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# WEAKLY Φ-CONTINUOUS FUNCTIONS IN GRILL TOPOLOGICAL SPACES

Ahmad Al-Omari $^{*\dagger}$  and Takashi Noiri $^{\ddagger}$ 

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#### Abstract

In this paper, we introduce and investigate the notion of a weakly  $\Phi$ continuous function in grill topological spaces and using this function we obtain a decomposition of continuity. Also, we investigate its relationship with other related functions.

**Keywords:** Grill topological space, Weak  $\Phi$ -continuity, Decomposition of continuity

2000 AMS Classification:  $54 \operatorname{A} 05, \, 54 \operatorname{C} 10$ 

# 1. Introduction

The idea of grills on a topological space was first introduced by Choquet [4]. The concept of grills has proved to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces, and the theory of compactifications and extension problems of different kinds (see [2], [3], [15] for details). In [14], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [6] defined new classes of sets and give a new decomposition of continuity in terms of grills. In this paper, we introduce and investigate the notion of a weakly  $\Phi$ -continuous function of a topological space into a grill topological space. By using weak  $\Phi$ -continuity, we obtain a decomposition of continuity which is analogous to the decomposition of continuity due to Levine [9].

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# 2. Preliminaries

Let  $(X, \tau)$  be a topological space with no separation properties assumed. For a subset A of a topological space  $(X, \tau)$ , Cl(A) and Int(A) denote the closure and the interior of A in  $(X, \tau)$ , respectively. The power set of X will be denoted by  $\mathcal{P}(X)$ . A subcollection  $\mathcal{G}$  of  $\mathcal{P}(X)$  is called a grill [4] on X if  $\mathcal{G}$  satisfies the following conditions:

(1)  $A \in \mathcal{G}$  and  $A \subseteq B$  implies that  $B \in \mathcal{G}$ ,

(2)  $A, B \subseteq X$  and  $A \cup B \in \mathcal{G}$  implies that  $A \in \mathcal{G}$  or  $B \in \mathcal{G}$ .

For any point x of a topological space  $(X, \tau)$ ,  $\tau(x)$  denotes the collection of all open neighborhoods of x.

**2.1. Definition.** [14] Let  $(X, \tau)$  be a topological space and  $\mathcal{G}$  be a grill on X. A mapping  $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$  is defined as follows:  $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G} \text{ for all } U \in \tau(x)\}$  for each  $A \in \mathcal{P}(X)$ . The mapping  $\Phi$  is called the *operator associated with the grill*  $\mathcal{G}$  and the topology  $\tau$ .

**2.2. Proposition.** [14] Let  $(X, \tau)$  be a topological space and  $\mathcal{G}$  a grill on X. Then for all  $A, B \subseteq X$ :

- (1)  $A \subseteq B$  implies that  $\Phi(A) \subseteq \Phi(B)$ ,
- (2)  $\Phi(A \cup B) = \Phi(A) \cup \Phi(B),$
- (3)  $\Phi(\Phi(A)) \subseteq \Phi(A) = \operatorname{Cl}(\Phi(A)) \subseteq \operatorname{Cl}(A).$

Let G be a grill on a space X. Then we define a map  $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$  by  $\Psi(A) = A \cup \Phi(A)$  for all  $A \in \mathcal{P}(X)$ . The map  $\Psi$  is a Kuratowski closure operator. Corresponding to a grill  $\mathcal{G}$  on a topological space  $(X, \tau)$ , there exists a unique topology  $\tau_{\mathcal{G}}$  on X given by  $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X-U) = X-U\}$ , where for any  $A \subseteq X, \Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}$ -Cl(A). For any grill  $\mathcal{G}$  on a topological space  $(X, \tau), \tau \subseteq \tau_{\mathcal{G}}$ . If  $(X, \tau)$  is a topological space with a grill,  $\mathcal{G}$  on X, then we call it a *grill topological space* and denote it by  $(X, \tau, \mathcal{G})$ .

The concept of ideals in topological spaces is treated in the classic text of Kuratowski [8] and Vaidyanathaswamy [16]. Janković and Hamlett [7] investigated further properties of ideal spaces. An ideal  $\mathfrak{I}$  on a topological space  $(X, \tau)$  is a non-empty collection of subsets of X which satisfies the following properties: (1)  $A \in \mathfrak{I}$  and  $B \subseteq A$  implies  $B \in \mathfrak{I}$ ; (2)  $A \in \mathfrak{I}$  and  $B \in \mathfrak{I}$  implies  $A \cup B \in \mathfrak{I}$ . An ideal topological space or simply an ideal space is a topological space  $(X, \tau)$  with an ideal  $\mathfrak{I}$  on X and is denoted by  $(X, \tau, \mathfrak{I})$ . For a subset  $A \subseteq X$ ,  $A^*(\mathfrak{I}, \tau) = \{x \in X : A \cap U \notin \mathfrak{I}$  for every  $U \in \tau(x)\}$ , where  $\tau(x) = \{U \in \tau : x \in U\}$ , is called the *local function of A with respect to*  $\mathfrak{I}$  and  $\tau$  [8]. We simply write  $A^*$  in case there is no chance for confusion. A Kuratowski closure operator  $\mathrm{Cl}^*(\cdot)$  for a topology  $\tau^*(\mathfrak{I}, \tau)$ , called the \*-topology finer than  $\tau$ , is defined by  $\mathrm{Cl}^*(A) = A \cup A^*$  [7].

The following lemma will be useful in the sequel.

**2.3. Lemma.** [12] Let  $(X, \tau)$  be a topological space. Then the following hold.

- (1)  $\mathcal{G}$  is a grill on X if and only if  $\mathcal{I} = \mathcal{P}(X) \mathcal{G}$  is an ideal on X,
- (2) The operators  $Cl^*$  on  $(X, \tau, \mathfrak{I})$ , where  $\mathfrak{I} = \mathfrak{P}(X) \mathfrak{G}$  and  $\Psi$  on  $(X, \tau, \mathfrak{G})$  are equal.

#### 3. $\Phi$ -continuous functions

**3.1. Definition.** [6] Let  $(X, \tau)$  be a topological space and  $\mathcal{G}$  be a grill on X. A subset A in X is said to be  $\Phi$ -open if  $A \subseteq \text{Int}(\Phi(A))$ . The complement of a  $\Phi$ -open set is said to be  $\Phi$ -closed.

**3.2. Lemma.** If a subset A of a grill topological space  $(X, \tau, \mathcal{G})$  is  $\Phi$ -closed, then  $\Phi(\text{Int}(A)) \subseteq A$ .

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*Proof.* Suppose that A is  $\Phi$ -closed. Then, we have  $X - A \subseteq \operatorname{Int}(\Phi(X - A)) \subseteq \operatorname{Int}(\operatorname{Cl}(X - A)) = X - \operatorname{Cl}(\operatorname{Int}(A))$ . Therefore  $\Phi(\operatorname{Int}(A)) \subseteq \operatorname{Cl}(\operatorname{Int}(A)) \subseteq A$ .

We denote by  $\Phi O(X, \tau) = \{A \subseteq X : A \subseteq \text{Int}(\Phi(A))\}$  or simply write  $\Phi O(X)$  for  $\Phi O(X, \tau)$  when there is no chance for confusion.

**3.3. Definition.** A subset A of a grill topological space  $(X, \tau, \mathcal{G})$  is said to be

- (1)  $\mathfrak{G}$ -dense-in-itself (rep.  $\mathfrak{G}$ -perfect) if  $A \subseteq \Phi(A)$  (resp.  $A = \Phi(A)$ ),
- (2)  $\mathfrak{G}$ -preopen [6] if  $A \subseteq \operatorname{Int}(\Psi(A))$ ,
- (3) preopen [11] if  $A \subseteq Int(Cl(A))$ .

**3.4. Theorem.** [6] Let  $(X, \tau, \mathcal{G})$  be a grill topological space. Then

- (1) Every  $\Phi$ -open set A is G-preopen.
- (2) Every  $\mathcal{G}$ -preopen set A is preopen.

**3.5. Theorem.** For a subset A of a grill topological space  $(X, \tau, \mathfrak{G})$ , the following conditions are equivalent:

- (1) A is  $\Phi$ -open;
- (2) A is *G*-preopen and *G*-dense-in-itself.

*Proof.* (1)  $\Longrightarrow$  (2) By Theorem 3.4 every  $\Phi$ -open set is  $\mathcal{G}$ -preopen. On the other hand  $A \subseteq \text{Int}(\Phi(A)) \subseteq \Phi(A)$ , which show that A is  $\mathcal{G}$ -dense-in-itself.

(2)  $\Longrightarrow$  (1) By assumption,  $A \subseteq Int(\Psi(A)) = Int(\Phi(A) \cup A) = Int(\Phi(A))$  and hence A is  $\Phi$ -open.

The following examples show that  $\mathcal{G}$ -preopen and  $\mathcal{G}$ -dense-in-itself are independent concepts.

**3.6. Example.** Let  $X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b, c\}\}$  and the grill  $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$ . Then  $A = \{a, c\}$  is a  $\mathcal{G}$ -preopen set which is not  $\mathcal{G}$ -dense-in-itself.

**3.7. Example.** Let  $X = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}\)$  and the grill  $\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}.$  Then  $B = \{a, b\}\)$  is a  $\mathcal{G}$ -dense-in-itself set which is not preopen and hence it is not  $\mathcal{G}$ -preopen.

3.8. Remark. It should be noted that:

(1) It is shown in Example 2.1 of [6] that  $\Phi$ -openness and openness are independent of each other.

(2) In [6], it is shown that  $\Phi$ -openness  $\implies$   $\Im$ -openness  $\implies$  preopenness and the converses are not true in general.

**3.9. Definition.** [6] A function  $f : (X, \tau, \mathfrak{G}) \to (Y, \sigma)$  is said to be  $\Phi$ -continuous if for each open set V in Y,  $f^{-1}(V)$  is  $\Phi$ -open in  $(X, \tau, \mathfrak{G})$ .

**3.10. Theorem.** For a function  $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ , the following properties are equivalent:

- (1) f is  $\Phi$ -continuous;
- (2) The inverse image of each closed set of Y is  $\Phi$ -closed;
- (3) For each  $x \in X$  and each  $V \in \sigma$  containing f(x), there exists  $W \in \Phi O(X)$  containing x such that  $f(W) \subseteq V$ ;
- (4) For each  $x \in X$  and each  $V \in \sigma$  containing f(x),  $\Phi(f^{-1}(V))$  is a neighborhood of x.

*Proof.*  $(1) \iff (2)$  Obvious.

(1)  $\implies$  (3) Since  $V \in \sigma$  contains f(x), by (1),  $f^{-1}(V) \in \Phi O(X)$ . By putting  $W = f^{-1}(V)$ , we have  $x \in W$  and  $f(W) \subseteq V$ .

(3)  $\Longrightarrow$  (4) Since  $V \in \sigma$  contains f(x), by (3), there exists  $W \in \Phi O(X)$  containing x such that  $f(W) \subseteq V$ . Thus,  $x \in W \subseteq \operatorname{Int}(\Phi(W)) \subseteq \operatorname{Int}(\Phi(f^{-1}(V))) \subseteq \Phi(f^{-1}(V))$ . Hence  $\Phi(f^{-1}(V))$  is a neighborhood of x.

(4)  $\Longrightarrow$  (1) Let V be any open set of Y and  $x \in f^{-1}(V)$ . Then  $f(x) \in V \in \sigma$ . By (4), there exists an open set of X such that  $x \in U \subseteq \Phi(f^{-1}(V))$ . Therefore,  $x \in U \subseteq \operatorname{Int}(\Phi(f^{-1}(V)))$ . This shows that  $f^{-1}(V) \subseteq \operatorname{Int}(\Phi(f^{-1}(V)))$ . Therefore, f is  $\Phi$ -continuous.

A function  $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$  is said to be  $\mathcal{G}$ -dense-continuous (resp.  $\mathcal{G}$ -precontinuous [6], precontinuous [11]) if the inverse image of every open set is  $\mathcal{G}$ -dense-in-itself (resp.  $\mathcal{G}$ -preopen, preopen).

Thus we have the following decomposition of  $\Phi$ -continuity.

**3.11. Theorem.** For a function  $f : (X, \tau, \mathfrak{G}) \to (Y, \sigma)$ , the following properties are equivalent:

- (1) f is  $\Phi$ -continuous;
- (2) f is  $\mathfrak{G}$ -precontinuous and  $\mathfrak{G}$ -dense-continuous.

The following two examples show that G-precontinuity and G-dense-continuity are independent of each other.

**3.12. Example.** Let  $X = \{a, b, c\}, \tau = \{\phi, X, \{a\}, \{b, c\}\}$  and  $\sigma = \{\phi, X, \{a\}, \{c\}\{a, c\}\}$ . Let  $\mathcal{G} = \{\{c\}, \{a, c\}, \{b, c\}, X\}$  be a grill on X. The identity function  $f : (X, \tau, \mathcal{G}) \rightarrow (X, \sigma)$  is  $\mathcal{G}$ -precontinuous but it is not  $\mathcal{G}$ -dense-continuous.

- (i) Let  $V = X \in \sigma$ , then  $f^{-1}(V)$  is  $\mathcal{G}$ -preopen.
- (ii) Let  $V = \{a\} \in \sigma$ , then  $f^{-1}(V)$  is  $\mathcal{G}$ -preopen.
- (iii) Let  $V = \{c\} \in \sigma, f^{-1}(V)$  is  $\mathcal{G}$ -preopen.

(iv) Let  $V = \{a, c\} \in \sigma$ , then  $f^{-1}(V)$  is  $\mathcal{G}$ -preopen set which is not  $\mathcal{G}$ -dense-in-itself. By (i), (ii, (iii) and (iv), f is a  $\mathcal{G}$ -precontinuous function which is not  $\mathcal{G}$ -dense-continuous.

**3.13. Example.** Let  $X = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$  and  $\mathcal{G} = \{\{a\}, \{b\}, \{a, c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}, \{c, b, d\}, \{a, c, d\}, \{b, c\}, \{b, d\}, \{b, c, d\}, X\}$  be a grill on X. Define a function  $f : (X, \tau, \mathcal{G}) \to (X, \tau)$  as follows: f(a) = c, f(b) = a, f(c) = b and f(d) = b. Then f is  $\mathcal{G}$ -dense-continuous but it is not  $\mathcal{G}$ -precontinuous.

- (i) Let  $V = X \in \sigma$ , then  $f^{-1}(V) = X$  is  $\mathcal{G}$ -dense-in-itself.
- (ii) Let  $V = \{a\} \in \sigma$ , then  $f^{-1}(V) = \{b\}$  is 9-dense-in-itself.
- (iii) Let  $V = \{c\} \in \sigma$ ,  $f^{-1}(V) = \{a\}$  is 9-dense-in-itself.
- (iv) Let  $V = \{a, c\} \in \sigma$ , then  $f^{-1}(V) = \{a, b\}$  is G-dense-in-itself set which is not G-preopen.

By (i), (ii, (iii) and (iv), f is a G-dense-continuous function which is not G-precontinuous.

## 4. Weakly $\Phi$ -continuous functions

Let  $(X, \tau)$  be a topological space and  $(Y, \sigma, \mathcal{G})$  a grill topological space. A function  $f: (X, \tau) \to (Y, \sigma)$  is said to be *weakly continuous* [9] if for each  $x \in X$  and each open set V in Y containing f(x), there exists an open set U containing x such that  $f(U) \subseteq \operatorname{Cl}(V)$ .

**4.1. Definition.** A function  $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$  is said to be *weakly*  $\Phi$ -continuous if for each  $x \in X$  and each open set V in Y containing f(x), there exists an open set U containing x such that  $f(U) \subseteq \Psi(V)$ .

Every weakly  $\Phi$ -continuous function is weakly continuous but the converse is not true (see Example 5.1).

**4.2. Definition.** [1] A function  $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$  is said to be *weakly*  $\mathcal{I}$ -continuous if for each  $x \in X$  and each open set V in Y containing f(x), there exists an open set U containing x such that  $f(U) \subseteq \operatorname{Cl}^*(V)$ .

**4.3. Remark.** Let  $(X, \tau)$ ,  $(Y, \sigma)$  be two topological spaces and  $\mathcal{G}$  a grill on Y. If  $\mathcal{I} = \mathcal{P}(Y) - \mathcal{G}$ , then by Lemma 2.3, a weakly  $\Phi$ -continuous function  $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$  coincides with the weakly  $\mathcal{I}$ -continuous function  $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ .

**4.4. Definition.** A grill topological space  $(X, \tau, \mathfrak{G})$  is called an  $A\mathfrak{G}$ -space if  $\operatorname{Cl}(A) \subseteq \Phi(A)$  for every open set  $A \subseteq X$ .

**4.5. Remark.** An ideal topological space  $(X, \tau, \mathcal{I})$  is called an  $FI^*$ -space [1] if  $Cl(A) \subseteq A^*$  for every  $A \in \tau$ . Let  $(X, \tau, \mathcal{G})$  be a grill topological space and  $\mathcal{I} = \mathcal{P}(X) - \mathcal{G}$  the ideal on X. If  $(X, \tau, \mathcal{G})$  is an A $\mathcal{G}$ -space, then  $(X, \tau, \mathcal{I})$  is the  $FI^*$ -space.

**4.6. Theorem.** For a grill topological space  $(Y, \sigma, \mathfrak{G})$ , the following properties are equivalent:

- (1)  $(Y, \sigma, \mathcal{G})$  is an A $\mathcal{G}$ -space;
- (2)  $\sigma \setminus \{\phi\} \subseteq \mathcal{G};$

(3)  $\Phi(V) = \operatorname{Cl}(V) = \Psi(V)$  for every  $V \in \sigma$ .

*Proof.* (1)  $\Longrightarrow$  (2) Let  $(Y, \sigma, \mathcal{G})$  be an  $A\mathcal{G}$ -space. Suppose that there exists  $U \in \sigma \setminus \{\phi\}$  such that  $U \notin \mathcal{G}$ . Then, there exist  $x \in U$  such that  $U \cap U = U \notin \mathcal{G}$ . Therefore,  $x \notin \Phi(U)$  and  $x \in \operatorname{Cl}(U) \setminus \Phi(U)$ . This shows that  $\operatorname{Cl}(U) \notin \Phi(U)$ . This is contrary that  $(Y, \sigma, \mathcal{G})$  is an  $A\mathcal{G}$ -space.

(2)  $\implies$  (3) For any  $V \in \sigma$ , in case  $V = \phi$  (3) is obvious. In case  $V \neq \phi$ ,  $V \in \mathcal{G}$ . Let  $x \in \operatorname{Cl}(V)$ . Then  $\phi \neq U \cap V \in \sigma$  for every  $U \in \sigma(x)$ . By (2),  $U \cap V \in \mathcal{G}$  and hence  $x \in \Phi(V)$ . Therefore,  $\operatorname{Cl}(V) \subseteq \Phi(V)$  for any  $V \in \sigma$ . Moreover, we have  $\operatorname{Cl}(V) \subseteq \Phi(V) \subseteq \Psi(V)$ . Since  $\sigma \subseteq \sigma_{\mathcal{G}}$ ,  $\operatorname{Cl}(V) \supseteq \Psi(V)$  and  $\Phi(V) = \operatorname{Cl}(V) = \Psi(V)$ .

 $(3) \Longrightarrow (1)$  This is obvious.

**4.7. Theorem.** A function  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$  is weakly  $\Phi$ -continuous if and only if for each open set  $V \subseteq Y$ ,  $f^{-1}(V) \subseteq \operatorname{Int}(f^{-1}(\Psi(V)))$ .

*Proof. Necessity.* Let V be any open set of Y and  $x \in f^{-1}(V)$ . Since f is weakly  $\Phi$ continuous, there exists an open set U such that  $x \in U$  and  $f(U) \subseteq \Psi(V)$ . Hence  $x \in U \subseteq$   $f^{-1}(\Psi(V))$  and  $x \in \text{Int}(f^{-1}(\Psi(V)))$ . Therefore, we obtain  $f^{-1}(V) \subseteq \text{Int}(f^{-1}(\Psi(V)))$ .

Sufficiency. Let  $x \in X$  and V be an open set of Y containing f(x). Then  $x \in f^{-1}(V) \subseteq \operatorname{Int}(f^{-1}(\Psi(V)))$ . Let  $U = \operatorname{Int}(f^{-1}(\Psi(V)))$ . Then  $x \in U$  and  $f(U) = f(\operatorname{Int}(f^{-1}(\Psi(V)))) \subseteq f(f^{-1}(\Psi(V))) \subseteq \Psi(V)$ . This shows that f is weakly  $\Phi$ -continuous.

**4.8. Definition.** A subset A of a topological space X is said to be *semi-open* [10] if  $A \subseteq Cl(Int(A))$ .

The following theorem gives characterizations of weakly  $\Phi$ -continuous functions.

**4.9. Theorem.** Let  $(Y, \sigma, \mathfrak{G})$  be an A $\mathfrak{G}$ -space. Then for a function  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$ , the following properties are equivalent:

- (1) f is weakly  $\Phi$ -continuous;
- (2) For every semi-open set V in Y, there exists an open set U in Y such that  $U \subseteq V$ and  $f^{-1}(U) \subseteq \text{Int}(f^{-1}(\Phi(V)));$

(3)  $f^{-1}(U) \subseteq \operatorname{Int}(f^{-1}(\Phi(U)))$  for every open set U in Y.

Proof. (1)  $\Longrightarrow$  (2) Suppose f is weakly  $\Phi$ -continuous and V is semi-open in  $(Y, \sigma)$ . Since V is semi-open in  $(Y, \sigma)$ , there exists an open set U in  $(Y, \sigma)$  such that  $U \subseteq V \subseteq \operatorname{Cl}(U)$ . By Theorem 4.6,  $\Phi(U) = \operatorname{Cl}(U) = \Psi(U)$ . Therefore  $U \subseteq V \subseteq \Phi(U)$  so that  $\Phi(U) = \Phi(V) = \Psi(U)$ . By Theorem 4.7,  $f^{-1}(U) \subseteq \operatorname{Int}(f^{-1}(\Psi(U))) = \operatorname{Int}(f^{-1}(\Phi(V)))$ , which proves (2).

 $(2) \Longrightarrow (3)$  Clear since every open set is semi-open.

(3)  $\Longrightarrow$  (1) Since  $\Phi(U) \subseteq \Psi(U)$ , the proof follows from Theorem 4.7.

**4.10. Theorem.** Let  $(Y, \sigma, \mathfrak{G})$  be an A $\mathfrak{G}$ -space. Then for a function  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$ , the following properties are equivalent:

- (1) f is weakly  $\Phi$ -continuous;
- (2)  $\operatorname{Cl}(f^{-1}(U)) \subseteq f^{-1}(\Psi(U))$  for each open set  $U \subseteq Y$ ;
- (3) f is weakly continuous.

*Proof.* (1) ⇒ (2) Suppose  $x \in Cl(f^{-1}(U)) - f^{-1}(\Psi(U))$  for some open set *U* of *Y*. Then  $x \in Cl(f^{-1}(U))$  and  $x \notin f^{-1}(\Psi(U))$ . Now  $x \notin f^{-1}(\Psi(U))$  implies that  $f(x) \notin \Psi(U) = Cl(U)$ , by Theorem 4.6. Therefore, there exists an open set *W* containing f(x) such that  $W \cap U = \phi$ ; hence  $Cl(W) \cap U = \phi$  and so  $\Psi(W) \cap U = \phi$ . Since *f* is weakly Φ-continuous, there is an open set *V* containing *x* in *X* such that  $f(V) \subseteq \Psi(W)$  and so  $f(V) \cap U = \phi$ . Now  $x \in Cl(f^{-1}(U))$  implies that  $V \cap f^{-1}(U) \neq \phi$  and so  $f(V) \cap U \neq \phi$  which is a contradiction. This completes the proof.

 $(2) \Longrightarrow (3)$  Since  $(Y, \sigma, \mathcal{G})$  is an A $\mathcal{G}$ -space, by Theorem 4.6 and (2) we have  $\operatorname{Cl}(f^{-1}(U)) \subseteq f^{-1}(\operatorname{Cl}(U))$  for every  $U \in \sigma$ . It follows from [13, Theorem 7] that f is weakly continuous.

(3)  $\Longrightarrow$  (1) It is shown in [9, Theorem 1] that  $f: (X, \tau) \to (Y, \sigma)$  is weakly continuous if and only if  $f^{-1}(V) \subseteq \operatorname{Int}(f^{-1}(\operatorname{Cl}(V)))$  for every  $V \in \sigma$ . Since  $(Y, \sigma, \mathfrak{G})$  is an Ag-space, by Theorem 4.6  $\operatorname{Cl}(V) = \Psi(V)$  for every  $V \in \sigma$  and by Theorem 4.7, f is weakly  $\Phi$ -continuous.

**4.11. Definition.** A grill topological space  $(X, \tau, \mathcal{G})$  is called an  $R\mathcal{G}$ -space if for each  $x \in X$  and each open neighbourhood V of x, there exists an open neighbourhood U of x such that  $x \in U \subseteq \Psi(U) \subseteq V$ .

**4.12. Remark.** Let  $(X, \tau, \mathfrak{G})$  be a grill topological space and  $\mathfrak{I} = \mathfrak{P}(X) - \mathfrak{G}$  the ideal on X. If  $(X, \tau, \mathfrak{G})$  is an RG-space, then  $(X, \tau, \mathfrak{I})$  is the RI-space given in [1, Definition 2.2].

**4.13. Theorem.** Let  $(Y, \sigma, \mathfrak{G})$  be an RG-space. Then  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$  is weakly  $\Phi$ -continuous if and only if f is continuous.

*Proof.* The sufficiency is clear.

*Necessity.* Let  $x \in X$  and V be an open set of Y containing f(x). Since Y is an  $R\mathfrak{G}$ -space, there exists an open set W of Y such that  $f(x) \in W \subseteq \Psi(W) \subseteq V$ . Since f is weakly  $\Phi$ -continuous, there exists an open set U such that  $x \in U$  and  $f(U) \subseteq \Psi(W)$ . Hence we obtain that  $f(U) \subseteq \Psi(W) \subseteq V$ . Thus, f is continuous.  $\Box$ 

We now introduce a complementary form of weak  $\Phi$ -continuity.

**4.14. Definition.** Let A be a subset of a grill topological space  $(Y, \sigma, \mathcal{G})$ . The  $\mathcal{G}$ -frontier of A is defined as  $\Phi(A) - \operatorname{Int}(A)$  and is denoted by  $fr_{\mathcal{G}}(A)$ .

**4.15. Definition.** A function  $f: (X, \tau) \to (Y, \sigma, \mathcal{G})$  is said to be *weakly*<sup>\*</sup>  $\Phi$ -continuous if for each open V in Y,  $f^{-1}(fr_G(V))$  is closed in X.

**4.16. Definition.** [1] A function  $f: (X, \tau) \to (Y, \sigma, \mathfrak{I})$  is said to be *weakly*<sup>\*</sup>  $\mathfrak{I}$ -continuous if for each open V in Y,  $f^{-1}(fr^*(V))$  is closed in X, where  $fr^*(V)$  is the \*-frontier of V defined by  $V^* - int(V)$ .

**4.17. Remark.** Let  $(X, \tau)$ ,  $(Y, \sigma)$  be two topological spaces and  $\mathcal{G}$  a grill on Y. If  $\mathcal{I} = \mathcal{P}(Y) - \mathcal{G}$ , then by Lemma 2.3, a weakly\*  $\Phi$ -continuous function  $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$  coincides with the weakly\*  $\mathcal{I}$ -continuous function  $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ .

**4.18. Theorem.** A function  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$  is continuous if and only if it is both weakly  $\Phi$ -continuous and weakly<sup>\*</sup>  $\Phi$ -continuous.

*Proof. Necessity.* The weak  $\Phi$ -continuity is clear. By Proposition 3.7,  $\Phi(A)$  is closed in  $(Y, \sigma, \mathcal{G})$  for every subset A of Y and  $fr_{\mathcal{G}}(A)$  is closed in Y. Therefore, f is weakly<sup>\*</sup>  $\Phi$ -continuous.

Sufficiency. Let  $x \in X$  and V be any open set of Y containing f(x). Since f is weakly  $\Phi$ -continuous, there exists an open set U containing x such that  $f(U) \subseteq \Psi(V)$ . Now  $fr_G(V) = \Phi(V) - \operatorname{Int}(V)$  and thus  $f(x) \notin fr_G(V)$ . Hence  $x \notin f^{-1}(fr_G(V))$  and  $U - f^{-1}(fr_G(V))$  is an open set containing x since f is weakly\*  $\Phi$ -continuous. The proof will be complete when we show  $f(U - f^{-1}(fr_G(V))) \subseteq V$ . To this end let  $z \in U - f^{-1}(fr_G(V))$ . Then  $z \in U$  and hence  $f(z) \in \Psi(V)$ . But  $z \notin f^{-1}(fr_G(V))$  and thus  $f(z) \notin fr_G(V) = \Phi(V) - V = \Psi(V) - V$ . This implies that  $f(z) \in V$ .

The following two examples show that weak  $\Phi$ -continuity and weak<sup>\*</sup>  $\Phi$ -continuity are independent of each other.

**4.19. Example.** Let  $X = \{a, b, c, d\}$ ,  $\tau = \{\phi, X, \{a\}, \{a, d\}\}$  and  $\sigma = \{\phi, X, \{b\}, \{b, d\}\}$ . Let  $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$  be a grill on X. The identity function  $f : (X, \tau) \to (X, \sigma, \mathcal{G})$  is weakly<sup>\*</sup>  $\Phi$ -continuous but it is not weakly  $\Phi$ -continuous.

- (i) Let  $V = X \in \sigma$ , then  $\Phi(V) = \{a, c\}$  and  $fr_{\mathfrak{I}}(V) = \Phi(V) \operatorname{Int}(V) = \phi$  and hence  $f^{-1}(fr_{\mathfrak{I}}(V))$  is closed.
- (ii) Let  $V = \{b\} \in \sigma$ , then  $\Phi(V) = \phi$  and  $fr_{\mathfrak{G}}(V) = \Phi(V) \operatorname{Int}(V) = \phi$  and hence  $f^{-1}(fr_{\mathfrak{G}}(V))$  is closed.
- (iii) Let  $V = \{b, d\} \in \sigma$ , then  $\Phi(V) = \phi$  and  $fr_{\mathfrak{g}}(V) = \Phi(V) \operatorname{Int}(V) = \phi$  and hence  $f^{-1}(fr_{\mathfrak{g}}(V))$  is closed.

By (i), (ii) and (iii), f is weakly<sup>\*</sup>  $\Phi$ -continuous. On the other hand, for  $V = \{b\} \in \sigma$ ,  $\Phi(V) = \phi$  and  $\Psi(V) = \{b\}$ . Then, there exists only one open set  $U = X \in \tau$  such that  $b \in U$ . Since  $f(U) = X \notin \Psi(V)$ , f is not weakly  $\Phi$ -continuous.

**4.20. Example.** Let  $X = \{a, b\}$  and  $\tau = \{\phi, X, \{a\}\}$ . Let  $\mathcal{G} = \{\{a\}, X\}$  be a grill on X. Define a function  $f : (X, \tau) \to (X, \tau, \mathcal{G})$  as f(a) = b and f(b) = a. Then f is weakly  $\Phi$ -continuous but it is not weakly<sup>\*</sup>  $\Phi$ -continuous.

- (i) Let  $a \in X$  and  $V \in \tau$  such that  $f(a) = b \in V = X$ , then there exists an open set  $U = \{a\}$  such that  $a \in U$  and  $f(U) = \{b\} \subseteq \Psi(V) = X$ .
- (ii) Let  $b \in X$  and  $V \in \tau$  such that  $f(b) = a \in V$ , then  $V = \{a\}$  or  $V = \{X\}$ , then there exists an open set U = X such that  $b \in U$  and  $f(U) = X \subseteq \Psi(V) = X$ .

By (i) and (ii), f is weakly  $\Phi$ -continuous. On the other hand, for  $V = \{a\} \in \tau$ , then  $\Phi(V) = \{a, b\}$  and  $fr_{\mathfrak{I}}(V) = \Phi(V) - \operatorname{Int}(V) = \{b\}$ , hence  $f^{-1}(fr_{\mathfrak{I}}(V)) = \{a\}$  which is not closed. Hence f is not weakly<sup>\*</sup>  $\Phi$ -continuous.

**4.21. Proposition.** If  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$  is precontinuous and  $\operatorname{Cl}(f^{-1}(V)) \subseteq f^{-1}(\Psi(V))$  for each open set  $V \subseteq Y$ , then f is weakly  $\Phi$ -continuous.

*Proof.* For any point  $x \in X$  and any open set  $V \subseteq Y$  containing f(x), by the hypothesis, we have  $\operatorname{Cl}(f^{-1}(V)) \subseteq f^{-1}(\Psi(V))$ . Since f is precontinuous,  $x \in f^{-1}(V) \subseteq \operatorname{Int}(\operatorname{Cl}(f^{-1}(V)))$  and hence there exists an open set  $U \subseteq X$  such that  $x \in U \subseteq \operatorname{Cl}(f^{-1}(V)) \subseteq f^{-1}(\Psi(V))$ . Thus  $f(U) \subseteq \Psi(V)$ . This implies that f is weakly  $\Phi$ -continuous.  $\Box$ 

A function  $f: (X, \tau) \to (Y, \sigma)$  is said to be  $\theta$ -continuous at  $x_0$  [5] if for each open set V of  $f(x_0)$ , there exists an open set U containing  $x_0$  such that  $f(\operatorname{Cl}(U)) \subseteq \operatorname{Cl}(V)$ . The function f is said to be  $\theta$ -continuous if it is  $\theta$ -continuous at each point in X.

**4.22. Theorem.** Let  $(Y, \sigma, \mathfrak{G})$  be an A $\mathfrak{G}$ -space. For a function  $f : (X, \tau) \to (Y, \sigma, \mathfrak{G})$ , the implications  $(1) \Longrightarrow (2) \iff (3)$  hold. If  $(X, \tau)$  is regular, they are all equivalent.

- (1) f is  $\theta$ -continuous;
- (2) f is weakly  $\Phi$ -continuous;
- (3) f is weakly continuous.

*Proof.* (1)  $\Longrightarrow$  (2) Let f be  $\theta$ -continuous,  $x \in X$  and V any open set of Y containing f(x). Since f is  $\theta$ -continuous, there exists an open set U containing x such that  $f(\operatorname{Cl}(U)) \subseteq \operatorname{Cl}(V)$ . Then since  $(Y, \sigma, \mathcal{G})$  is an  $A\mathcal{G}$ -space, by Theorem 4.6  $f(U) \subseteq f(\operatorname{Cl}(U)) \subseteq \operatorname{Cl}(V) = \Psi(V)$ . Thus f is weakly  $\Phi$ -continuous.

 $(2) \iff (3)$  This follows from Theorem 4.10.

(3)  $\Longrightarrow$  (1) Suppose that  $(X, \tau)$  is regular. Let f be weakly continuous,  $x \in X$  and V any open set of Y containing f(x). Then, there exists an open set U of X containing x such that  $f(U) \subseteq \operatorname{Cl}(V)$ . Since  $(X, \tau)$  is a regular space, there exists an open set H of x such that  $x \in H \subseteq \operatorname{Cl}(H) \subseteq U$ . Then  $f(\operatorname{Cl}(H)) \subseteq \operatorname{Cl}(V)$ . Thus f is  $\theta$ -continuous.  $\Box$ 

**4.23. Theorem.** Let  $(Y, \sigma, \mathfrak{G})$  be a grill topological space such that  $Y - V \subseteq \Phi(V)$  for every  $V \in \sigma$ . Then

- (1) Every function  $f: (X, \tau) \to (Y, \sigma, \mathfrak{G})$  is  $\theta$ -continuous and weakly  $\Phi$ -continuous.
- (2) A function  $f: (X, \tau) \to (Y, \sigma, \mathfrak{G})$  is continuous if and only if it is weakly<sup>\*</sup>  $\Phi$ -continuous.

*Proof.* (1) By hypothesis  $\Psi(V) = Y$  for every  $V \in \sigma$  and every function f is weakly  $\Phi$ -continuous. Furthermore  $\operatorname{Cl}(V) = Y$  for every  $V \in \sigma$  since  $\Psi(V) \subseteq \operatorname{Cl}(V)$ . Thus every function f is  $\theta$ -continuous.

(2) By (1), every function  $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$  is weakly  $\Phi$ -continuous and by Theorem 4.18, f is continuous.

# 5. Examples

It is well known that continuity implies both  $\theta$ -continuity and precontinuity and also  $\theta$ -continuity implies weak continuity. Therefore, we have the following diagram.



**5.1. Example.** Let  $X = \{a, b, c, d\}, \tau = \{\phi, X, \{a\}, \{a, d\}\}$  and  $\sigma = \{\phi, X, \{b\}, \{b, d\}\}$ . Let  $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$  be a grill on X. The identity function  $f : (X, \tau) \to (X, \sigma, \mathcal{G})$  is  $\theta$ -continuous but it is not weakly  $\Phi$ -continuous (for more details see [1, Example 2.1] and Example 4.19). **5.2. Example.** Let  $X = \{a, b\}$  and  $\tau = \{\phi, X, \{a\}\}$ . Let  $\mathcal{G} = \{\{a\}, X\}$  be a grill on X. Define a function  $f : (X, \tau) \to (X, \tau, \mathcal{G})$  as follows: f(a) = b and f(b) = a. Then f is weakly  $\Phi$ -continuous but it is not continuous (for more details see Example 4.20).

**5.3. Example.** Let  $X = \{a, b\}$  and  $\tau = \{\phi, X, \{a\}\}$ . Let  $\mathcal{G} = \{\{a\}, X\}$  be a grill on X. Define a function  $f : (X, \tau) \to (X, \tau, \mathcal{G})$  as follows: f(a) = b and f(b) = a. Then f is weakly  $\Phi$ -continuous but it is not  $\Phi$ -continuous, since  $f^{-1}(\{a\}) = \{b\}$  is not  $\Phi$ -open (for more details see Example 4.20).

**5.4. Example.** Let  $X = \{a, b, c, d\}$ ,  $\tau = \{\phi, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ ,  $Y = \{a, b\}$  and  $\sigma = \{\phi, Y, \{a\}\}$ . Let  $\mathcal{G} = \{\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$  be a grill on X. Define a function  $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$  as follows: f(a) = f(b) = f(d) = a and f(c) = b. Then f is precontinuous since  $f^{-1}(\{a\}) = \{a, b, d\}$  is preopen. But it is not  $\Phi$ -continuous, since  $f^{-1}(\{a\}) = \{a, b, d\}$  is not  $\Phi$ -open.

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