Hacettepe Journal of Mathematics and Statistics
Volume 39 (2) (2010), 205 – 217

SOFT SETS AND SOFT BCH-ALGEBRAS

O. Kazancı* †, Ş. Yılmaz* and S. Yamak*

Received 16 : 06 : 2009 : Accepted 16 : 12 : 2009

Abstract

In this paper, the concept of soft BCH-algebra is introduced and in the meantime, some of their properties and structural characteristics are discussed and studied. The bi-intersection, extended intersection, restricted union, $\vee$-union, $\wedge$-intersection and cartesian product of the family of soft BCH-algebras and soft BCH-subalgebras are established. Also, the theorems of homomorphic image and homomorphic pre-image of soft sets are given. Moreover, the notion of soft BCH-homomorphism is introduced and its basic properties are studied.

Keywords: Soft sets, BCH-algebras, Soft BCH-algebras, Soft BCH-subalgebras.

2000 AMS Classification: 06 F 35.

1. Introduction

In 1992, Molodtsov [19] introduced the concept of soft set, which can be seen a new mathematical tool for dealing with uncertainty. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields. For example, the study of smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability, the theory of measurement and so on. At present, work on soft set theory is progressing rapidly. Maji *et al.* [18] described the application of soft set theory to a decision making problem. In theoretical aspects, Maji *et al.* [17] defined several operations on soft sets. Chen *et al.* [7] presented a new definition of soft set parameterization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. Some results on an application of fuzzy-soft-sets in a decision making problem have been given by Roy *et al.* [22]. Also, some new operations in soft set theory have been given by Irfan Ali *et al.* [2].

In 1966, Imai and Iséki [12] and Iséki [13] introduced two classes of abstract algebras, BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper

*Department of Mathematics, Karadeniz Technical University, 61080 Trabzon, Turkey.
E-mail: (O. Kazancı) kazancio@yahoo.com (Ş. Yılmaz) syilmaz@ktu.edu.tr
(S. Yamak) syamak@ktu.edu.tr
†Corresponding Author.

subclass of the class of BCI-algebras. In 1983, Hu and Li [10,11] introduced the notion of a BCH-algebra, which is a generalization of the notions of BCK and BCI-algebras. They have studied a few properties of these algebras. Certain other properties have been studied by Ahmad [1], Chaudhry [5], Chaudhry *et. al.* [4,6], Dudek and Thomys [8], and Roh *et. al.* [20,21]. The algebraic structure of soft sets has been studied by some authors. Aktaş *et al.* [3] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets. They also discussed the notion of soft groups. Jun [15] introduced and investigated the notion of soft BCK/BCI-algebras. Jun and Park [16] discussed the applications of soft sets in the ideal theory of BCK/BCI-algebras. Feng *et. al.* [9] introduced the notions of soft ideals and idealistic soft semirings.

In this paper, we apply these new definitions to BCH-algebras for the first time. Using the algebraic structure of soft sets, introduced in [2,9,15,16,19], the concept of soft BCH-algebra is introduced and in the meantime, some of their properties and structural characteristics are discussed and studied. We investigate relations between soft BCH-algebras and soft BCH-subalgebras. The bi-intersection, extended intersection, restricted union $\vee$-union, $\wedge$-intersection and cartesian product of families of soft BCH-algebras and soft BCH-subalgebras are established.

2. Preliminaries

Molodtsow [19] defined the notion of a soft set in the following way: Let U be an initial universe set and E a set of parameters. The power set of U is denoted by $\mathcal{P}(U)$ and A is a subset of E.

2.1. Definition. [19] A pair (F, A) is called a *soft set* over U, where F is a mapping $F : A \to \mathcal{P}(U)$.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-approximate elements of the soft set (F, A). Clearly, a soft set is not just a subset of U. Maji *et al.* [18] and Feng *et al.* [9] introduced and investigated several binary operations on soft sets.

2.2. Definition. [19] Let $(F, A), (G, B)$ be soft sets over a common universe U.

(i) (F, A) is said to be a *soft subset* of (G, B), denoted by

$(F, A)\widetilde{\subseteq}(G, B)$,

if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$,

(ii) (F, A) and (G, B) are said to be *soft equal*, denoted by

$(F, A) = (G, B)$,

if $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$.

2.3. Definition. [2,9]

(i) The *bi(restricted)-intersection* of two soft sets (F, A) and (G, B) over a common universe U is defined as the soft set

$(H, C) = (F, A)\widetilde{\sqcap}(G, B)$,

where $C = A \cap B \neq \emptyset$, and $H(c) = F(c) \cap G(c)$ for all $c \in C$.

(ii) The *bi(restricted)-intersection* of a nonempty family soft sets $\{(F_i, A_i) \mid i \in \Lambda\}$ over a common universe U is defined as the soft set

$(H, B) = \widetilde{\sqcap}_{i\in\Lambda}(F_i, A_i)$,

where $B = \bigcap_{i\in\Lambda} A_i \neq \emptyset$, and $H(x) = \bigcap_{i\in\Lambda} F_i(x)$ for all $x \in B$.

2.4. Definition. [2]

(i) The *extended intersection* of two soft sets (F, A) and (G, B) over a common universe U is defined as the soft set

$$(H, C) = (F, A)\widetilde{\cap}(G, B),$$

where $C = A \cup B$, and for all $c \in C$,

$$H(c) = \begin{cases} F(c) & \text{if } c \in A\backslash B \\ G(c) & \text{if } c \in B\backslash A \\ F(c) \cap G(c) & \text{if } c \in A \cap B \end{cases}$$

(ii) The *extended intersection* of a nonempty family soft sets $\{(F_i, A_i) \mid i \in \Lambda\}$ over a common universe U is defined as the soft set

$$(H, B) = \widetilde{\bigcap}_{i\in\Lambda}(F_i, A_i),$$

where $B = \bigcup_{i\in\Lambda} A_i$ and $H(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$, and $\Lambda(x) = \{i \mid i \in A_i\}$ for all $x \in B$.

2.5. Definition. [2] The *restricted union* of two soft sets (F, A) and (G, B) over a common universe U is defined as the soft set

$$(H, C) = (F, A)\widetilde{\cup}(G, B),$$

where $C = A \cap B \neq \emptyset$, and $H(c) = F(c) \cup G(c)$ for all $c \in C$.

As a generalization of the restricted union of two soft sets, we define the restricted union of a nonempty family of soft sets in the following way.

2.6. Definition. The *restricted union* of a nonempty family soft sets $\{(F_i, A_i) \mid i \in \Lambda\}$ over a common universe U is defined as the soft set

$$(H, B) = \widetilde{\bigcup}_{i\in\Lambda}(F_i, A_i),$$

where $B = \bigcap_{i\in\Lambda} A_i \neq \emptyset$ and $H(x) = \bigcup_{i\in\Lambda} F_i(x)$ for all $x \in B$.

2.7. Definition. [9,17]

(i) The $\wedge$-*intersection* of two soft sets (F, A) and (G, B) over a common universe U is defined as the soft set

$$(H, C) = (F, A)\widetilde{\wedge}(G, B),$$

where $C = A \times B$, and $H(a, b) = F(a) \cap G(b)$ for all $(a, b) \in A \times B$;

(ii) The $\wedge$-*intersection* of a nonempty family soft sets $\{(F_i, A_i) \mid i \in \Lambda\}$ over a common universe U is defined as the soft set

$$(H, B) = \widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i),$$

where $B = \prod_{i\in\Lambda} A_i$ and $H(x) = \bigcap_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

2.8. Definition. [9,17]

(i) The $\vee$ *union* of two soft sets (F, A) and (G, B) over a common universe U is defined as the soft set

$$(H, C) = (F, A)\widetilde{\vee}(G, B),$$

where $C = A \times B$, and $H(a, b) = F(a) \cup G(b)$ for all $(a, b) \in A \times B$;

(ii) The $\vee$ *union* of a nonempty family soft sets $\{(F_i, A_i) \mid i \in \Lambda\}$ over a common universe U is defined as the soft set

$$(H, B) = \widetilde{\bigvee}_{i\in\Lambda}(F_i, A_i),$$

where $B = \prod_{i\in\Lambda} A_i$, and $H(x) = \bigcup_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

2.9. Definition. [17] Let (F, A) and (G, B) be two soft sets over U and V, respectively. The *cartesian product* of the two soft sets (F, A) and (G, B) is defined as the soft set

$$(C, A \times B) = (F, A) \times (G, B),$$

where $C(x, y) = F(x) \times G(y)$ for all $(x, y) \in A \times B$.

As a generalization of the cartesian product of two soft sets, we define the cartesian product of a nonempty family of soft sets in the following way.

2.10. Definition. Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft sets over U_i, $i \in \Lambda$. The *cartesian product* of the nonempty family of soft sets $\{(F_i, A_i) \mid i \in \Lambda\}$ over the universes U_i is defined as the soft set

$$(H, B) = \widetilde{\prod}_{i\in\Lambda}(F_i, A_i),$$

where $B = \prod_{i\in\Lambda} A_i$ and $H(x) = \prod_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

Now, we describe certain definitions, known results and examples that will be used in the sequel.

2.11. Definition. [12,13] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCH-algebra* if it satisfies the following conditions:

(i) $x * x = 0$,
(ii) $x * y = 0 = y * x$ implies $x = y$,
(iii) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

A BCH-algebra X is called a BCI-algebra if it satisfies the identity:

$(BCI\,1)$: $((x * y) * (x * z)) * (z * y) = 0$, for all $x, y, z \in X$.

If a BCI-algebra X satisfies the following condition:

$(BCK\,1)$: $0 * x = 0$ for all $x \in X$,

then X is called a BCK-algebra.

A BCH-algebra X is called *non-negative* if it satisfies the condition $(BCK\,1)$. A BCH-algebra X is called *proper* if it does not satisfies the condition $(BCI\,1)$. A BCI-algebra X is called a *proper* if it does not satisfies condition $(BCK\,1)$.

In any BCH-algebra X, the following hold: (see [4,5]):

(H1) $x * 0 = x$,
(H2) $x * 0 = 0$ implies $x = 0$,
(H3) $0 * (x * y) = (0 * x) * (0 * y)$,
(H4) $x * (x * y) \leq y$,

where $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$.

A non-empty subset S of a BCH-algebra X is called a *BCH-subalgebra* of X if $x*y \in S$ for all $x, y \in S$.

A mapping f from a BCH-algebra X to a BCH-algebra Y is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

It is known that every BCI-algebra is a BCH-algebra but the following example shows that the converse is not true.

2.12. Example. (see [4, 10]) Let $X = \{0, 1, 2, 3\}$, on which $*$ is defined by:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

Then $(X,\ *,\ 0)$ is a proper BCH-algebra, but it is not a BCI-algebra because

$$(2 * 3) * (2 * 1) = 2 * 0 = 2 \not\leq 1 * 3 = 3.$$

2.13. Example. Let $\mathbb{Z}$ be the set of all integer numbers with the operation $*$ defined by $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Then $(\mathbb{Z},\ *,\ 0)$ is a BCH-algebra, but it is not a BCK-algebra because $0 * x \neq 0$ for all $x \in \mathbb{Z} - \{0\}$.

3. Soft BCH-algebras

If X is a BCH-algebra and A a nonempty set, a set-valued function $F : A \to \mathcal{P}(X)$ can be defined by $F(x) = \{y \in X \mid (x, y) \in R\}$, $x \in A$, where R is an arbitrary binary relation from A to X, that is a subset of $A \times X$. The pair (F, A) is then a soft set over X. The soft sets in the examples that follow are obtained by making an appropriate choice for the relation R.

For a soft set (F, A), the set $\operatorname{Supp}(F, A) = \{x \in A \mid F(x) \neq \emptyset\}$ is called the *support* of the soft set (F, A), and the soft set (F, A) is called a *non-null* if $\operatorname{Supp}(F, A) \neq \emptyset$ [9].

3.1. Definition. Let (F, A) be a non-null soft set over X. Then (F, A) is called a *soft BCH-algebra* over X if $F(x)$ is a *BCH-subalgebra* of X for all $x \in \operatorname{Supp}(F, A)$.

3.2. Example. Let $X = \{0, 1, 2, 3\}$ be the proper BCH-algebra with the following Cayley table [11]:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

(i) Let (F, A) be a soft set over X, where $A = X$ and $F : A \to \mathcal{P}(X)$ the set-valued function defined by

$$F(x) = \{y \in X \mid y * (y * x) = 0\}$$

for all $x \in A$. Then $F(0) = X$, $F(1) = F(3) = \{0, 1\}$, and $F(2) = \{0\}$ are BCH-subalgebras of X for all $x \in \operatorname{Supp}(F, A) = A$. Therefore (F, A) is a soft BCH-algebra over X.

(ii) Let (F, A) be a soft set over X, where $A = \{1, 2, 3\}$ and $F : A \to \mathcal{P}(X)$ the set-valued function defined by

$$F(x) = \begin{cases} \{y \in X \mid y * (y * x) = 0\}, & \text{if } x \in \{1, 2\}, \\ \emptyset, & \text{if } x = 3. \end{cases}$$

Then $F(1) = \{0, 1\}$, and $F(2) = \{0\}$ are BCH-subalgebras of X for all $x \in \operatorname{Supp}(F, A) = \{1, 2\} \subset A$.

This example shows that $\operatorname{Supp}(F, A)$ can be a proper subset of A.

3.3. Example. Let $X = \{0, 1, 2, 3, 4\}$ be the proper BCH-algebra with the following Cayley table [10]:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Let (F, A) be a soft set over X, where $A = \{0, 1, 2\}$ and $F : A \to \mathcal{P}(X)$ is the set-valued function defined by

$$F(x) = \{y \in X \mid y * x \in \{1, 4\}\}$$

for all $x \in A$. Then $F(0) = \{1, 4\}$, $F(1) = \{3, 4\}$ and $F(2) = \{1\}$ are not BCH-subalgebras of X. Therefore, (F, A) is not a soft BCH-algebra over X.

This shows that there can exist set-valued functions $F : A \to \mathcal{P}(X)$ such that the soft set (F, A) is not a soft BCH-algebra over X.

3.4. Theorem. *Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. Then the bi-intersection $\widetilde{\sqcap}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X if it is non-null.*

Proof. Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. By Definition 2.3 (ii), we can write $\widetilde{\sqcap}_{i\in\Lambda}(F_i, A_i) = (H, B)$, where $B = \bigcap_{i\in\Lambda} A_i$, and $H(x) = \bigcap_{i\in\Lambda} F_i(x)$ for all $x \in B$.

Let $x \in \operatorname{Supp}(H, B)$. Then $\bigcap_{i\in\Lambda} F_i(x) \neq \emptyset$, and so we have $F_i(x) \neq \emptyset$ for all $i \in \Lambda$. Since $\{(F_i, A_i) \mid i \in \Lambda\}$ is a nonempty family of soft BCH-algebras over X, it follows that $F_i(x)$ is a BCH-subalgebra of X for all $i \in \Lambda$, and its intersection is also a BCH-subalgebra of X, that is, $H(x) = \bigcap_{i\in\Lambda} F_i(x)$ is a BCH-subalgebra of X for all $x \in \operatorname{Supp}(H, B)$. Hence $(H, B) = \widetilde{\sqcap}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X. □

3.5. Corollary. *Let $\{(F_i, A) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. Then the bi-intersection $\widetilde{\sqcap}_{i\in\Lambda}(F_i, A)$ is a soft BCH-algebra over X if it is non-null.*

Proof. Straightforward. □

3.6. Theorem. *Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. Then the extended intersection $\widetilde{\bigcap}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebras over X.*

Proof. Assume that $\{(F_i, A_i) \mid i \in \Lambda\}$ is a nonempty family of soft BCH-algebras over X. By Definition 2.4 (ii), we can write $\widetilde{\bigcap}_{i\in\Lambda}(F_i, A_i) = (H, B)$, where $B = \bigcup_{i\in\Lambda} A_i$, and $H(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$ for all $x \in B$.

Let $x \in \operatorname{Supp}(H, B)$. Then $\bigcap_{i\in\Lambda(x)} F_i(x) \neq \emptyset$ and so we have $F_i(x) \neq \emptyset$ for all $i \in \Lambda(x)$. Since (F_i, A_i) is a soft BCH-algebras over X for all $i \in \Lambda$, we deduce that the nonempty set $F_i(x)$ is a BCH-algebras of X for all $i \in \Lambda$. It follows that $H(x) = \bigcap_{i\in\Lambda(x)} F_i(x)$ is a BCH-subalgebra of X for all $x \in \operatorname{Supp}(H, B)$. Hence, the extended intersection $\widetilde{\bigcap}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X. □

3.7. Theorem. *Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in \Lambda$, $x_i \in A_i$, then the restricted union $\widetilde{\bigcup}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X.*

Proof. Assume that $\{(F_i, A_i) \mid i \in \Lambda\}$ is a nonempty family of soft BCH-algebra over X. By Definition 2.6, we can write $\widetilde{\bigcup}_{i\in\Lambda}(F_i, A_i) = (H, B)$, where $B = \bigcap_{i\in\Lambda} A_i$, and $H(x) = \bigcup_{i\in\Lambda} F_i(x)$ for all $x \in B$.

Let $x \in \text{Supp}\,(H, B)$. Since $\text{Supp}\,(H, B) = \bigcup_{i\in\Lambda} \text{Supp}\,(F_i, A_i) \neq \emptyset$ we have $F_{i_0}(x) \neq \emptyset$ for some $i_0 \in \Lambda$. By assumption, $\bigcup_{i\in\Lambda} F_i(x_i)$ is a BCH-subalgebra of X for all $x \in \text{Supp}\,(H, B)$. Hence the restricted union $\widetilde{\bigcup}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X. □

3.8. Theorem. *Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. Then the $\wedge$-intersection $\widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X if it is non-null.*

Proof. By Definition 2.7 (ii), we can write $\widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i) = (H, B)$, where $B = \prod_{i\in\Lambda} A_i$, and $H(x) = \bigcap_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

Suppose that the soft set (H, B) is non-null. If $x = (x_i)_{i\in\Lambda} \in \text{Supp}\,(H, B)$, then $H(x) = \bigcap_{i\in\Lambda} F_i(x_i) \neq \emptyset$. Since (F_i, A_i) is a soft BCH-algebra over X for all $i \in \Lambda$, we deduce that the nonempty set $F_i(x_i)$ is a BCH-subalgebra of X for all $i \in \Lambda$. It follows that $H(x) = \bigcap_{i\in\Lambda} F_i(x_i)$ is a BCH-subalgebra of X for all $x = (x_i)_{i\in\Lambda} \in \text{Supp}\,(H, B)$.

Hence, the $\wedge$-intersection $\widetilde{\bigwedge}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X. □

3.9. Theorem. *Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a nonempty family of soft BCH-algebras over X. If $F_i(x_i) \subseteq F_j(x_j)$ or $F_j(x_j) \subseteq F_i(x_i)$ for all $i, j \in \Lambda, x_i \in A_i$, then the $\vee$-union $\widetilde{\bigvee}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X.*

Proof. Assume that $\{(F_i, A_i) \mid i \in \Lambda\}$ is a nonempty family of soft BCH-algebra over X. By Definition 2.8 (ii), we can write $\widetilde{\bigvee}_{i\in\Lambda}(F_i, A_i) = (H, B)$, where $B = \prod_{i\in\Lambda} A_i$ and $H(x) = \bigcup_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

Let $x = (x_i)_{i\in\Lambda} \in \text{Supp}\,(H, B)$. Then $H(x) = \bigcup_{i\in\Lambda} F_i(x_i) \neq \emptyset$, and so we have $F_{i_0}(x_{i_0}) \neq \emptyset$ for some $i_0 \in \Lambda$. By assumption, $\bigcup_{i\in\Lambda} F_i(x_i)$ is a BCH-subalgebra of X for all $x = (x_i)_{i\in\Lambda} \in \text{Supp}\,(H, B)$. Hence the $\vee$-union $\widetilde{\bigvee}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over X. □

3.10. Theorem. *Let $\{(F_i, A_i) \mid i \in \Lambda\}$ be a non-empty family of soft BCH-algebras over X_i. Then the cartesian product $\widetilde{\prod}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over $\prod_{i\in\Lambda} X_i$.*

Proof. By Definition 2.10, we can write $\widetilde{\prod}_{i\in\Lambda}(F_i, A_i) = (H, B)$, where $B = \prod_{i\in\Lambda} A_i$ and $H(x) = \prod_{i\in\Lambda} F_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

Let $x = (x_i)_{i\in\Lambda} \in \text{Supp}\,(H, B)$. Then $H(x) = \prod_{i\in\Lambda} F_i(x_i) \neq \emptyset$, and so we have $F_i(x_i) \neq \emptyset$ for all $i \in \Lambda$. Since $\{(F_i, A_i) \mid i \in \Lambda\}$ is a soft BCH-algebras over X_i for all $i \in \Lambda$, we have that $F_i(x_i)$ is a BCH-subalgebra of X_i, so $\prod_{i\in\Lambda} F_i(x_i)$ is a BCH-subalgebra of $\prod_{i\in\Lambda} X_i$ for all $x = (x_i)_{i\in\Lambda} \in \text{Supp}\,(H, B)$. Hence, the cartesian product $\widetilde{\prod}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-algebra over $\prod_{i\in\Lambda} X_i$. □

3.11. Definition. [15] Let (F, A) be a soft BCH-algebra over X.

(i) (F, A) is called the *trivial* soft BCH-algebra over X if $F(x) = \{0\}$ for all $x \in A$.

(ii) (F, A) is called the *whole* soft BCH-algebra over X if $F(x) = X$ for all $x \in A$.

3.12. Example. Consider the proper BCH-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table [4]:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	3	0	3
3	3	0	0	0

Let (F, A) be a soft set over X, where $A = \{1, 2\}$ and $F : A \to \mathcal{P}(X)$ is the set-valued function defined by

$$F(x) = \{y \in X \mid y * x \in \{0, 3\}\}$$

for all $x \in A$. Then $F(x) = X$ for all $x \in A$, so (F, A) is a whole soft BCH-algebra over X.

3.13. Definition. Let X, Y be two BCH-algebras and $f : X \to Y$ a mapping of BCH-algebras. If (F, A) and (G, B) are soft sets over X and Y respectively, then $(f(F), A)$ is a soft set over Y where $f(F) : A \to \mathcal{P}(Y)$ is defined by $f(F)(x) = f(F(x))$ for all $x \in A$ and $(f^{-1}(G), B)$ is a soft set over X where $f^{-1}(G) : B \to \mathcal{P}(X)$ is defined by $f^{-1}(G)(y) = f^{-1}(G(y))$ for all $y \in B$.

3.14. Lemma. *Let $f : X \to Y$ be an onto homomorphism of BCH-algebras.*

(i) *If (F, A) is a soft BCH-algebra over X, then $(f(F), A)$ is a soft BCH-algebra over Y.*
(ii) *If (G, B) is a soft BCH-algebra over Y, then $(f^{-1}(G), B)$ is a soft BCH-algebra over X if it is non-null.*

Proof. (i) Since (F, A) is a soft BCH-algebra over X, it is clear that $(f(F), A)$ is a non-null soft set over Y.

For every $x \in \operatorname{Supp}(f(F), A)$, we have $f(F)(x) = f(F(x)) \neq \emptyset$. Since the nonempty set $F(x)$ is a BCH-subalgebra of X, its onto homomorphic image $f(F(x))$ is a BCH-subalgebra of Y. Hence $f(F(x))$ is a BCH-subalgebra of Y for all $x \in \operatorname{Supp}(f(F), A)$. That is, $(f(F), A)$ is a soft BCH-algebra over Y.

(ii) It is easy to see that $\operatorname{Supp}(f^{-1}(G), B) \subseteq \operatorname{Supp}(G, B)$. Let $y \in \operatorname{Supp}(f^{-1}(G), B)$. Then $G(y) \neq \emptyset$. Since the nonempty set $G(y)$ is a BCH-subalgebra of Y, its homomorphic inverse image $f^{-1}(G(y))$ is also a BCH-subalgebra of X. Hence $f^{-1}(G(y))$ is a BCH-subalgebra of Y for all $y \in \operatorname{Supp}(f^{-1}(G), B)$. That is, $(f^{-1}(G), B)$ is a soft BCH-algebra over X. □

3.15. Theorem. *Let $f : X \to Y$ be a homomorphism of BCH-algebras. Let (F, A) and (G, B) be two soft BCH-algebras over X and Y respectively.*

(i) *If $F(x) = \ker(f)$ for all $x \in A$, then $(f(F), A)$ is the trivial soft BCH-algebra over Y.*
(ii) *If f is onto and (F, A) is whole, then $(f(F), A)$ is the whole soft BCH-algebra over Y.*
(iii) *If $G(y) = f(X)$ for all $y \in B$, then $(f^{-1}(G), B)$ is the whole soft BCH-algebra over X.*
(iv) *If f is injective and (G, B) is trivial, then $(f^{-1}(G), B)$ is the trivial soft BCH-algebra over X.*

Proof. (i) Assume that $F(x) = \ker(f)$ for all $x \in A$. Then, $f(F)(x) = f(F(x)) = \{0_Y\}$ for all $x \in A$. Hence $(f(F), A)$ is soft BCH-algebra over Y by Lemma 3.14 and Definition 3.11.

(ii) Suppose that f is onto and that (F, A) is whole. Then, $F(x) = X$ for all $x \in A$, and so $f(F)(x) = f(F(x)) = f(X) = Y$ for all $x \in A$. It follows from Lemma 3.14 and Definition 3.11 that $(f(F), A)$ is the whole soft BCH-algebra over Y.

(iii) Assume that $G(y) = f(X)$ for all $y \in B$. Then, $f^{-1}(G)(y) = f^{-1}(G(y)) = f^{-1}(f(X)) = X$ for all $y \in B$. Hence, $(f^{-1}(G), B)$ is the whole soft BCH-algebra over X by Lemma 3.14 and Definition 3.11.

(iv) Suppose that f is injective and (G, B) is trivial. Then, $G(y) = \{0\}$ for all $y \in B$, and so $f^{-1}(G)(y) = f^{-1}(G(y)) = f^{-1}(\{0\}) = \ker(f) = \{0_X\}$ for all $y \in B$. It follows from Lemma 3.14 and Definition 3.11 that $(f^{-1}(G), B)$ is the trivial soft BCH-algebra over X. □

4. Soft subalgebras

4.1. Definition. Let (F, A) and (G, B) be two soft BCH-algebras over X. Then (G, B) is called a *soft BCH-subalgebra* of (F, A), denoted by $(G, B) \widetilde{<}_s (F, A)$, if it satisfies the following conditions:

(i) $B \subseteq A$,
(ii) $G(x)$ is a BCH-subalgebra of $F(x)$ for all $x \in \operatorname{Supp}(G, B)$.

From the above definition, one easily deduces that if (G, B) is a soft BCH-subalgebra of (F, A), then $\operatorname{Supp}(G, B) \subset \operatorname{Supp}(F, A)$.

4.2. Example. Consider the proper BCH-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table [5]:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

Let (F, A) be a soft set over X, where $A = \{0, 1, 2\}$ and $F : A \to \mathcal{P}(X)$ is the set-valued function defined by

$$F(x) = \{y \in X \mid y * (y * x) \in \{0, 1\}\}$$

for all $x \in A$. Then (F, A) is a soft BCH-algebra over X for all $x \in \operatorname{Supp}(F, A) = A$.

Let (G, B) be a soft set over X, where $B = \{0, 2\}$ and $G : A \to \mathcal{P}(X)$ is the set-valued function defined by

$$G(x) = \{y \in X \mid y * (y * x) \in \{0, 4\}\}$$

for all $x \in B$. It is easy to see that (G, B) is a soft BCH-algebra over X for all $x \in \operatorname{Supp}(G, B) = B$, and $G(0) = X = F(0)$, $G(2) = \{0, 1, 3\} = F(2)$. Hence (G, B) is a soft BCH-subalgebra of (F, A).

4.3. Theorem. *Let (F, A) and (G, B) be two soft BCH-algebras over X and $(G, B) \widetilde{\subseteq} (F, A)$. Then $(G, B) \widetilde{<}_s (F, A)$*

Proof. Straightforward. □

4.4. Theorem. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A_i) \mid i \in \Lambda\}$ a nonempty family of soft BCH-subalgebras of (F, A). Then the bi-intersection $\widetilde{\sqcap}_{i \in \Lambda}(H_i, A_i)$ is a soft BCH-subalgebra of (F, A) if it is non-null.*

Proof. Similar to the proof of Theorem 3.4. □

4.5. Corollary. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A) \mid i \in \Lambda\}$ a nonempty family of soft BCH-subalgebras of (F, A). Then the bi-intersection $\widetilde{\sqcap}_{i \in \Lambda}(H_i, A)$ is a soft BCH-subalgebra of (F, A) if it is non-null.*

Proof. Straightforward. □

4.6. Theorem. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A_i) \mid i \in \Lambda\}$ a nonempty family of soft BCH-subalgebras of (F, A). Then the extended intersection $\widetilde{\bigcap}_{i\in\Lambda}(H_i, A_i)$ is a soft BCH-subalgebras of (F, A).*

Proof. Similar to the proof of Theorem 3.6. □

4.7. Theorem. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A_i) \mid i \in \Lambda\}$ a nonempty family of soft BCH-subalgebras of (F, A). If $H_i(x_i) \subseteq H_j(x_j)$ or $H_j(x_j) \subseteq H_i(x_i)$ for all $i, j \in \Lambda$, $x_i \in A_i$, then the restricted union $\widetilde{\bigcup}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-subalgebra of (F, A).*

Proof. Assume that $\{(H_i, A_i) \mid i \in \Lambda\}$ is a nonempty family of soft BCH-subalgebra of (F, A). By Definition 2.6 (ii), we can write $\widetilde{\bigcup}_{i\in\Lambda}(H_i, A_i) = (H, B)$, where $B = \bigcap_{i\in\Lambda} A_i$, and $H(x) = \bigcup_{i\in\Lambda} H_i(x)$ for all $x \in B$.

Let $x \in \mathrm{Supp}\,(H, B)$. Then $H(x) = \bigcup_{i\in\Lambda} H_i(x) \neq \emptyset$, and so we have $H_{i_0}(x_{i_0}) \neq \emptyset$ for some $i_0 \in \Lambda$. Since $H_i(x_i) \subseteq H_j(x_j)$ or $H_j(x_j) \subseteq H_i(x)$ for all $i, j \in \Lambda$, $x_i \in A_i$, clearly $\bigcup_{i\in\Lambda} H_i(x_i)$ is a subalgebra of $F(x)$ for all $x \in \mathrm{Supp}\,(H, B)$. Hence the restricted union $\widetilde{\bigcup}_{i\in\Lambda}(H_i, A_i)$ is a soft subalgebra of (F, A). □

4.8. Theorem. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A_i) \mid i \in \Lambda\}$ a nonempty family of soft subalgebras of (F, A). Then the $\wedge$-intersection $\widetilde{\bigwedge}_{i\in\Lambda}(H_i, A_i)$ is a soft subalgebra of $\widetilde{\bigwedge}_{i\in\Lambda}(F, A)$.*

Proof. Similar to the proof of Theorem 3.8. □

4.9. Theorem. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A_i) \mid i \in \Lambda\}$ a nonempty family of soft BCH-subalgebras of (F, A). If $H_i(x_i) \subseteq H_j(x_j)$ or $H_j(x_j) \subseteq H_i(x_i)$ for all $i, j \in \Lambda$, $x_i \in A_i$, then the $\vee$-union $\widetilde{\bigvee}_{i\in\Lambda}(H_i, A_i)$ is a soft BCH-subalgebra of $\widetilde{\bigvee}_{i\in\Lambda}(F, A)$.*

Proof. The proof follows from Theorem 3.9. □

4.10. Theorem. *Let (F, A) be a soft BCH-algebra over X and $\{(H_i, A_i) \mid i \in \Lambda\}$ a nonempty family of soft BCH-subalgebra of (F, A). Then the cartesian product of the family $\widetilde{\prod}_{i\in\Lambda}(H_i, A_i)$ is a soft BCH-subalgebra of $\widetilde{\prod}_{i\in\Lambda}(F, A)$.*

Proof. By Definition 2.10, we can write $\widetilde{\prod}_{i\in\Lambda}(H_i, A_i) = (H, B)$, where $B = \prod_{i\in\Lambda} A_i$ and $H(x) = \prod_{i\in\Lambda} H_i(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in B$.

Let $x = (x_i)_{i\in\Lambda} \in \mathrm{Supp}\,(H, B)$. Then $H(x) = \prod_{i\in\Lambda} H_i(x_i) \neq \emptyset$, and so we have $H_i(x_i) \neq \emptyset$ for all $i \in \Lambda$. Since $\{(H_i, A_i) \mid i \in \Lambda\}$ is a soft subalgebras of (F, A), we have that $H_i(x_i)$ is a BCH-subalgebra of $F(x_i)$, from which we obtain that $\prod_{i\in\Lambda} H_i(x_i)$ is a BCH-subalgebra of $\prod_{i\in\Lambda} F(x_i)$ for all $x = (x_i)_{i\in\Lambda} \in \mathrm{Supp}\,(H, B)$. Hence, the cartesian product of the family $\widetilde{\prod}_{i\in\Lambda}(F_i, A_i)$ is a soft BCH-subalgebra of $\widetilde{\prod}_{i\in\Lambda}(F, A)$. □

4.11. Theorem. *Let $f : X \to Y$ be a homomorphism of BCH-algebras and (F, A), (G, B) two soft BCH-algebras over X. If $(G, B) \mathrel{\widetilde{<}_s} (F, A)$, then $(f(G), B) \mathrel{\widetilde{<}_s} (f(F), A)$.*

Proof. Assume that $(G, B) \mathrel{\widetilde{<}_s} (F, A)$. Let $x \in \mathrm{Supp}\,(G, B)$. Then $x \in \mathrm{Supp}\,(F, A)$. By Definition 4.1, $A \subseteq B$ and $G(x)$ is a BCH-subalgebra of $F(x)$ for all $x \in \mathrm{Supp}\,(G, B)$. Since f is a homomorphism, $f(G)(x) = f(G(x))$ is a BCH-subalgebra of $f(F(x)) = f(F)(x)$ and, therefore, $(f(G), B) \mathrel{\widetilde{<}_s} (f(F), A)$. □

4.12. Theorem. *Let $f : X \to Y$ be a homomorphism of BCH-algebras and (F, A), (G, B) two soft BCH-algebras over Y.*

If $(G, B) \widetilde{<}_s (F, A)$, then $(f^{-1}(G), B) \widetilde{<}_s (f^{-1}(F), A)$.

Proof. Assume that $(G, B) \widetilde{<}_s (F, A)$. Let $y \in \mathrm{Supp}\,(f^{-1}(G), B)$. By Definition **??**, $B \subseteq A$ and $G(y)$ is a BCH-subalgebra of $F(y)$ for all $y \in B$. Since f is a homomorphism, $f^{-1}(G)(y) = f^{-1}(G(y))$ is a BCH-subalgebra of $f^{-1}(F(y)) = f^{-1}(F)(y)$ for all $y \in \mathrm{Supp}\,(f^{-1}(G), B)$. Hence, $(f^{-1}(G), B) \widetilde{<}_s (f^{-1}(F), A)$. □

4.13. Definition. Let X, Y be two BCH-algebras and (F, A), (G, B) two soft BCH-algebras over X and Y, respectively. Let $f : X \to Y$ and $g : A \to B$ be two mappings. Then the pair (f, g) is called a *soft function* from (F, A) to (G, B).

A pair (f, g) is called a *soft homomorphism* from X to Y if it satisfies the following conditions:

(i) f is a homomorphism,
(ii) g is a mapping,
(iii) $f(F(x)) = G(g(x))$ for all $x \in A$.

We say that (F, A) is *softly homomorphic* to (G, B) under the soft homomorphism (f, g), if (f, g) is a soft homomorphism and f, g are both surjective.

In this definition, if f is an isomorphism from X to Y and g a one-to-one and onto mapping from A to B, then we say that (f, g) is a *soft isomorphism* and that (F, A) is *softly isomorphic* to (G, B) under the soft homomorphism (f, g). This is denoted by $(F, A) \simeq (G, B)$.

4.14. Proposition. *The relation $\simeq$ is an equivalence relation on soft BCH-algebras.*

Proof. Straightforward. □

4.15. Example. Consider the proper BCH-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table [5]:

*	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	1	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Let $(F, \mathbb{N})$ be a soft set over X, where $\mathbb{N}$ is the set of natural numbers and $F : \mathbb{N} \to \mathcal{P}(X)$ the set-valued function defined by $F(n) = \begin{cases} \{0, 3, 4\}, & \text{if } 2 \mid n, \\ \{0, 4\}, & \text{otherwise} \end{cases}$ for all $x \in \mathbb{N}$. Then $(F, \mathbb{N})$ is a soft BCH-algebra over X.

Let $(G, \mathbb{N})$ be a soft set over X, where $\mathbb{N}$ is the set of natural numbers and $G : \mathbb{N} \to \mathcal{P}(X)$ the set-valued function defined by $G(n) = \begin{cases} \{0, 4\}, & \text{if } 2 \mid n, \\ \emptyset, & \text{otherwise} \end{cases}$ for all $x \in \mathbb{N}$. Then $(G, \mathbb{N})$ is a soft BCH-algebra over X.

Let $f : X \to X$ be the mapping defined by $f(x) = \begin{cases} 4, & \text{if } x = 4, \\ 0, & \text{if } x \neq 4. \end{cases}$ It is clear that f is a BCH-homomorphism. Consider the mapping $g : \mathbb{N} \to \mathbb{N}$ given by $g(x) = 2x$. Then one can easily verify that $f(F(x)) = G(x) = G(g(x))$ for all $x \in \mathbb{N}$. Hence (f, g) is a soft homomorphism from X to X by Definition 4.13.

4.16. Theorem. *Let $f : X \to Y$ be an onto homomorphism of BCH-algebras and (F, A), (G, B) two soft BCH-algebras over X and Y respectively.*

(i) *The soft function (f, I_A) from (F, A) to (H, A) is a soft homomorphism from X to Y, where $I_A : A \to A$ is the identity mapping and the set-valued function $H : A \to \mathcal{P}(Y)$ is defined by $H(x) = f(F(x))$ for all $x \in A$.*

(ii) *If $f : X \to Y$ is an isomorphism, then the soft function (f^{-1}, I_B) from (G, B) to (K, B) is a soft homomorphism from Y to X, where $I_B : B \to B$ is the identity mapping and the set-valued function $K : B \to \mathcal{P}(X)$ is defined by $K(x) = f^{-1}(G(x))$ for all $x \in B$.*

Proof. Follows from Lemma 3.14. □

4.17. Proposition. *Let X, Y and Z be BCH algebras and (F, A), (G, B) and (H, C) soft BCH-algebras over X, Y, and Z respectively. Let the soft function (f, g) from (F, A) to (G, B) be a soft homomorphism from X to Y, and the soft function (f', g') from (G, B) to (H, C) a soft homomorphism from Y to Z. Then the the soft function $(f' \circ f, g' \circ g)$ from (F, A) to (H, C) is a soft homomorphism from X to Z.*

Proof. Straightforward. □

4.18. Theorem. *Let X and Y be BCH algebras and (F, A), (G, B) soft sets over X and Y respectively. If (F, A) is a soft BCH-algebra over X and $(F, A) \simeq (G, B)$, then (G, B) is a soft BCH-algebra over Y.*

Proof. Follows from Definition 4.13 and Lemma 3.14. □

Acknowledgements. The authors would like to thank the referees and Professor Yücel Tıraş, Editor-in-Chief, for their valuable comments and suggestions which have helped immensely in improving the quality of the paper.

References

[1] Ahmad, B. *On classification of BCH-algebras*, Math. Japon. **35**, 801–804, 1990.

[2] Irfan Ali, M., Feng, F., Liu, X., Minc, W. K. and Shabir, M. *On some new operations in soft set theory*, Computers and Mathematics with Applications **57**, 1547–1553, 2009.

[3] Aktaş, H. and Çağman, N. *Soft sets and soft groups*, Information Sciences **177**, 2726–2735, 2007.

[4] Chaudhry, M. A. and Fakhar-Ud-Din, H. *Ideals and filters in BCH-algebras*, Math. Japon. **44**, 101–111, 1996.

[5] Chaudhry, M. A. *On BCH-algebras*, Math. Japon. **36**, 665–676, 1991.

[6] Chaudhary, M. A. and Fakhar-Ud-Din, H. *On some classes of BCH-algebras*, Int. J. Math. Math. Sci. **27**, 1739–1750, 2003.

[7] Chen, D., Tsang, E. C. C., Yeung, D. S. and Wang, X. *The parameterization reduction of soft sets and its applications*, Computers and Mathematics with Applications **49**, 757–763, 2005.

[8] Dudek, W. A. and Thomys, J. *On decompositions of BCH-algebras*, Math. Japon. **35**, 1131–1138, 1990.

[9] Feng, F., Jun, Y. B. and Zhao, X. *Soft semirings*, Computers and Mathematics with Applications **56**, 2621–2628, 2008.

[10] Hu, Q. P. and Li, X. *On BCH-algebras*, Math. Seminar Notes **11**, 313–320, 1983.

[11] Hu, Q. P. and Li, X. *On proper BCH-algebras*, Math. Japonica **30**, 659–661, 1985.

[12] Imai, Y. and Iséki, K. *On axiom systems of propositional calculi*, XIV, Proc. Japan Acad. **42**, 19–22, 1966.

[13] Iséki, K. *An algebra related with a propositional calculus*, Proc. Japan Acad. **42**, 26–29, 1966.

[14] Jun, Y. B., Roh, E. H. and Kim, H. S. *On fuzzy B-algebras*, Czechoslovak Math. J. **52** (127), 375–384, 2002.

[15] Jun, Y. B. *Soft BCK/BCI-algebras*, Computers and Mathematics with Applications **56**, 1408–1413, 2008.

[16] Jun, Y. B. and Park, C. H. *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Information Sciences**178**, 2466–2475, 2008.

[17] Maji, P. K., Biswas, R. and Roy, A. R. *Soft set theory*, Computers and Mathematics with Applications **45**, 555–562, 2003.

[18] Maji, P. K., Roy, A. R. and Biswas, R. *An application of soft sets in a decision making problem*, Comput. Math. Appl. **44**, 1077–1083, 2002.

[19] Molodtsov, D. *Soft set theory - first results*, Computers and Mathematics with Applications **37**, 19–31, 1999.

[20] Roh, E. H., Jun, Y. B. and Zhang, Q. *Special subset in BCH-algebras*, Far East J. Math. Sci. (FJMS) **3**, 723–729, 2001.

[21] Roh, E. H., Kim, S. Y. and Jun, Y. B. *On a problem in BCH-algebras*, Math. Japon. **52** 279–283, 2000.

[22] Roy, A. R. and Maji, P. K. *A fuzzy soft set theoretic approach to decision making problems*, Journal of Computational and Applied Mathematics **203**, 412–418, 2007.