to see that K is a strongly large extension of N, let $k \in K \setminus N$ and $s \in R$ with $ks \neq 0$. Let $\overline{k}$ denote the image of $k \in K$ under the natural map from K to K/N. Since K/N is torsion and $\overline{k}$ is nonzero in K/N, there exists $0 \neq r \in R$ such that $\overline{k}r = \overline{0} \in K/N$. So $kr \in N$. By hypothesis $rs \neq 0$. If $krs = 0$, then k would be a nonzero torsion element of M. Hence $krs \neq 0$. Therefore N is a strongly large submodule of K. □

3.10. Example. Let M denote the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}$. By Proposition 3.9, M is strongly extending. For this reason every SL-closed submodule of M is a direct summand. Also by Proposition 2.8 and Proposition 2.9 it can be checked that N is a direct summand of M if and only if N has the form $N = (a, b)\mathbb{Z}$ for some integers a, b with the property that the greatest common divisor of a and b is 1. Now, in addition to Corollary 2.10, we determine all SL-closed submodules of M.

3.11. Proposition. *Let R be a principal ideal domain. Then every finitely generated flat R-module is strongly extending.*

Proof. This follows from Proposition 3.9 and the fact that a module over a principal ideal domain is flat if and only if it is torsion-free. □

3.12. Proposition. *Every finitely generated torsion-free module over a Prüfer ring is strongly extending.*

Proof. Let M be a finitely generated torsion-free module over a Prüfer ring R and N an SL-closed submodule of M. We claim that M/N is also torsion-free. For, if otherwise, there exists $m \in M \setminus N$ such that $mr \in N$ for some nonzero element r of R. To see that N is a proper strongly large submodule of $N + mR$, let $n + ma \in N + mR$ and $s \in R$ with $(n + ma)s \neq 0$. Then $(n + ma)r \in N$ and $(n + ma)rs \neq 0$. But this contradicts that N is SL-closed in M. Since M/N is finitely generated and R is a Prüfer ring, M/N is a projective R-module. Therefore N is a direct summand of M. So M is strongly extending. □

3.13. Proposition. *Every finitely generated flat module over a Prüfer ring is strongly extending.*

Proof. We have known that if R is a Prüfer ring, then R-modules are flat if and only if they are torsion-free. Hence Proposition 3.12 completes the proof. □

In some cases extending modules are strongly extending.

3.14. Proposition. *Let M be a torsion-free module. Then M is strongly extending if and only if it is extending.*

Proof. The necessity is clear. For the sufficiency, let N be an SL-closed submodule of M and K a large extension of N in M. Now we prove that N is strongly large in K. Let $k \in K$ and $s \in R$ with $ks \neq 0$. So there exists $r \in R$ such that $0 \neq kr \in N$. Since M is torsion-free, $krs \neq 0$. Hence $N = K$ and so N is also closed in M. Therefore N is a direct summand of M because M is extending. □

3.15. Proposition. *Let M be a flat module over a commutative domain. Then M is strongly extending if and only if it is extending.*

Proof. Obvious by Proposition 3.14. □

3.16. Proposition. *Let M be a prime module. Then M is strongly extending if and only if it is extending.*

Proof. For the sufficiency, let N be an SL-closed submodule of M and we prove that N is also closed in M. Let K be a submodule of M and N large in K. Consider $k \in K$ and $s \in R$ such that $ks \neq 0$. Then we have $r \in R$ with $0 \neq kr \in N$. Since M is prime, $krRs \neq 0$. And so there exists $r_1 \in R$ with $krr_1 \in N$ and $krr_1s \neq 0$. Hence N is strongly large in K and $N = K$. Then N is a direct summand by the extending property of M. □

We have known the following lemma from [3].

3.17. Lemma. *A free $\mathbb{Z}$-module F is extending if and only if F has finite rank.*

3.18. Proposition. *Let F be a free $\mathbb{Z}$-module. Then F is strongly extending if and only if F has finite rank. Moreover, F is strongly extending if and only if F is extending.*

Proof. Let F be a strongly extending free $\mathbb{Z}$-module. Clearly F is an extending module. Therefore F has finite rank by Lemma 3.17. Conversely, let F be a free $\mathbb{Z}$-module with finite rank. Then F is isomorphic to finite direct sum of copies of $\mathbb{Z}$. Hence F is a finitely generated torsion-free $\mathbb{Z}$-module. And so F is a strongly extending module by Proposition 3.9. □

Let M be a module. The module M is called *SL-injective relative to a module N*, if each homomorphism $f : K \to M$, where K is strongly large in N, can be extended to N, i.e., there exists a homomorphism $g : N \to M$ such that $g(k) = f(k)$ for all $k \in K$. If M is SL-injective relative to every module, then it is called *SL-injective*. Clearly every injective module is SL-injective. But the converse is not true in general. For example, as a $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is SL-injective relative to $\mathbb{Z}/4\mathbb{Z}$, while it is not injective relative to $\mathbb{Z}/4\mathbb{Z}$. It is well known that every injective module is an extending module. Unfortunately, this is not the case when we deal with strongly largeness, as the following example shows.

3.19. Example. Let Π denote the set of positive prime integers. Consider the injective $\mathbb{Z}$-module $\bigoplus\limits_{p\in\Pi} \mathbb{Z}_{p^\infty}$. Since every direct summand of an injective module is also injective, $\mathbb{Z}_{p^\infty}$ is injective for any prime number p. Now consider a submodule $K = \left(1/p^k + \mathbb{Z}\right)\mathbb{Z}$ of $\mathbb{Z}_{p^\infty}$. For every $n \in \mathbb{N}$ with $k \leq n$ the submodule $N = (1/p^n + \mathbb{Z})\,\mathbb{Z}$ of $\mathbb{Z}_{p^\infty}$ contains K. Let I denote the right ideal $p^k\mathbb{Z}$ of $\mathbb{Z}$. Then we have $KI = 0$ and $NI \neq 0$. Therefore K has no proper strongly large extensions in $\mathbb{Z}_{p^\infty}$. Accordingly every submodule of $\mathbb{Z}_{p^\infty}$ is SL-closed but none of them are direct summand. Consequently, $\mathbb{Z}_{p^\infty}$ is not strongly extending.

The following lemma is proved in [7].

3.20. Lemma. *Let a module $M = \bigoplus_{i\in I} M_i$ be a direct sum of submodules M_i $(i \in I)$ and let N be a fully invariant submodule of M. Then $N = \bigoplus_{i\in I}(N \cap M_i)$.*

3.21. Proposition. *Let M be a strongly extending module and N a fully invariant submodule of M. Then N is a strongly extending module.*

Proof. Let A be a submodule of N. Since M is strongly extending, there exists a decomposition $M = K \oplus L$ such that $A \leq K$ and $A \oplus L$ is strongly large in M. Then by Lemma 3.20, $N = (N \cap K) \oplus (N \cap L)$. On the other hand, $A \leq K \cap N$ and by Lemma 2.7, $(A \oplus L) \cap N = A \oplus (L \cap N)$ is strongly large in N. Therefore N is a strongly extending module. □

3.22. Theorem. *Let M be a strongly extending module. Then every submodule of M is strongly extending if M satisfies any of the following conditions:*

(1) *M is a multiplication module (that is, every submodule of M is of the form MI, for some ideal I of R).*
(2) *M is a duo module (that is, every submodule is fully invariant).*
(3) *M is a distributive module (that is, for any submodules K, L and N of M, $K \cap (N + L) = (K \cap N) + (K \cap L)$).*

Proof. (1) Let N be a submodule of M. Since M is a multiplication module, there exists an ideal I of R such that $N = MI$. Then for any $f \in \mathrm{End}_R(M)$, $f(N) = f(MI) = f(M)I \leq MI = N$. And so N is a fully invariant submodule of M. Therefore the proof is completed by Proposition 3.21.

(2) Clear from Proposition 3.21.

(3) Let N be a submodule of M and L an SL-closed submodule of N. Accordingly Proposition 2.11, there exists an SL-closed submodule U of M containing L as a strongly large submodule. Then $M = U \oplus K$ for some submodule K of M. Since M is a distributive module, we have $N = (U \cap N) \oplus (K \cap N)$. Also by Lemma 2.7 L is a strongly large submodule of $U \cap N$. Hence $L = U \cap N$. It follows that L is a direct summand of N. This completes the proof. □

For any module M and $N \leq L$ submodules of M such that N is closed in L and L is closed in M, it is well known that N is also closed in M. But "SL-closed" version of this statement is an open question. So we have the following definition.

3.23. Definition. Let M be a module and N, L submodules of M with $N \leq L$. The module M is called *SL-c-transitive* if N is SL-closed in L and L is SL-closed in M, then N is SL-closed in M.

The answer of the aforementioned question is affirmative in the next lemma.

3.24. Lemma. *Let M be a module and N a submodule of M. If N is SL-closed in a direct summand of M, then N is SL-closed in M.*

Proof. Let $M = M_1 \oplus M_2$ with N SL-closed in M_1 and K a strongly large extension of N in M. Let π denote the projection of M on M_1. Since $N \leq M_1$, we have $N = \pi(N) \leq \pi(K) \leq M_1$. Now we show that $\pi(N)$ is strongly large in $\pi(K)$. Let $k \in K$ and $s \in R$ such that $\pi(k)s \neq 0$. Hence $ks \neq 0$ and so there exists $r \in R$ with $kr \in N$ and $krs \neq 0$. Also $k = m_1 + m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Since $kr = m_1r + m_2r \in N$ and $M_1 \cap M_2 = 0$, we have $m_2r = 0$ and $m_1r \in N$, also $m_1rs \neq 0$. Thus $\pi(k)r = m_1r \in N = \pi(N)$ and $\pi(k)rs = m_1rs \neq 0$. Then $\pi(K)$ is a strongly large extension of N in M_1. But N is SL-closed in M_1, so $N = \pi(K) \leq K$. Hence $(1 - \pi)(K) \leq K$. Let $n = (1 - \pi)(x) \in N \cap (1 - \pi)(K)$ with $x = x_1 + x_2$ for some $n \in N, x \in K, x_1 \in M_1$ and

$x_2 \in M_2$. Then $n = x_2 \in M_1 \cap M_2 = 0$. So $N \cap (1-\pi)(K) = 0$. Since N is large in K, $(1-\pi)(K) = 0$. Hence $K \leq \pi(K) \leq M_1$, and $N = K$ because N is SL-closed in M_1. Therefore N is also SL-closed in M. $\square$

3.25. Theorem. *Let M be a strongly extending module. Then every direct summand of M is strongly extending.*

Proof. Let N be a direct summand of M and K an SL-closed submodule of N. Then K is also SL-closed in M by Lemma 3.24. Since M is strongly extending, $M = K \oplus L$ for some $L \leq M$. Thus $N = K \oplus (N \cap L)$. Therefore N is strongly extending. $\square$

3.26. Proposition. *Let $M = M_1 \oplus M_2$ be an SL-c-transitive module. Then M is strongly extending if and only if every SL-closed submodule K of M with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a direct summand of M.*

Proof. The necessity is clear. For the sufficiency, let L be an SL-closed submodule of M. By Proposition 2.11, there exists an SL-closed submodule H of L in which $L \cap M_1$ is strongly large. Because M is SL-c-transitive, H is SL-closed in M. Since $H \cap M_2 = 0$, by hypothesis H is a direct summand of M. Let $M = H \oplus H'$. By modularity condition $L = H \oplus (L \cap H')$. Hence $L \cap H'$ is SL-closed in L and also in M. Clearly $(L \cap H') \cap M_1 = 0$. By hypothesis $M = (L \cap H') \oplus H''$ for some submodule H'' of M. So $H' = (L \cap H') \oplus (H' \cap H'')$. We have $M = H \oplus (L \cap H') \oplus (H' \cap H'') = L \oplus (H' \cap H'')$. Thus M is strongly extending. $\square$

If M_1 and M_2 are strongly extending modules, then the module $M_1 \oplus M_2$ need not be a strongly extending module, as the following example shows.

3.27. Example. Consider $\mathbb{Z}$-module $M = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Q}$ for any prime p. It is clear that $\mathbb{Z}/p\mathbb{Z}$ is strongly extending. Now we show that $\mathbb{Q}$ is also strongly extending. Let N be a nonzero submodule of $\mathbb{Q}$, $a/b \in \mathbb{Q}$ and $s \in \mathbb{Z}$ with $(a/b)s \neq 0$. Let $0 \neq x/y \in N$. Then $ax \in N$. Hence $(a/b)bx \in N$ and $(a/b)bxs \neq 0$ for $bx \in \mathbb{Z}$. Thus N is a strongly large submodule of $\mathbb{Q}$. Since $\mathbb{Q}$ is a module in which every nonzero submodule is strongly large, $\mathbb{Q}$ is strongly extending. Now consider the submodule $N = (1 + p\mathbb{Z}, 1)\mathbb{Z}$ of M. For the submodule $K = (p - 1 + p\mathbb{Z}, 0)\mathbb{Z}$ of M, $N \cap K = 0$. Then N is not large in M, so N is not strongly large in M. Also N has no proper strongly large extensions in M and then N is an SL-closed submodule of M. Next we show N is not a direct summand. Otherwise we assume that there exists a submodule L of M such that $M = N \oplus L$. For $(1 + p\mathbb{Z}, 1/(p+1)) \in M$ there exist $a \in \mathbb{Z}$ and $(u + p\mathbb{Z}, v) \in L$ with $(1 + p\mathbb{Z}, 1/(p+1)) = (1 + p\mathbb{Z}, 1)a + (u + p\mathbb{Z}, v)$. Then $(1 + p\mathbb{Z}, 1) = (1 + p\mathbb{Z}, 1)a(p+1) + (u + p\mathbb{Z}, v(p+1))$, so $(u + p\mathbb{Z}, v(p+1)) = (1 + p\mathbb{Z}, 1) - (1 + p\mathbb{Z}, 1)a(p+1) \in N \cap L = 0$. So $1 = a(p+1)$. This is a contradiction. Hence M does not have a submodule L with $M = N \oplus L$. Therefore M is not strongly extending.

3.28. Remark. It is not necessary that any factor module of a strongly extending module is strongly extending. For example $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module is not strongly extending, however, we have seen in Example 3.27 that $\mathbb{Q}$ is a strongly extending $\mathbb{Z}$-module. We show that $\mathbb{Q}/\mathbb{Z}$ is not strongly extending. Assume that it is strongly extending and reach a contradiction. Let Π denote the set of positive prime integers. $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p\in\Pi} \mathbb{Z}_{p^\infty}$ and let M denote the module $\bigoplus_{p\in\Pi} \mathbb{Z}_{p^\infty}$. Let $p \in \Pi$ and consider the submodule $N = (0, \ldots, 0, 1/p + \mathbb{Z}, 0, \ldots)\,\mathbb{Z}$ of M. We claim that N is SL-closed in M but not a direct summand. Let K be a strongly large extension of N in M. If K contains an element which the p-th component is zero, then we contradict with strongly largeness of N in K. So all elements of K have p components nonzero. Now if K contains an element which the q-th component is nonzero with $q \neq p$ for some $q \in \Pi$, some like $k = \big(\dots, x/q^i + \mathbb{Z}, \dots, a/p^j + \mathbb{Z}, \dots\big)$ such that $0 \neq x/q^i + \mathbb{Z} \in \mathbb{Z}_{q^\infty}$ and $0 \neq a/p^j + \mathbb{Z} \in \mathbb{Z}_{p^\infty}$, then $kp^{j-1} \in K$. Also $k' = (0, \dots, 0, a/p + \mathbb{Z}, 0, \dots) \in K$, and hence the p-th component of $kp^{j-1} - k' \in K$ is zero, but this is a contradiction. Accordingly, K must be a submodule of $\mathbb{Z}_{p^\infty} \oplus 0$. However N is SL-closed in $\mathbb{Z}_{p^\infty} \oplus 0$, so $K = N$. Therefore N is also SL-closed in M. By assumption N is a direct summand of M and $\mathbb{Z}_{p^\infty} \oplus 0$. But N is a large submodule of $\mathbb{Z}_{p^\infty} \oplus 0$. This is the desired contradiction. So N is not a direct summand of M.

Recall that any R-modules $\{M_i \mid i \in I\}$ are called *relatively injective* if M_i is M_j-injective for all distinct $i, j \in I$. The following lemma is well known [4].

3.29. Lemma. *Let M_1 and M_2 be R-modules and let $M = M_1 \oplus M_2$. Then M_1 is M_2-injective if and only if for every submodule N of M such that $N \cap M_1 = 0$ there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $N \leq M'$.*

We have already observed that the direct sum of strongly extending modules need not be strongly extending. Note the following fact.

3.30. Theorem. *Let $M = M_1 \oplus M_2$ be an SL-c-transitive module where M_1 and M_2 are strongly extending modules. If M_1 and M_2 are both relatively injective modules, then M is also a strongly extending module.*

Proof. Let N be an SL-closed submodule of M with $N \cap M_1 = 0$. By Lemma 3.29, there exists a submodule M' of M such that $M = M_1 \oplus M'$ and $N \leq M'$. Since M_2 is strongly extending and $M' \cong M_2$, M' is strongly extending. Clearly, N is SL-closed in M' and hence N is a direct summand of M', whence a direct summand of M. Now let K be SL-closed in M with $K \cap M_2 = 0$. Similarly, K is a direct summand of M. Therefore M is strongly extending by Proposition 3.26. $\square$

3.31. Corollary. *Let $M = \bigoplus_{i=1}^{n} M_i$ $(n \in \mathbb{N})$ be an SL-c-transitive module where every M_i is relatively injective. Then M is strongly extending if and only if all M_i are strongly extending.*

Proof. Theorem 3.25 and Theorem 3.30 complete the proof by induction on n. $\square$

The following example is noteworthy in order to show that the sufficiency of Corollary 3.31 does not hold for infinite direct sum of strongly extending modules.

3.32. Example. The $\mathbb{Z}$-module $\mathbb{Z}^{(\mathbb{N})}$ is SL-c-transitive by Lemma 4.8 and it is well known that $\mathbb{Z}$ is strongly extending. However $\mathbb{Z}^{(\mathbb{N})}$ is not strongly extending since it is not an extending $\mathbb{Z}$-module.

4. The Set $Z_S(M)$

Now we remind of some information in [9]. Let M be a module. M is said to satisfy condition (*) in case of, for each $0 \neq m \in M$ and $r_1, r_2 \in R$, if $r_i \notin r_R(m)$ for some $i = 1, 2$ and $r_1 R r_2 \subseteq r_R(m)$, then $r_j = 0$ for $j \neq i$. Examples of modules with the condition (*) include any faithful prime module, any free module over a domain and any ring with no nonzero divisors of zero as a module over itself.

We set $Z_S(M) = \{m \in M \mid r_R(m)$ is strongly large in $R_R\}$. Then $Z_S(M)$ is always an abelian subgroup of M. If M is a module with the condition (*), then $Z_S(M)$ is a submodule of M. If M is a module over a commutative ring or a semiprime ring, then $Z_S(M)$ is a submodule of M. In this case $Z_S(M)$ is called *a strongly singular submodule.* Also M is called *a strongly singular module* if $Z_S(M) = M$, and M is called *a non-strongly singular module* if $Z_S(M) = 0$. Note that $Z_S(M) \subseteq Z(M)$ since strongly largeness implies largeness, but there is not the other inclusion in general. The next two lemmas are proved in [9] we record for future use.

4.1. Lemma. *Let M be a module with the condition (*). If K is a strongly large submodule of M, then M/K is a strongly singular R-module.*

4.2. Lemma. *If M is a non-strongly singular R-module, then* $\mathrm{Hom}_R(N, M) = 0$ *for all strongly singular modules N. The converse holds if $Z_S(M)$ is a submodule of M.*

Let M be a module and assume that $Z_S(M)$ is a submodule. We denote the residue class of an $m \in M$ in $M/Z_S(M)$ by $\overline{m}$. Consider $Z_S^2(M) = \{m \in M : \overline{m} \in Z_S(M/Z_S(M))\}$. We provide some properties of $Z_S^2(M)$.

4.3. Proposition. *Let M be a module with the condition (*). Then $Z_S^2(M)$ is a submodule of M and $Z_S(M/Z_S(M)) = Z_S^2(M)/Z_S(M)$.*

Proof. Let $m \in Z_S^2(M)$ and $x \in R$. We show that $r_R(\overline{mx})$ is a strongly large right ideal in R. Let $t \in R \setminus r_R(\overline{mx})$ and $s \in R$ such that $ts \neq 0$. Then $mxt \neq 0$. Hence xt does not belong to $r_R(m)$. Consider the following cases:

(1) If $xts \neq 0$, there exists $t_1 \in R$ such that $xtt_1 \in r_R(\overline{m})$ and $xtt_1s \neq 0$ because $r_R(\overline{m})$ is strongly large in R. Thus $tt_1 \in r_R(\overline{mx})$ and $tt_1s \neq 0$.

(2) If $xts = 0$, there exists $t_2 \in R$ such that $mxtt_2s \neq 0$ by the condition (*). Hence $xtt_2s \neq 0$. Since $r_R(\overline{m})$ is strongly large in R, there exists $t_3 \in R$ such that $xtt_2t_3 \in r_R(\overline{m})$ and $xtt_2t_3s \neq 0$. Thus $tt_2t_3 \in r_R(\overline{mx})$ and $tt_2t_3s \neq 0$.

Therefore $mx \in Z_S^2(M)$, and so $Z_S^2(M)$ is a submodule of M. Thus $Z_S(M/Z_S(M)) = Z_S^2(M)/Z_S(M)$. □

As well as Proposition 4.3, $Z_S^2(M)$ is a submodule of a module M over the ring which mentioned in the following proposition.

4.4. Proposition. *$Z_S^2(M)$ is a submodule of M if M is a module over any of the following rings:*

(1) *commutative ring,*

(2) *semiprime ring,*

Proof. Let M be a module, $m \in Z_S^2(M)$ and $x \in R$. For each case of R we prove that $r_R(\overline{mx})$ is strongly large in R_R. Assume that $t, s \in R$ with $ts \neq 0$.

(1) Let R be a commutative ring. Since $r_R(\overline{m})$ is strongly large in R_R, there exists $t_1 \in R$ such that $tt_1 \in r_R(\overline{m})$ and $tt_1s \neq 0$. So $mtt_1 \in Z_S(M)$ and then $mtt_1x = mxtt_1 \in Z_S(M)$. Thus $tt_1 \in r_R(\overline{mx})$ and the proof is completed in this case.

(2) Let R be a semiprime ring and consider the following situations:

(2a) Assume that $\overline{mx}ts \neq \overline{0}$. Then $xts \neq 0$, and so there exists $t_2 \in R$ such that $xtt_2 \in r_R(\overline{m})$ and $xtt_2s \neq 0$. Hence $tt_2 \in r_R(\overline{mx})$, and $tt_2s \neq 0$.

(2b) Assume that $\overline{mx}ts = \overline{0}$. So $mxts \in Z_S(M)$. On the other hand, there exists $t_3 \in R$ with $tst_3ts \neq 0$, because R is semiprime. Also $mxtst_3t \in Z_S(M)$, and then $tst_3t \in r_R(\overline{mx})$.

Therefore $r_R(\overline{mx})$ is strongly large in R_R. So $Z_S(M/Z_S(M)) = Z_S^2(M)/Z_S(M)$. □

4.5. Remark. The second singular submodule $Z_2(M)$ of an R-module M contains $Z_S^2(M)$ when $Z_S^2(M)$ is a submodule of M. In order to see this inclusion let $m \in Z_S^2(M)$. Also $Z_S(M) \leq Z(M)$, then $f : M/Z_S(M) \longrightarrow M/Z(M)$ with $f(x + Z_S(M)) = x + Z(M)$ is a homomorphism. Since $r_R(m+Z_S(M)) \leq r_R(f(m+Z_S(M))) = r_R(m + Z(M))$ and $r_R(m + Z_S(M))$ is strongly large in R, $r_R(m + Z(M))$ is large in R. Therefore $m \in Z_2(M)$.

4.6. Proposition. *Let M be a module with the condition (*). Then $Z_S(M)$ is strongly large in $Z_S^2(M)$.*

Proof. Let $m \in Z_S^2(M) \setminus Z_S(M)$ and $s \in R$ with $ms \neq 0$. Since $r_R(\overline{m})$ is strongly large in R_R, there exists a nonzero element x in R such that $x \in r_R(\overline{m})$ and $xs \neq 0$. It follows that $mx \in Z_S(M)$. If $mxs \neq 0$, then there is nothing to prove. Now suppose that $mxs = 0$. Since $s \notin r_R(m)$ and $x \neq 0$, there exists $y \in R$ such that $mxys \neq 0$ in accordance with the condition (*) and $mxy \in Z_S(M)$. Hence $Z_S(M)$ is a strongly large submodule of $Z_S^2(M)$. □

4.7. Proposition. *Let M be a module with the condition (*). Then $Z_S^2(M)$ is an SL-closed submodule of M.*

Proof. Let N be a proper submodule of M in which $Z_S^2(M)$ is strongly large. Then $Z_S(M)$ is a strongly large submodule of N by Lemma 2.6. Since M is a module with the condition (*), also N satisfies the condition (*). It follows that $N/Z_S(M)$ is a strongly singular module from Lemma 4.1. Hence $Z_S(N/Z_S(M)) = N/Z_S(M) \leq Z_S(M/Z_S(M)) = Z_S^2(M)/Z_S(M)$. Therefore $N = Z_S^2(M)$, and so $Z_S^2(M)$ is SL-closed in M. □

The answer of the mentioned open question is also affirmative in the next lemma.

4.8. Lemma. *Let M be a module with the condition (*). Then M is an SL-c-transitive module.*

Proof. Let N and K be submodules of M with N SL-closed in K and K SL-closed in M. By assumption, every submodule of M satisfies the condition (*). And so largeness implies strongly largeness in every submodule of M. It follows that N is closed in K and K is closed in M. Then N is also closed in M. Therefore N is an SL-closed submodule of M. □

In the extending case we have known from [5] that a module M is extending if and only if $M = Z_2(M) \oplus N$, where the second singular submodule $Z_2(M)$ of M and N are extending and $Z_2(M)$ is N-injective. Now we present the analogue of this statement for the strongly extending case.

4.9. Theorem. *Let M be a module with the condition (*). Then M is strongly extending if and only if $M = Z_S^2(M) \oplus M'$ for some submodule M' of M such that $Z_S^2(M)$ and M' are both strongly extending and $Z_S^2(M)$ is M'-injective.*

Proof. Assume that M is a strongly extending module. We have known that $Z_S^2(M)$ is SL-closed in M by Proposition 4.7, so there exists a submodule M' of M with $M = Z_S^2(M) \oplus M'$. By Theorem 3.25, $Z_S^2(M)$ and M' are both strongly extending. On the other hand, let N be a submodule of M with $N \cap Z_S^2(M) = 0$. Since M is strongly extending, there exists a decomposition $M = L_1 \oplus L_2$ such that N is strongly large in L_1. It follows that $L_1 \cap Z_S^2(M) = 0$. Then L_1 is a non-strongly singular module since $Z_S(L_1) = L_1 \cap Z_S(M) \leq L_1 \cap Z_S^2(M) = 0$. Hence $Z_S(M) = Z_S(L_2) \leq Z_S^2(L_2) \leq Z_S^2(M)$. So $Z_S^2(M)$ is a strongly large extension of $Z_S^2(L_2)$. Since L_2 satisfies the condition (*), $Z_S^2(L_2)$ is an SL-closed submodule of L_2. Also $Z_S^2(L_2)$ is SL-closed in M by Lemma 4.8. Thus we have $Z_S^2(M) = Z_S^2(L_2)$, and so $Z_S^2(M) \leq L_2$. Hence $M = Z_S^2(M) \oplus (M' \cap L_2) \oplus L_1$ and $N \leq (M' \cap L_2) \oplus L_1$. Then by Lemma 3.29, $Z_S^2(M)$ is M'-injective.

Conversely, assume that $M = Z_S^2(M) \oplus M'$ where $Z_S^2(M)$ and M' are both strongly extending and $Z_S^2(M)$ is M'-injective. Let $x \in Z_S(M')$. Then $x + Z_S(M) \in Z_S^2(M)/Z_S(M)$, and so $x = 0$. Hence M' is a non-strongly singular module. Now we show that $\mathrm{Hom}_R(Z_S^2(M), M') = 0$. Let $f \in \mathrm{Hom}_R(Z_S^2(M), M')$ and $m \in Z_S(M)$. Since $r_R(m)$ is strongly large in R_R, $r_R(f(m))$ is also strongly large in R_R from Lemma 2.6. Thus

$f(m) \in Z_S(M) \cap M' = 0$ and so $m \in Kerf$. Then there exists a homomorphism $g : Z_S^2(M)/Z_S(M) \longrightarrow M'$ such that $g\pi = f$ where π is the natural epimorphism of $Z_S^2(M)$ onto $Z_S^2(M)/Z_S(M)$. But $g = 0$ due to $g\left(Z_S^2(M)/Z_S(M)\right) = g\left(Z_S\left(M/Z_S(M)\right)\right) \leq Z_S(M') = 0$, and hence $f = 0$. Therefore M' is $Z_S^2(M)$-injective. Then Lemma 4.8 and Theorem 3.30 complete the proof. □

4.10. Remark. For any module M with the condition (*), M is strongly extending if and only if M is extending. So the submodules $Z_2(M)$ and $Z_S^2(M)$ are both direct summands of M, however they are unequal. In addition, $Z_S^2(M)$ is also a direct summand of $Z_2(M)$.

5. Strongly Extending and Singularities

In this section we investigate some relations between strongly extending modules and Baer modules. In [8], an R-module M with $S = \mathrm{End}_R(M)$ is called *Baer* if for any submodule N of M, $l_S(N) = Se$ with $e^2 = e \in S$. A module M is said to be $\mathcal{K}$-*nonsingular* if Kerφ is large in M for all $\varphi \in S$, implies $\varphi = 0$. The module M is said to be $\mathcal{K}$-*cononsingular* if for any submodule N of M, $l_S(N) = 0$ implies N is large in M. Since every strongly extending module is extending, every strongly extending module is $\mathcal{K}$-cononsingular, and every $\mathcal{K}$-nonsingular strongly extending module is Baer from [8]. In general, a Baer module need not be strongly extending, as shown in the following example.

5.1. Example. Consider $\mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$ as a $\mathbb{Z}$-module. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0$, the endomorphism ring S of $\mathbb{Q}\oplus(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\begin{bmatrix} \mathbb{Q} & 0 \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{bmatrix}$. Let N be a submodule of $\mathbb{Q} \oplus (\mathbb{Z}/2\mathbb{Z})$. If $N = 0$, then $l_S(N) = S$. Assume that N is nonzero. Let $(x, y+2\mathbb{Z})$ be a nonzero element of N. Then either $x \neq 0$ or $y+2\mathbb{Z} = 1+2\mathbb{Z}$. We have three cases:
Case 1. All elements of N have first components zero, that is, $N = \{(0, 0+2\mathbb{Z}), (0, 1+2\mathbb{Z})\}$. Then $l_S(N) = \begin{bmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & 0+2\mathbb{Z} \end{bmatrix}$.
Case 2. If $x \neq 0$ and $y + 2\mathbb{Z} = 0 + 2\mathbb{Z}$, then $l_S(N) = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}/2\mathbb{Z} \end{bmatrix} = S \begin{bmatrix} 0 & 0 \\ 0 & 1+2\mathbb{Z} \end{bmatrix}$.
Case 3. If $x \neq 0$ and $y + 2\mathbb{Z} = 1 + 2\mathbb{Z}$, then $l_S(N) = 0$.
As a result $\mathbb{Q}\oplus(\mathbb{Z}/2\mathbb{Z})$ is a Baer module, however it is not strongly extending by Example 3.27.

5.2. Proposition. *Let M be a finitely generated module over a principal ideal domain. If M is Baer, then M is strongly extending.*

Proof. Let M be a Baer module. We know from [8] that M is semisimple or torsion-free. So M is strongly extending by Example 3.3 or Proposition 3.9. □

5.3. Definition. Let M be an R-module with $S = \mathrm{End}_R(M)$. Then M is called *weakly $\mathcal{K}$-nonsingular* if for any $f \in S$, Ker$f$ is strongly large in M implies $f = 0$. The module M is said to be *strongly $\mathcal{K}$-cononsingular* if for any submodule N of M, $l_S(N) = 0$ implies N is strongly large in M.

Clearly, every $\mathcal{K}$-nonsingular module is weakly $\mathcal{K}$-nonsingular and every strongly $\mathcal{K}$-cononsingular module is $\mathcal{K}$-cononsingular.

5.4. Proposition. *Let R be a ring. Then the following hold.*

(1) *Every strongly extending R-module is strongly $\mathcal{K}$-cononsingular.*

(2) *Every weakly $\mathcal{K}$-nonsingular strongly extending R-module is Baer.*
(3) *Every Baer R-module is weakly $\mathcal{K}$-nonsingular.*

Proof. (1) Let M be a strongly extending R-module with $S = \mathrm{End}_R(M)$ and N a submodule of M with $fN \neq 0$ for all $0 \neq f \in S$. Assume that N is not strongly large in M. Then N is strongly large in eM for some $1 \neq e^2 = e \in S$, and so $1 - e \neq 0$ and $(1 - e)N = 0$. Hence we have a contradiction. Therefore N is strongly large in M.
(2) Let M be a weakly $\mathcal{K}$-nonsingular strongly extending R-module with $S = \mathrm{End}_R(M)$ and $f \in S$ with Ker$f$ large in M. Then Ker$f$ is strongly large in eM for some $e^2 = e \in S$. This implies that eM is large in M, and so $e = 1$. Hence Ker$f$ is strongly large in M. It follows that $f = 0$. Thus M is $\mathcal{K}$-nonsingular. Also M is an extending module. Therefore M is Baer by [8, Lemma 2.14].
(3) Clear from [8, Lemma 2.15]. □

5.5. Proposition. *Let M be a Baer and strongly $\mathcal{K}$-cononsingular module. If every proper direct summand of M is strongly extending, then M is also strongly extending.*

Proof. Let $S = \mathrm{End}_R(M)$ and N be an SL-closed submodule of M. Then $l_S(N) = Sf$ for some $f^2 = f \in S$. If $f = 0$, then N is strongly large in M, and so $N = M$. Let $f \neq 0$. We have $N \leq r_M(l_S(N)) = (1-f)M$. If N is strongly large in $(1-f)M$, then $N = (1-f)M$. Now assume that N is not strongly large in $(1 - f)M$. Due to $f \neq 0$, $(1 - f)M \neq M$. By assumption $(1 - f)M$ is strongly extending, and by Proposition 3.1 N is not large in $(1 - f)M$. Hence there exists a submodule P of $(1 - f)M$ such that $N \cap P = 0$. Let K be a complement of P in M with $N \leq K$. By strongly $\mathcal{K}$-cononsingularity of M, we have $l_S(K) \neq 0$ since K is not strongly large in M. There exists $0 \neq s \in l_S(K)$, and so $s \in l_S(N)$. Hence $s(K \oplus P) = 0$ because $sP \leq s(1 - f)M = 0$. Thus $K \oplus P \leq$ Ker$s$. Since $K \oplus P$ is a large submodule of M, Ker$s$ is also large in M. On the other hand, Ker$s$ is a direct summand of M because M is Baer. Hence Ker$s = M$, so $s = 0$. This is a contradiction. Therefore N is a direct summand of M, and then M is strongly extending. □

Theorem 5.6 is an immediate consequence of Proposition 5.4 and Proposition 5.5.

5.6. Theorem. *Let M be a module. If M is strongly extending and weakly $\mathcal{K}$-nonsingular, then it is Baer and strongly $\mathcal{K}$-cononsingular. The converse holds if every proper direct summand of M is a strongly extending module.*

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