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THE LIFTINGS OF R-MODULES TO COVERING GROUPOIDS

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Abstract

In this paper we prove that the group structure of a group object in the category of groupoids lifts to a covering groupoid. We also prove similar results for a R-module object in the category of groupoids.

Keywords: Group-groupoid, Covering groupoid, Topological R-module, R-module groupoid.

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1. Introduction

The theory of covering spaces is one of the most interesting theories in algebraic topology. Covering groupoids play an important role in the applications of groupoids (see for example [2] and [7]). The fundamental groupoid functor gives an equivalence of categories between the category of covering spaces of a reasonably nice space X and the category of covering groupoids of $\pi_1(X)$.

We know from [2, Proposition 10.4.3] that if G is a transitive groupoid, x is an object of G and C is a subgroup of the object group $G(x)$, then there is a covering morphism $p\colon (\widetilde{G}_C, \widetilde{x}) \to (G, x)$ of groupoids with characteristic group C.

In this paper using this existence of covering groupoids we prove that if G is a group object in the category of groupoids which is also called a *group-groupoid*, the underlying groupoid of G is transitive and $p\colon \widetilde{G} \to G$ is a covering morphism of groupoids, then $\widetilde{G}$ also becomes a group-groupoid. This result gives an easy way of proving that the group structure of a topological group X lifts to its simply connected covering space, i.e., if X is an additive topological group, $p\colon \widetilde{X} \to X$ is a simply connected covering map, $0 \in X$ is the identity element and $\widetilde{0} \in \widetilde{X}$ is such that $p(\widetilde{0}) = 0$, then $\widetilde{X}$ becomes a topological group with identity $\widetilde{0}$ such that p is a morphism of topological groups.

We also prove similar results for R-module objects in the category of groupoids.

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The problem of universal covers of non-connected topological groups was first studied by Taylor in [14]. He proved that a topological group X determines an obstruction class k_X in $H^3(\pi_0(X), \pi_1(X, e))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure to a universal cover. In [9] an analogous algebraic result is given in terms of crossed modules and group objects in the category of groupoids (see also [4] for a revised version, which generalizes these results and shows the relation with the theory of obstructions to extensions for groups).

2. Preliminaries on covering groupoids

We assume that all topological spaces X are locally path connected and semi-locally 1-connected, so that each path component of X admits a simply connected cover. Recall that a covering map $p\colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every cover of X in the sense that if $q\colon \widetilde{Y} \to X$ is another cover of X then there exists a map $r\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a cover). A covering map $p\colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. So a simply connected cover is a universal cover.

A subset V of X is called *liftable* if it is open, path connected and V lifts to each cover of X, that is, if $p\colon \widetilde{X} \to X$ is a covering map, $\imath\colon V \to X$ is the inclusion map, and $\widetilde{x} \in \widetilde{X}$ satisfies $p(\widetilde{x}) = x \in V$, then there exists a map (necessarily unique) $\hat{\imath}\colon V \to \widetilde{X}$ such that $p\hat{\imath} = \imath$ and $\imath(x) = \widetilde{x}$.

It is easy to see that V is liftable if and only if it is open, path connected and for each $x \in V$ the fundamental group $\pi_1(V, x)$ is mapped to the singleton by the morphism induced by the inclusion map $\imath\colon V \to X$.

Note that if X is a semi-locally simply connected topological space, then each point $x \in X$ has a liftable neighbourhood. So if X is a semi-locally simply connected topological space then each $x \in X$ has a liftable neighbourhood.

A *groupoid* is a small category in which each morphism is an isomorphism [2]. So a groupoid G has a set G of morphisms, which we call just *elements* of G, a set $\mathrm{Ob}(G)$ of *objects* together with maps $s, t\colon G \to \mathrm{Ob}(G)$ and $\epsilon\colon \mathrm{Ob}(G) \to G$ such that $s\epsilon = t\epsilon = 1_{\mathrm{Ob}(G)}$. The maps s, t are called *initial* and *final* point maps respectively and the map ϵ is called *object inclusion*. If $a, b \in G$ and $t(a) = s(b)$, then the *composite* ab exists such that $s(ab) = s(a)$ and $t(ab) = t(b)$. So there exists a partial composition defined by the map $G_t \times_s G \to G, (a, b) \mapsto ab$, where $G_t \times_s G$ is the pullback of t and s. Further, this partial composition is associative, for $x \in \mathrm{Ob}(G)$ the element $\epsilon(x)$ denoted by 1_x acts as the identity and each element a has an inverse a^{-1} such that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $aa^{-1} = (\epsilon s)(a)$, $a^{-1}a = (\epsilon t)(a)$. The map $G \to G$, $a \mapsto a^{-1}$ is called the *inversion*.

In a groupoid G for $x, y \in \mathrm{Ob}(G)$, we write $G(x, y)$ for $s^{-1}(x) \cap t^{-1}(y)$ and say that G is *transitive* if for all $x, y \in \mathrm{Ob}(G)$, $G(x, y)$ is not empty. For $x \in \mathrm{Ob}(G)$ we write G_x for $s^{-1}(x)$ and call G_x the *star* of G at x. The set $s^{-1}(x) \cap t^{-1}(x)$ is a group called the *object group* at x, and denoted by $G(x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f\colon H \to G$ and $f_0\colon \mathrm{Ob}(H) \to \mathrm{Ob}(G)$ such that $s_G \circ f = f_0 \circ s_H$, $t_G \circ f = f_0 \circ t_H$ and $f(ab) = f(a)f(b)$ for all $(a, b) \in H_t \times_s H$. For such a morphism we simply write $f\colon H \to G$.

A morphism $p\colon \widetilde{G} \to G$ of groupoids is called a *covering morphism* and $\widetilde{G}$ a *covering groupoid* of G if for each $\widetilde{x} \in \mathrm{Ob}(\widetilde{G})$ the restriction $(\widetilde{G})_{\widetilde{x}} \to G_{p(\widetilde{x})}$ of p is bijective. A covering morphism $p\colon \widetilde{G} \to G$ is called *transitive* if both groupoids $\widetilde{G}$ and G are transitive.

A transitive covering morphism $p\colon \widetilde{G} \to G$ is called *universal* if $\widetilde{G}$ covers every cover of G, i.e., if for every covering morphism $q\colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\widetilde{q}\colon \widetilde{G} \to \widetilde{H}$ such that $q\widetilde{q} = p$ (and hence $\widetilde{q}$ is also a covering morphism), this is equivalent to that for $\widetilde{x}, \widetilde{y} \in \mathrm{Ob}(\widetilde{G})$ the set $\widetilde{G}(\widetilde{x}, \widetilde{y})$ has not more than one element.

A morphism $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ of pointed groupoids is called a *covering morphism* if the morphism $p\colon \widetilde{G} \to G$ is a covering morphism.

2.1. Theorem. [2, 10.6.1] *Let X be a topological space whose underlying space has a simply connected cover. Then the slice category TCov/X of covering spaces of X is equivalent to the category $\mathsf{GpdCov}/\pi_1(X)$ of the covering groupoids of $\pi_1(X)$.*

Let $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ be a covering morphism of groupoids. We say a morphism $f\colon (H, z) \to (G, x)$ *lifts* to p if there exists a unique morphism $\widetilde{f}\colon (H, z) \to (\widetilde{G}, \widetilde{x})$ such that $f = p\widetilde{f}$. For any groupoid morphism $p\colon \widetilde{G} \to G$ and object $\widetilde{x}$ of $\widetilde{G}$ we call the subgroup $p(\widetilde{G}(\widetilde{x}))$ of $G(p\widetilde{x})$ the *characteristic group* of p at $\widetilde{x}$.

The following result gives a criterion on the lifting of morphisms [2, 10.3.3].

2.2. Theorem. *Let $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ be a covering morphism of groupoids and $f\colon (H, z) \to (G, x)$ a morphism of pointed groupoids such that H is transitive. Then the morphism $f\colon (H, z) \to (G, x)$ lifts to a morphism $\widetilde{f}\colon (H, z) \to (\widetilde{G}, \widetilde{x})$ if and only if the characteristic group of f is contained in that of p; and if this lifting exists, then it is unique.* □

As a result of this Theorem we have the following corollary

2.3. Corollary. *Let $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ and $q\colon (\widetilde{H}, \widetilde{z}) \to (G, x)$ be transitive covering morphisms with characteristic groups C and D respectively. If $C \subseteq D$, then there is a unique covering morphism $r\colon (\widetilde{G}, \widetilde{x}) \to (\widetilde{H}, \widetilde{z})$ such that $p = qr$. If $C = D$, then r is an isomorphism.* □

For the existence of the covering groupoid we need the idea of an action groupoid. Let G be a groupoid. An *action* of G on a set consists of a set X, a function $\omega\colon X \to \mathrm{Ob}(G)$ and a partial function $X_\omega \times_s G \to X, (x, a) \longmapsto xa$ defined on the pullback $X_\omega \times_s G$ of ω and p such that

i. $\omega(xa) = t(a)$
ii. $x(ab) = (xa)b$
iii. $x1_{\omega(x)} = x$.

As an example if $p\colon \widetilde{G} \to G$ is a covering morphism of groupoids, $X = \mathrm{Ob}(\widetilde{G})$ and $\omega = O_p$, then we obtain an action of G on X via ω by assigning to $x \in X$ and $a \in G_{p(x)}$ the target of the unique lift of a with source x.

Given such an action, the *action groupoid* $G \ltimes X$ is defined to be the groupoid with object set X and elements of $(G \ltimes X)(x, y)$ the pairs (a, x) such that $a \in G(\omega(x), \omega(y))$ and $xa = y$. The groupoid composite is defined to be

$$(a, x) \circ (b, y) = (ab, x).$$

The following result is from [2, 10.4.3]. We need some details of the proof for later.

2.4. Theorem. *Let x be an object of a transitive groupoid G, and let C be a subgroup of the object group $G(x)$. Then there exists a covering morphism $q\colon (\widetilde{G}_C, \widetilde{x}) \to (G, x)$ with characteristic group C.*

Proof. Let X be the set of (left) cosets $Ca = \{Ca \mid c \in C\}$ for a in G_x and $\omega\colon X \to \mathrm{Ob}(G)$ map Ca to the final point of a. Then G acts on X by $(Ca)g = Cag$. The required groupoid $\widetilde{G}_C$ is taken to be the action groupoid $G \ltimes X$. Then the projection $q\colon \widetilde{G}_C \to G$ given on objects by $\omega\colon X \to \mathrm{Ob}(G)$ and on elements by $(g, Ca) \mapsto g$, is a covering morphism of groupoids with the characteristic group C. The required object $\widetilde{x} \in \widetilde{G}_C$ is the coset C. $\square$

3. Covering groupoids of group-groupoids

A *group-groupoid*, which is also known in the literature as a 2-*group*, is a group object in the category of groupoids. This is an internal category in the category of groups (Porter [12]). The category of group-groupoids is equivalent to the category of crossed modules (Brown and Spencer [5]). There are a large number of papers in the literature under the name of 2-groups. Recently the ring object in the category of groupoids and their coverings have been developed by Mucuk in [10].

The formal definition of a group-groupoid we use is given by Brown and Spencer in [5] under the name $\mathcal{G}$-*groupoid* as follows:

3.1. Definition. A *group-groupoid* G is a groupoid endowed with a group structure such that the following maps which are called respectively addition, inverse and unit, are morphisms of groupoids:

i. $m\colon G \times G \to G$, $(a, b) \mapsto a + b$;
ii. $u\colon G \to G$, $a \mapsto -a$;
iii. $0\colon \{\star\} \to G$, where $\{\star\}$ is singleton.

In a group-groupoid G, for $a, b \in G$ the groupoid composite is denoted by ab when $s(b) = t(a)$ and the group addition by $a + b$.

Note that the condition (i) is equivalent to the usual interchange law

$$(ac + bd) = (a + b)(c + d)$$

for $a, b, c, d \in G$ whenever ac and bd are defined, and the condition (iii) means that if 0 is the identity element of $\mathrm{Ob}(G)$, then 1_0 is the identity of G.

From Definition 3.1, the following properties, which we need in some detail, follow.

3.2. Proposition. *Let G be a group groupoid:*

i. *if $a \in G(x, y)$ and $b \in G(u, v)$, then $a + b \in G(x + u, y + v)$;*
ii. *$(a + b)^{-1} = a^{-1} + b^{-1}$ for $a, b \in G$;*
iii. *$-(ab) = (-a)(-b)$ for $a, b \in G$ such that ab is defined ;*
iv. *if $a \in G(x, y)$, then $-a \in G(-x, -y)$;*
v. *$(-a)^{-1} = -a^{-1}$ for $a \in G$;*
vi. *$1_x + 1_y = 1_{x+y}$ for $x, y \in \mathrm{Ob}(G)$;*
vii. *$s(a + b) = s(a) + s(b)$ for $a, b \in G$;*
viii. *$t(a + b) = t(a) + t(b)$ for $a, b \in G$.*

Proof. (i) Since in a group-groupoid G, the group addition $m\colon G \times G \to G$, $(a, b) \mapsto a + b$ is a morphism of groupoids, if $a \in G(x, y)$ and $b \in G(u, v)$, then we have that $a + b \in G(x + u, y + v)$.

(ii) By the interchange law for $a, b \in G$

$$(a + b)(a^{-1} + b^{-1}) = (aa^{-1}) + (bb^{-1}) = 1_{s(a)} + 1_{s(b)} = 1_{s(a+b)}$$

and

$$(a^{-1} + b^{-1})(a + b) = (a^{-1}a) + (b^{-1}b) = 1_{t(a)} + 1_{t(b)} = 1_{t(a+b)}.$$

Therefore it follows that $(a + b)^{-1} = a^{-1} + b^{-1}$ for $a, b \in G$.

(iii), (iv) and (v) follow from the fact that the map $G \to G, a \mapsto -a$ is a morphism of groupoids.

(vi), (vii) and (viii) follow from the fact that the addition $+ \colon G \times G \to G$ is a morphism of groupoids. □

Let $\widetilde{G}$ and G be two group-groupoids. A *morphism* $f \colon \widetilde{G} \to G$ of group-groupoids is a morphism of the underlying groupoids preserving also the group structure. A morphism $f \colon \widetilde{G} \to G$ of group-groupoids is called a *cover* (resp. *universal cover*) if it is a covering morphism (resp. a universal cover) on underlying groupoids.

The following example appears in [5]. Brown and Danesh-Naruie proved in [3] that if X is a semi-locally simply connected topological space, then $\pi_1(X)$ is a topological groupoid.

3.3. Example. If X is a topological group, then the fundamental groupoid $\pi_1(X)$ is a group-groupoid.

3.4. Example. [8, 4.3] Let X be an additive group. Then the groupoid $G = X \times X$ is also a group-groupoid with object set X: A pair (x, y) is a morphism from x to y and the groupoid composition is defined by $(x, y)(z, u) = (x, u)$ whenever $y = z$. Here, for an object $x \in X$ the identity morphism at x is $1_x = (x, x)$ and for a morphism $(x, y) \in G$ the groupoid inverse of (x, y) is (y, x). The group addition on G is defined by $(x, y) + (u, v) = (x + u, y + v)$.

If $a = (x, y)$, $c = (y, z)$, $b = (u, v)$ and $d = (v, w)$ are the morphisms in G so that the compositions ac and bd are defined, then we have $(ac) + (bd) = (x + u, z + w)$ and $(a + b)(c + d) = (x + u, z + w)$. Hence the interchange law

$$(ac) + (bd) = (a + b)(c + d).$$

is satisfied.

For the morphisms $a = (x, y)$ and $b = (y, z)$ in G we have $-(ab) = (-x, -z)$ and $(-a)(-b) = (-x, -z)$ and therefore $-(ab) = (-a)(-b)$. For $x \in X$, $-1_x = (-x, -x) = 1_{-x}$. In addition to these if $0 \in X$ is the identity element of the group X, then $1_0 = (0, 0)$ is the identity element of G. From all this, we deduce that G is a group-groupoid.

The following result appears in [4, 9].

3.5. Theorem. *If X is a topological group whose underlying space has a simply connected cover, then the category* TGCov/X *of topological group covers of X is equivalent to the category* $\mathsf{GpGdCov}/\pi_1(X)$ *of group-groupoid covers of $\pi_1(X)$.* □

3.6. Definition. Suppose that G is a group-groupoid and 0 is the identity of $\mathrm{Ob}(G)$. Let $\widetilde{G}$ be a groupoid, $p \colon \widetilde{G} \to G$ a covering morphism of groupoids and $\widetilde{0} \in \mathrm{Ob}(\widetilde{G})$ is such that $p(\widetilde{0}) = 0$. We say the group structure of G *lifts* to $\widetilde{G}$ if there exists a group structure on $\widetilde{G}$ with the identity element $\widetilde{0} \in \mathrm{Ob}(\widetilde{G})$ such that $p \colon \widetilde{G} \to G$ is a morphism of group-groupoids.

We now use Theorem 2.4 to prove that the group structure of a group-groupoid lifts to a covering groupoid.

3.7. Theorem. *Let $\widetilde{G}$ be a groupoid and G a group-groupoid whose underlying groupoid is transitive. Let $0 \in \mathrm{Ob}(G)$ be the identity element of the additive group. Suppose that $p \colon (\widetilde{G}, \widetilde{0}) \to (G, 0)$ is a covering morphism of underlying groupoids such that the characteristic group C of p is a subgroup of the additive group of G. Then the group structure of G lifts to $\widetilde{G}$ with identity $\widetilde{0}$.*

Proof. Let C be the characteristic group of $p\colon (\widetilde{G},\widetilde{0}) \to (G,0)$. Then by Theorem 2.4 we have a covering morphism $q\colon (\widetilde{G}_C,\widetilde{x}) \to (G,0)$ with characteristic group C. So by Corollary 2.3 the covering morphisms p and q are equivalent. Therefore it is sufficient to prove that the group structure of G lifts to $\widetilde{G}_C$ by the covering morphism $q\colon (\widetilde{G}_C,\widetilde{x}) \to (G,0)$.

Let $m\colon G\times G \to G, (a,b) \mapsto a+b$ be the group addition of the group-groupoid G. Now define a group addition on $X = \mathrm{Ob}(\widetilde{G}_C)$ by

$$(Ca)+(Cb) = C(a+b)$$

for $Ca, Cb \in X$. Here note that $a+b \in G_0$ when $a,b\in G_0$ and so $C(a+b)\in X$. We now prove that this addition is well defined, i.e., if $Ca = Ca'$ and $Cb = Cb'$, then $C(a+b) = C(a'+b')$. For if $Ca = Ca'$ and $Cb = Cb'$ then $a'a^{-1}, b'b^{-1} \in C$ and by the interchange law we have that

$$(a'+b')(a+b)^{-1} = (a'+b')(a^{-1}+b^{-1}) = (a'a^{-1}) + (b'b^{-1}).$$

Since C is a subgroup of the additive group of G, we have $(a'+b')(a+b)^{-1}\in C$ and therefore $C(a+b) = C(a'+b')$.

Define a group addition on the morphisms of $\widetilde{G}_C$ by

$$(g,Ca)+(h,Cb) = (g+h, C(a+b)).$$

It is straightforward to see that $\widetilde{G}_C$ is a group-groupoid. For the interchange law when the necessary groupoid compositions are possible we have

$$\begin{aligned}(g,Ca)(k,Cc)+(h,Cb)(t,Cd) &= (gk,Ca)+(ht,Cb)\\ &= (gk+ht, C(a+b)).\\ ((g,Ca)+(h,Cb))((k,Cc)+(t,Cd)) &= (g+h,C(a+b))(k+t,C(c+d))\\ &= ((g+h)(k+t), C(a+b)).\end{aligned}$$

Since G is a group-groupoid $gk+ht = (g+h)(k+t)$, and therefore

$$(g,Ca)(k,Cc)+(h,Cb)(t,Cd) = ((g,Ca)+(h,Cb))((k,Cc)+(t,Cd))$$

i.e., the interchange law is satisfied.

Further the morphism q preserves the group structure as follows:

$$q((g,Ca)+(h,Cb)) = q(g+h, C(a+b)) = g+h = q(g,Ca)+q(h,Cb). \qquad \square$$

As a result of Theorem 3.7 we obtain a proof for a result in the theory of covering spaces [13, 6] (see also [11] for a similar result on topological rings).

3.8. Corollary. *Let X be a path connected topological group with identity 0 and $p\colon (\widetilde{X},\widetilde{0}) \to (X,0)$ a covering map such that $\widetilde{X}$ is simply connected. Then the group structure of X lifts to $\widetilde{X}$, i.e., $\widetilde{X}$ has a group structure with identity $\widetilde{0}$ such that $\widetilde{X}$ is a topological group and p is a morphism of topological groups.*

Proof. Since X is a topological group, by Example 3.3 the fundamental groupoid $\pi_1(X)$ is a group-groupoid and since $p\colon \widetilde{X} \to X$ is a covering map, the induced morphism $\pi_1(p)\colon \pi_1(\widetilde{X}) \to \pi_1(X)$ becomes a covering morphism of groupoids with trivial characteristic group and by [2, 10.5.5] the topology on $\widetilde{X}$ is the lifted topology. Further since X is path connected the groupoid $\pi_1(X)$ is transitive. Therefore by Theorem 3.7 the group structure of $\pi_1(X)$ lifts to $\pi_1(\widetilde{X})$ and so we have a morphism of groupoids

$$\widetilde{m}\colon \pi_1(\widetilde{X})\times\pi_1(\widetilde{X}) \to \pi_1(\widetilde{X})$$

such that $\pi_1(p) \circ \widetilde{m} = \pi_1(m) \circ (\pi_1(p) \times \pi_1(p))$, where m is the group addition on X and $\widetilde{m}$ is a group structure on $\pi_1(\widetilde{X})$. By [2, 10.5.5] $\widetilde{m}$ induces a continuous additive map on $\widetilde{X}$. The fact that this is a group structure follows from the fact that $\widetilde{m}$ is a group structure. □

4. Covering groupoids of R-module groupoids

We now apply these methods to topological R-modules.

4.1. Definition. Let R be a topological ring with identity 1_R. A *topological (left) R-module* is an additive abelian topological group M together with a continuous function $\delta\colon R \times M \to M, (r, a) \mapsto ra$ called an *action* of R on M such that for $r, s \in R$ and $a, b \in M$

i. $r(a + b) = ra + rb$;
ii. $(r + s)a = ra + sa$;
iii. $(rs)a = r(sa)$;
iv. $1_R a = a$.

In [1, Theorem 3.1] the following theorem is proved.

4.2. Theorem. *If R is a countable, Noetherian ring and M is any R-module, then the underlying abelian group M_G of M is isomorphic to the fundamental group $\pi_1(T(M))$ for some path connected topological R-module $T(M)$.* □

This result enables to one to find examples of topological R-modules which are not simply connected and so have non-trivial covering spaces.

As a result of Theorem 4.2, taking $R = \mathbb{Z}$ the following corollary is obtained.

4.3. Corollary. *Every abelian group is isomorphic to the fundamental group of some topological group.* □

4.4. Definition. Let R be a topological ring with identity 1_R and M, N be topological left R-modules. A *morphism* of topological left R-modules is a group morphism $f\colon N \to M$ which is continuous and $f(ra) = rf(a)$ for $a \in N$ and $r \in R$. A morphism $f\colon N \to M$ of topological left R-modules is called a *cover* if f is a covering map on the underlying topological spaces.

We now give the definition of an R-module object in the theory of categories as follows.

4.5. Definition. Let R be a ring with identity 1_R. An *R-module groupoid*, denoted by G_M, is a groupoid in which G and $\mathrm{Ob}(G)$ are both R-modules and; the initial and final point maps $s, t\colon G_M \to \mathrm{Ob}(G_M)$, object inclusion map $\epsilon\colon \mathrm{Ob}(G_M) \to G_M$, partial composite map $(G_M)_t \times_s (G_M) \to G_M, (a, b) \mapsto ab$ and the inversion $G_M \to G_M, a \mapsto a^{-1}$ are all R-module morphisms.

So, an R-module groupoid G_M is a group-groupoid and; for $r \in R$, $x \in \mathrm{Ob}(G_M)$ and $a, b \in G_M$ such that the composite ab is defined, we have $s(ra) = rs(a)$, $t(ra) = rt(a)$, $(ra)^{-1} = r(a^{-1})$, $\epsilon(rx) = r\epsilon(x) = r1_x$ and $(ra)(rb) = r(ab)$. Therefore G_M is an R-module groupoid.

Let R be a ring with identity 1_R. In an R-module groupoid G_M the groupoid composite is denoted by ab when $s(b) = t(a)$, the group addition by $a + b$ for $a, b \in G_M$.

Let $\widetilde{G}_M$ and G_M be two R-module groupoids. A *morphism* of R-module groupoids is a morphism $f\colon \widetilde{G}_M \to G_M$ of group-groupoids preserving the R-module structure. A morphism $f\colon \widetilde{G}_M \to G_M$ of R-module groupoids is called a *cover* if it is a covering morphism on the underlying groupoids. We can give the following example which is similar to Example 3.3.

4.6. Example. If R is a topological ring with identity 1_R and M is a topological R-module, then the fundamental groupoid $\pi_1(M)$ of M is an R-module groupoid: If M is a topological R-module, with a continuous group addition

$$m\colon M \times M \to M, (a,b) \mapsto a+b,$$

a continuous inverse map

$$u\colon M \to M, a \mapsto -a$$

and a continuous action $\delta\colon R\times M \to M,\ (r,a) \mapsto ra$. Then we have the following induced maps

$$\pi_1(m)\colon \pi_1(M) \times \pi_1(M) \to \pi_1(M),\ ([a],[b]) \mapsto [a+b],$$

$$\pi_1(u)\colon \pi_1(M) \to \pi_1(M),\ [a] \mapsto [-a] = -[a],$$

$$R \times \pi_1(M) \to \pi_1(M),\ (r,[a]) \mapsto r[a] = [ra],$$

where the path ra is defined by $(ra)(t) = ra(t)$ for $t \in [0,1]$.

We know from Example 3.3 that $\pi_1(M)$ is a group-groupoid. Further $\pi_1(M)$ becomes an R-module groupoid with this action, as required.

4.7. Example. If M is an R-module, the groupoid $G_M = M \times M$ on M defined as in Example 3.4 is a group-groupoid. Further for $r \in R$, $x \in M$ and $a = (x,y)$, $b = (y,z)$ we have that $s(ra) = rs(a)$, $t(ra) = rt(a)$, $(ra)^{-1} = r(a^{-1})$, $1_{rx} = r1_x$ and $(ra)(rb) = r(ab)$. Therefore G_M is an R-module groupoid.

Let M be a topological R-module. So $\pi_1(X)$ is an R-module groupoid. Then we have a slice category $\mathsf{TModCov}/M$ of topological R-module covers of M and a category $\mathsf{GdModCov}/\pi_1(M)$ of covering R-module groupoids.

4.8. Theorem. *Let R be a topological ring with identity 1_R and M a topological R-module. Suppose that the underlying topology of M has simply connected covers. Then the categories $\mathsf{TModCov}/M$ and $\mathsf{GdModCov}/\pi_1(M)$ are equivalent.*

Proof. Define a functor

$$\pi_1\colon \mathsf{TModCov}/M \to \mathsf{GdModCov}/\pi_1(M)$$

as follows: Suppose that $p\colon \widetilde{M} \to M$ is a covering morphism of topological R-modules. Then the induced morphism $\pi_1(p)\colon \pi_1(\widetilde{M}) \to \pi_1(M)$ is a morphism of group-groupoids and a covering morphism on the underlying groupoids. Further for $[\widetilde{a}] \in \pi_1(\widetilde{M})$ and $r \in R$ we have that

$$\pi_1(p)[r\widetilde{a}] = [p(r\widetilde{a})] = [r(p\widetilde{a})] = r[p\widetilde{a}] = r\pi_1(p)[\widetilde{a}].$$

Therefore $\pi_1 p\colon \pi_1(\widetilde{M}) \to \pi_1(M)$ becomes a covering morphism of R-module groupoids.

We now define another functor

$$\eta\colon \mathsf{GdModCov}/\pi_1(M) \to \mathsf{TModCov}/M$$

as follows: Suppose that $q\colon \widetilde{G}_M \to \pi_1(M)$ is a covering morphism of R-module groupoids. By [2, 10.5.5] there is a lifted topology on $\widetilde{M} = \mathrm{Ob}(\widetilde{G}_M)$ and an isomorphism $\alpha\colon \widetilde{G}_M \to \pi_1(\widetilde{M})$ such that $p = O_q\colon \widetilde{M} \to M$ is a covering map and $q = \pi_1(p)\ \alpha$. Hence the R-module structure on $\widetilde{G}_M$ transports via α to $\pi_1(\widetilde{M})$. So we have a morphism of groupoids

$$\widetilde{m}\colon \pi_1(\widetilde{M}) \times \pi_1(\widetilde{M}) \to \pi_1(\widetilde{M})$$

such that $\pi_1(p) \circ \widetilde{m} = m \circ (\pi_1(p) \times \pi_1(p))$ and an action

$$\widetilde{\delta}\colon R \times \pi_1(\widetilde{M}) \to \pi_1(\widetilde{M}),\ (r,[\widetilde{a}]) \mapsto r[\widetilde{a}]$$

such that $\delta \circ (1 \times \pi_1(p)) = \pi_1(p) \circ \widetilde{\delta}$, where δ is the continuous action $R \times M \to M$. Therefore these maps induce a topological R-module structure on $\widetilde{M}$.

Since by Theorem 3.5 the category of topological group covers is equivalent to the category of group-groupoid covers, by the following diagram the proof is completed

$$\begin{array}{ccc} \mathsf{TModCov}/M & \xrightarrow{\pi_1} & \mathsf{GdModCov}/\pi_1(M) \\ \downarrow & & \downarrow \\ \mathsf{TGCov}/M & \xrightarrow[\pi_1]{} & \mathsf{GpGdCov}/\pi_1(M). \end{array}$$ □

4.9. Definition. Let R be a ring with identity 1_R, G_M a groupoid R-module and 0 the identity element of the group of $\mathrm{Ob}(G_M)$. Suppose that $\widetilde{G}$ is a groupoid, $p\colon \widetilde{G} \to G_M$ is a covering morphism of groupoids and $\widetilde{0} \in \mathrm{Ob}(\widetilde{G})$ such that $p(\widetilde{0}) = 0$. Then we say that the R-module structure of G_M *lifts* to $\widetilde{G}$ if there exists an R-module groupoid structure on $\widetilde{G}$ such that $\widetilde{0}$ is the identity element of the group structure of $\widetilde{G}$ and $p\colon \widetilde{G} \to G_M$ is a morphism of groupoid R-modules.

4.10. Theorem. *Let R be a ring with identity 1_R. Suppose that G_M is a R-module groupoid whose groupoid is transitive, 0 is the identity element of the additive group $\mathrm{Ob}(G_M)$ and $\widetilde{G}$ is a groupoid. Let $p\colon (\widetilde{G}, \widetilde{0}) \to (G_M, 0)$ be a covering morphism of groupoids. Suppose that the characteristic group C of p at $\widetilde{0}$ is a submodule of the R-module G_M. Then the R-module structure of G_M lifts to $\widetilde{G}$.*

Proof. Let C be the characteristic group of $p\colon \widetilde{G} \to G_M$ at $\widetilde{0}$ and let $q\colon \widetilde{G}_C \to G_M$ be the covering map corresponding to C as in Theorem 2.4. As in the proof of Theorem 3.7, it is sufficient to prove that the R- module structure lifts to $\widetilde{G}_C$.

We know from Theorem 3.7 that $\widetilde{G}_C$ is a group-groupoid. Let

$$\delta\colon R \times G_M \to G_M,\ (r, g) \mapsto rg$$

be the given R-module action on the groupoid R-module G_M. Now define an R-module action on $\widetilde{G}_C$ by

$$\widetilde{\delta}\colon R \times \widetilde{G}_C \to \widetilde{G}_C,\ (r, (g, Ca)) \mapsto (rg, C(ra))$$

and an action on $X = \mathrm{Ob}(\widetilde{G}_C)$ by $r(Ca) = C(ra)$. Since C is a submodule these actions are well defined. This action gives a groupoid R-module structure on $\widetilde{G}_C$ as required.

Further the morphism q preserves the R-module structure as follows:

$$q(r(g, Ca)) = q(rg, C(ra)) = rg = rq(g, Ca).$$ □

From Theorem 4.10 we obtain the following corollary.

4.11. Corollary. *Let R be a connected topological ring with identity 1_R and M a topological R-module whose underlying space is connected. Suppose that $\widetilde{M}$ is a simply connected topological space and $p\colon \widetilde{M} \to M$ is a covering map from $\widetilde{M}$ to the underlying topology of M. Let 0 be the identity element of the additive group of M and $\widetilde{0} \in \widetilde{M}$ be such that $p(\widetilde{0}) = 0$. Then $\widetilde{M}$ becomes a topological R-module such that $\widetilde{0}$ is the identity element of the group structure of $\widetilde{M}$ and p is a morphism of topological R-modules.*

Proof. Since $p\colon \widetilde{M} \to M$ is a covering map of topological R-modules, the induced morphism $\pi_1(p)\colon \pi_1(\widetilde{M}) \to \pi_1(M)$ becomes a covering morphism of groupoids with trivial characteristic group. Since M is a topological R-module, by Example 4.6 $\pi_1(M)$ is an R-module groupoid and since M is path connected, the groupoid $\pi_1(M)$ is transitive.

So by Theorem 4.10, the R-module structure of G_M lifts to $\pi_1(\widetilde{M})$. Hence $\widetilde{M}$ has an R-module structure. Similar to the proof of Corollary 3.8, $\widetilde{M}$ becomes a topological R-module as required. □

5. Conclusion

Group-groupoids and R-module groupoids are internal categories respectively in the category of groups and the category of R-modules. So it would be interesting to develop these results in terms of groups with operations and internal categories rather than special categories.

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