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ABSTRACT | FULL TEXT

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EXISTENCE OF ENTIRE RADIALLY SYMMETRIC SOLUTIONS FOR A QUASILINEAR SYSTEM WITH d-EQUATIONS

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Abstract

The studies developed within this article will be focused on achieving results related to the existence and qualitative properties of entire radially symmetric solutions for a Schrodinger problem of type $\Delta_p u_i + h_i(r) |\nabla u_i|^{p-1} = a_i(r) f_i(u_{i+1})$ for $i = \overline{1, d-1}$ and $\Delta_p u_d + h_d(r) |\nabla u_d|^{p-1} = a_d(r) f_d(u_1)$ on $\mathbb{R}^N$, where $p > 1$, $d \geq 2$, h_i and a_i are nonnegative radial continuous functions and f_i are nonnegative increasing continuous functions on $[0, \infty)$.

1. Introduction

In this article we generalize existence results for systems such as

$$
(1.1) \quad
\begin{cases}
\Delta_p u_1 + h_1(r) |\nabla u_1|^{p-1} = a_1(r) f_1(u_2) \ \text{in} \ \mathbb{R}^N, \\
\ldots\ldots \\
\Delta_p u_i + h_i(r) |\nabla u_i|^{p-1} = a_i(r) f_i(u_{i+1}) \ \text{in} \ \mathbb{R}^N, \\
\ldots\ldots \\
\Delta_p u_d + h_d(r) |\nabla u_d|^{p-1} = a_d(r) f_d(u_1) \ \text{in} \ \mathbb{R}^N,
\end{cases}
$$

where $r := |x| \geq 0$, $N \geq 3$, $d \geq 2$ is an integer, $i = \overline{1, d-1}$, Δ_p is the p-Laplacian operator defined by

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2} \nabla u), \ 1 < p < \infty,$$

$h_{j_{j=\overline{1,d}}}$, $a_{j_{j=\overline{1,d}}} : [0, \infty) \to [0, \infty)$ are radial continuous functions and $f_{j_{j=\overline{1,d}}}$ satisfy the following hypotheses

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(C1) $f_{j_{j=\overline{1,d}}} : [0, \infty) \to [0, \infty)$ are continuous;

(C2) $f_{j_{j=\overline{1,d}}}$ are increasing on $[0, \infty)$.

Answers to questions related to the existence of radially symmetric solutions for problem like (1.1) have been gradually achieved by several authors over many years (see [1]-[13] and references therein).

Before we start describing our results let us mention that the problem of the existence of solutions to (1.1) has received an increased interest with Alan Lair's recent paper [6]. In [6] the author considered problem (1.1) under hypotheses $p = d = 2$, $h_{i_{i=\overline{1,2}}} = 0$, $a_1(x) f_1(u_2(x)) := a_1(|x|) u_2^\alpha(|x|)$ $(\alpha \in (0,1])$ and $a_2(x) f_2(u_1(x)) := a_2(|x|) u_1^\beta(|x|)$ $(\beta \in (0,1])$.

The main result achieved by Alan Lair may be summarized as follows:

1.1. Proposition. *Problem* (1.1) *has an explosive radial symmetric solution on* $\mathbb{R}^N$ *if and only if the continuous radially symmetric functions* $a_{i_{i=\overline{1,2}}} : [0, \infty) \to [0, \infty)$ *simultaneously meet the following conditions*

$$\int_0^\infty t a_1(t) \left(t^{2-N} \int_0^t s^{N-3} \int_0^s \tau a_2(\tau)\, d\tau\, ds \right)^\alpha dt = \infty$$

$$\int_0^\infty t a_2(t) \left(t^{2-N} \int_0^t s^{N-3} \int_0^s \tau a_1(\tau)\, d\tau\, ds \right)^\beta dt = \infty.$$

Moreover, the author proposes the following problem:

"*It remains unknown whether an analogous result holds for the system*

$$\begin{cases} \Delta u_1(|x|) = a_1(|x|) f_1(u_2(|x|)) \ \text{for } x \in \mathbb{R}^N, \\ \Delta u_2(|x|) = a_2(|x|) f_2(u_1(|x|)) \ \text{for } x \in \mathbb{R}^N, \end{cases}$$

where f_1 *and* f_2 *meet, for example*

$$(1.2) \qquad \int_1^\infty \left[\int_0^s f_i(t)\, dt \right]^{-1/2} ds = \infty, \ i = \overline{1,2},$$

or satisfy the stronger condition

$$(1.3) \qquad \int_1^\infty [f_i(t)]^{-1} dt = \infty, \ i = \overline{1,2}.\text{"}$$

When $h_{j_{j=\overline{1,d}}} = 0$ and $p = 2$ we mention that in the paper [3] the author established the existence results for the problems (1.1) under conditions of type (1.2).

The starting point for the analysis and research of the more general problems (1.1) in order to discover the novelty which is in fact the scientific task of the article, is generated by the proposed problem in [6]. In this sense, we establish results related to the existence and various qualitative properties of solutions for problems (1.1) under conditions of the (1.3) type.

The principal difficulty in the treatment of (1.1) is due to the nonlinear gradient term combined with multiple equations of the system. The solving methods that we will use for the problem are based on results of nonlinear analysis, the above cited articles as well as other techniques that will be discovered during the research.

Throughout this paper we use the notations

$$j = \overline{1,d} \qquad\qquad H_j(r) := r^{N-1} e^{\int_0^r h_j(t)\, dt}$$

$$A_j(\infty) := \lim_{r \to \infty} A_j(r), \quad A_j(r) = \int_0^r \left(\frac{1}{H_j(t)} \int_0^t H_j(s) a_j(s)\, ds \right)^{1/(p-1)} dt$$

and

$$F\left(\infty\right) := \lim_{r\to\infty} F\left(r\right), F\left(r\right) = \int_a^r (\sum_{j=1}^d f_j\left(s\right))^{-1/(p-1)} \, ds; \ r \geq a > 0.$$

We see that

$$F'\left(r\right) = \left(\sum_{j=1}^d f_j\left(r\right)\right)^{-1/(p-1)} > 0 \text{ for all } r > a$$

and F has the inverse function F^{-1} on $[a, \infty)$.

Our main results will be stated in what follows.

1.2. Theorem. *Assume that* (C1)–(C2) *hold and that*

(C3) $F\left(\infty\right) = \infty$.

Then the system (1.1) *possesses at least one positive radial solution* $(u_1, \ldots, u_d)$. *If, in addition,* $A_{j_{j=\overline{1,d}}}\left(\infty\right) < \infty$, *the positive radial solution* $(u_1, \ldots, u_d)$ *is bounded. Moreover, when* $A_{j_{j=\overline{1,d}}}\left(\infty\right) = \infty$, *the positive solution* $(u_1, \ldots, u_d)$ *is an entire large solution, i.e.*

$$\lim_{r\to\infty} u_1\left(r\right) = \cdots = \lim_{r\to\infty} u_d\left(r\right) = \infty.$$

1.3. Theorem. *Assume that* (C1)–(C2) *hold and that*

(C4) $F\left(\infty\right) < \infty$;
(C5) $A_{j_{j=\overline{1,d}}}\left(\infty\right) < \infty$;
(C6) *there exists* $\beta > \frac{a}{d}$ *such that*

$$\sum_{j=1}^d A_j\left(\infty\right) < F\left(\infty\right) - F\left(d\beta\right).$$

Then, the system (1.1) *possesses at least one positive bounded radial solution* $(u_1, \ldots, u_d)$ *satisfying*

$$\beta + f_j^{1/(p-1)}\left(\beta\right) A_j\left(r\right) \leq u_j\left(r\right) \leq F^{-1}\left(F\left(d\beta\right) + \sum_{j=1}^d A_j\left(r\right)\right), \ j = \overline{1,d}.$$

2. Proof of the theorems

2.1. Proof of Theorem 1.2. We note that radial solutions of (1.1) are positive solutions $(u_1, \ldots, u_d)$ of the integral equations

$$\begin{cases} u_1\left(r\right) = \beta + \int_0^r \left(\frac{1}{H_1(t)} \int_0^t H_1\left(s\right) a_1\left(s\right) f_1\left(u_2\left(s\right)\right) \, ds\right)^{1/(p-1)} dt, \\ \ldots\ldots \\ u_i\left(r\right) = \beta + \int_0^r \left(\frac{1}{H_i(t)} \int_0^t H_i\left(s\right) a_i\left(s\right) f_i\left(u_{i+1}\left(s\right)\right) \, ds\right)^{1/(p-1)} dt, \\ \ldots\ldots \\ u_d\left(r\right) = \beta + \int_0^r \left(\frac{1}{H_d(t)} \int_0^t H_d\left(s\right) a_d\left(s\right) f_d\left(u_1\left(s\right)\right) \, ds\right)^{1/(p-1)} dt, \end{cases}$$

where β may be any non-negative number.

Define sequences $\left\{u_j^k\right\}_{j=\overline{1,d}}^{k\geq 1}$ on $[0,\infty)$ by

$$
\begin{cases}
u_1^0 = \cdots = u_d^0 = \beta, & r \geq 0, \\
u_1^k(r) = \beta + \int_0^r \left(\frac{1}{H_1(t)}\int_0^t H_1(s)\, a_1(s)\, f_1\left(u_2^{k-1}(s)\right)\, ds\right)^{1/(p-1)} dt, & \\
\ldots\ldots & \\
u_i^k(r) = \beta + \int_0^r \left(\frac{1}{H_i(t)}\int_0^t H_i(s)\, a_i(s)\, f_i\left(u_{i+1}^{k-1}(s)\right)\, ds\right)^{1/(p-1)} dt, & i = \overline{1,d-1} \\
\ldots\ldots & \\
u_d^k(r) = \beta + \int_0^r \left(\frac{1}{H_d(t)}\int_0^t H_d(s)\, a_d(s)\, f_d\left(u_1^k(s)\right)\, ds\right)^{1/(p-1)} dt, & d \geq 2
\end{cases}
$$

We remark that, for all $r \geq 0$, $j = \overline{1,d}$ and $k \in N$,

$$u_j^k(r) \geq \beta.$$

We claim that $\left\{u_j^k\right\}_{j=\overline{1,d}}^{k\geq 1}$ are increasing sequences on $[0,\infty)$. Since

$$u_j^0 = \beta \leq u_j^1, \ j = \overline{1,d},$$

it follows that

$$f_j\left(u_{j+1}^0\right) \leq f_j\left(u_{j+1}^1\right), \ j = \overline{1,d-1},$$

and so

$$
\begin{aligned}
u_j^1(r) &= \beta + \int_0^r \left(\frac{1}{H_j(t)}\int_0^t H_j(s)\, a_j(s)\, f_j\left(u_{j+1}^0(s)\right)\, ds\right)^{1/(p-1)} dt \\
&\leq \beta + \int_0^r \left(\frac{1}{H_j(t)}\int_0^t H_j(s)\, a_j(s)\, f_j\left(u_{j+1}^1(s)\right)\, ds\right)^{1/(p-1)} dt \\
&= u_j^2(r), \ j = \overline{1,d-1}.
\end{aligned}
$$

From the above relation we obtain

$$
\begin{aligned}
u_d^1(r) &= \beta + \int_0^r \left(\frac{1}{H_d(t)}\int_0^t H_d(s)\, a_d(s)\, f_d\left(u_1^1(s)\right)\, ds\right)^{1/(p-1)} dt \\
&\leq \beta + \int_0^r \left(\frac{1}{H_d(t)}\int_0^t H_d(s)\, a_d(s)\, f_d\left(u_1^2(s)\right)\, ds\right)^{1/(p-1)} dt \\
&= u_d^2(r).
\end{aligned}
$$

Consequently,

$$u_j^2 \leq u_j^3, \ j = \overline{1,d-1},$$

which yield

$$u_d^2 \leq u_d^3.$$

Continuing this line of reasoning, we obtain that $\left\{u_j^k\right\}_{j=\overline{1,d}}^{k\geq 1}$ are increasing sequences.

By conditions (C1) and (C2) we obtain

$$\left(u_1^k(r)\right)' = \left(\frac{1}{H_1(r)}\int_0^r H_1(s)\,a_1(s)\,f_1\left(u_2^{k-1}(s)\right)\,ds\right)^{1/(p-1)}$$
$$\leq f_1^{1/(p-1)}\left(u_2^k(r)\right)A_1'(r)$$
$$\leq \left(\sum_{j=1}^d f_j\left(\sum_{j=1}^d u_j^k(r)\right)\right)^{1/(p-1)}A_1'(r),$$

$$\cdots\cdots\cdots$$

$$(2.1)\quad \left(u_i^k(r)\right)' = \left(\frac{1}{H_i(r)}\int_0^r H_i(s)\,a_i(s)\,f_i\left(u_{i+1}^{k-1}(s)\right)\,ds\right)^{1/(p-1)}$$
$$\leq f_i^{1/(p-1)}\left(u_{i+1}^k(r)\right)A_i'(r)$$
$$\leq \left(\sum_{j=1}^d f_j\left(\sum_{j=1}^d u_j^k(r)\right)\right)^{1/(p-1)}A_i'(r),\ i=\overline{1,d-1}$$

$$\cdots\cdots\cdots$$

$$\left(u_d^k(r)\right)' = \left(\frac{1}{H_d(r)}\int_0^r H_d(s)\,a_d(s)\,f_d\left(u_1^k(s)\right)\,ds\right)^{1/(p-1)}$$
$$\leq f_d^{1/(p-1)}\left(u_1^k(r)\right)A_d'(r)$$
$$\leq \left(\sum_{j=1}^d f_j\left(\sum_{j=1}^d u_j^k(r)\right)\right)^{1/(p-1)}A_d'(r),\ d\geq 2.$$

Summing up gives

$$\left(\sum_{j=1}^d f_j\left(\sum_{j=1}^d u_j^k(t)\right)\right)^{-1/(p-1)}\cdot\left(\sum_{j=1}^d u_j^k(t)\right)' \leq \sum_{j=1}^d A_j'(r).$$

Integrating the above equation between 0 and r, we have

$$\int_0^r \left(\sum_{j=1}^d f_j\left(\sum_{j=1}^d u_j^k(t)\right)\right)^{-1/(p-1)}\cdot\left(\sum_{j=1}^d u_j^k(t)\right)'\,dt \leq \sum_{j=1}^d A_j(r)\ \text{ for each } r>0,$$

which is equivalent to

$$\int_0^r F'\left(\sum_{j=1}^d u_j^k(t)\right)\,dt \leq \sum_{j=1}^d A_j(r)\ \text{ for each } r>0.$$

Consequently,

$$(2.2)\quad F\left(\sum_{j=1}^d u_j^k(r)\right) - F(d\beta) \leq \sum_{j=1}^d A_j(r)$$

for all $r\geq 0$.

Since F^{-1} is increasing on $[0,\infty)$, it follows that

$$(2.3)\quad \sum_{j=1}^d u_j^k(r) \leq F^{-1}\left(F(d\beta) + \sum_{j=1}^d A_j(r)\right)$$

for all $r\geq 0$.

Since (C3) holds, we can see that

$$(2.4) \qquad F^{-1}(\infty) = \infty.$$

It follows that the sequences $\left\{u_j^k\right\}_{j=\overline{1,d}}^{k\geq 1}$ are bounded and increaing on $[0, c]$ for $c > 0$.

Thus,

$$\left(u_1^k, \ldots, u_d^k\right) \text{ converges to } (u_1, \ldots, u_d) \text{ on } [0, c]^d.$$

Consequently, $(u_1, \ldots, u_d)$ is the positive entire radial solution of system (1.1). Moreover, when

$$A_{j_{j=\overline{1,d}}}(\infty) < \infty,$$

we see by (2.3) that

$$\sum_{j=1}^d u_j(r) \leq F^{-1}\left(F(d\beta) + \sum_{j=1}^d A_j(\infty)\right) \text{ for all } r \geq 0.$$

When

$$A_j(\infty) = \infty \text{ for } j = \overline{1,d}$$

then by (C2) and the monotonicity of $\left\{u_j^k\right\}_{j=\overline{1,d}}^{k\geq 1}$ it follows that

$$u_j(r) \geq \beta + f_j^{1/(p-1)}(\beta) A_j(r), \text{ for all } r \geq 0 \text{ and } j = \overline{1,d}.$$

Thus

$$\lim_{r\to\infty} u_1(r) = \cdots = \lim_{r\to\infty} u_d(r) = \infty,$$

ending the proof of the Theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2. Proof of Theorem 1.3. We proceed as in the proof of the Theorem 1.2. Then, by (2.2) we see that

$$(2.5) \qquad F\left(\sum_{j=1}^d u_j^k(r)\right) \leq F(d\beta) + \sum_{j=1}^d A_j(\infty) < F(\infty) < \infty.$$

Since F^{-1} is strictly increasing on $[0, \infty)$, we have

$$(2.6) \qquad \sum_{j=1}^d u_j^k(r) \leq F^{-1}\left(F(d\beta) + \sum_{j=1}^d A_j(\infty)\right) < \infty \text{ for all } r \geq 0.$$

Therefore, since the sequence $\left\{u_j^k(r)\right\}$ is monotone it converges to some function $\left\{u_j(r)\right\}_{j=\overline{1,d}}$ on $\mathbb{R}^N$ that in fact is a solution to (1.1). This concludes the proof. $\qquad\square$

2.1. Remark. If (C1), (C2), (C3) are satisfied then

$$\int_a^\infty \frac{ds}{f_j^{1/(p-1)}(s)} = \infty \text{ for all } j = \overline{1,d}.$$

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