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PAGES: 173-183

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/86827>

ON MEROMORPHIC HARMONIC STARLIKE FUNCTIONS WITH MISSING COEFFICIENTS

Hakan Bostancı^{*†} and Metin Öztürk[‡]

Received 11:02:2009 : Accepted 26:03:2009

Abstract

In this paper, we introduce a new class of meromorphic harmonic starlike functions with missing coefficients in the punctured unit disk $U^* = \{z : 0 < |z| < 1\}$. We obtain coefficient inequalities, a distortion theorem and a closure theorem. In addition, we investigate some properties of this class.

Keywords: Harmonic functions, Meromorphic functions, Starlike functions.

2000 AMS Classification: 30C45, 30C50, 30C55.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D . In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]). In [5], Hengartner and Schober investigated functions harmonic in the exterior of the unit disc $\tilde{U} = \{z : |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k},$$

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$0 \leq |\beta| < |\alpha|$, $A \in \mathbb{C}$. Also, Jahangiri and Silverman [11], Jahangiri [7] and Murugusundaramoorthy [9, 10] have studied classes of meromorphic harmonic functions.

Let MH_p denote the class of functions harmonic, univalent, sense-preserving and meromorphic in $U^* = \{z : 0 < |z| < 1\}$ and which have the representation

$$(1.1) \quad f(z) = h(z) + \overline{g(z)} = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} + \overline{\sum_{k=1}^{\infty} b_{p+k} z^{p+k}},$$

where $a_{-1} \neq 0$, $p \in \mathbb{N} = \{1, 2, \dots\}$.

Notice that, if we take $p = 0$, then we are not missing any coefficients. Also, if we substitute $p = 0$ and $a_{-1} = 1$ in the above representation (1.1), then we have

$$(1.2) \quad f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k},$$

therefore $f \in MH$. The class MH is defined and some subclasses are studied by Bostancı *et. al.* in [2]. Otherwise, if we define the composite function $w(z) = f(1/z)$, then $w(z)$ is a harmonic, sense preserving, univalent mapping in $\tilde{U}$ and $w \in \Sigma''$. This class Σ'' is investigated by Hengartner and Schober in [5].

A function $f(z) \in MH_p$ is said to be in the subclass $MHS_p^*(\alpha)$ of meromorphic harmonic α -starlike functions in U^* if it satisfies the condition

$$(1.3) \quad \Re \left\{ -\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} > \alpha,$$

where $0 \leq \alpha < 1$. If $\alpha = 0$ then this class is called the class of meromorphic harmonic starlike functions. We denote the meromorphic harmonic starlike functions by MHS^* . This classification (1.3) for harmonic univalent functions was first used by Jahangiri [6].

The class of meromorphic functions has been studied by various authors, including Joshi and Sangle [7], Darwish [5], Uraleggaddi and Somanatha [12], and Aouf and Hossen[4]. In this paper, we define a new operator H for meromorphic harmonic functions. Also, we define the classes $MH_p^n(\alpha)$ and $\overline{MH}_p^n(\alpha)$. Then, we investigate some properties of these classes, such as coefficient estimates and a distortion theorem.

We define the new operator H for harmonic functions as follows:

$$H^0 f(z) = f(z), H^1 f(z) = Hf(z) = \frac{(z^2 h(z))'}{z} - \overline{z^3 \left(\frac{g(z)}{z^2} \right)'},$$

and for $n = 2, \dots$,

$$H^n f(z) = H(H^{n-1} f(z)).$$

Hence, we obtain for $n = 0, 1, \dots$,

$$H^n f(z) = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} (p+k+2)^n a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} (p+k-2)^n b_{p+k} z^{p+k}}.$$

Using the operator H , we now make the following definition:

For $0 \leq \alpha < 1$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $MH_p^n(\alpha)$ denotes the class consisting of those functions in MH_p satisfying

$$(1.4) \quad \Re \left\{ \frac{H^{n+1} f(z)}{H^n f(z)} - 2 \right\} < -\alpha$$

for $z \in U^*$.

Also, let $\overline{MH}_p^n(\alpha)$ be the subclass of $MH_p^n(\alpha)$ which consisting of meromorphic harmonic functions of the form

$$(1.5) \quad f_n(z) = h(z) + (-1)^n \overline{g(z)} = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k} z^{p+k}}$$

where $a_{-1} > 0, a_{p+k} \geq 0, b_{p+k} \geq 0, p \in \mathbb{N}$.

2. Some results for the class MH_p

2.1. Theorem. *If $f \in MH_p$, then the diameter D_f of $\mathbb{C} \setminus f(U^*)$ satisfies*

$$D_f \geq 2|a_{-1}|.$$

This estimate is sharp for $f(z) = a_{-1}z^{-1}$.

Proof. Let $D_f(r)$ be the diameter of $f(|z| = r)$, $0 < r < 1$, and let $D_f^*(r) = \max_{|z|=r} |f(z) - f(-z)|$. Then $D_f(r) \searrow D_f$ as $r \rightarrow 1$ and $D_f(r) \geq D_f^*(r)$. Since

$$\begin{aligned} [D_f^*(r)]^2 &\geq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta}) - f(re^{-i\theta})|^2 d\theta \\ &= \begin{cases} 4 \left[\frac{|a_{-1}|^2}{r^2} + \sum_{k=1}^{\infty} (|a_{p+2k-1}|^2 + |b_{p+2k-1}|^2) r^{2(p+2k-1)} \right], & p \text{ is odd} \\ 4 \left[\frac{|a_{-1}|^2}{r^2} + \sum_{k=1}^{\infty} (|a_{p+2k}|^2 + |b_{p+2k}|^2) r^{2(p+2k)} \right], & p \text{ is even} \end{cases} \\ &\geq 4|a_{-1}|^2 r^{-2}, \end{aligned}$$

we conclude that $D_f \geq 2|a_{-1}|$. $\square$

2.2. Corollary. *If we substitute $p = 0$, $a_{-1} = 1$ and $w(z) = f(1/z)$ in the above theorem, then we have $w \in \Sigma''$ and*

$$D_w \geq 2|1 + b_1|. \quad \square$$

2.3. Theorem. *If $f \in MH_p$ has expansion (1.1) then*

$$\sum_{k=1}^{\infty} (p+k)(|a_{p+k}|^2 - |b_{p+k}|^2) \leq |a_{-1}|^2.$$

Equality occurs if and only if $\mathbb{C} \setminus f(U^)$ has zero area.*

Proof. The area of the omitted set is

$$\begin{aligned} \lim_{r \rightarrow 1} \frac{1}{2i} \int_{|z|=r} \bar{f} df &= \lim_{r \rightarrow 1} \left[\frac{1}{2i} \int_{|z|=r} \bar{h} h' dz + \frac{1}{2i} \int_{|z|=r} g \bar{g}' d\bar{z} \right] \\ &= \pi \left[-|a_{-1}|^2 + \sum_{k=1}^{\infty} (p+k)(|a_{p+k}|^2 - |b_{p+k}|^2) \right] \\ &\leq 0. \end{aligned}$$

$\square$

2.4. Corollary. *If we substitute $p = 0$, $a_{-1} = 1$ and $w(z) = f(1/z)$ in the above theorem, then we have $w \in \Sigma''$ and*

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2\Re b_1. \quad \square$$

3. Coefficient inequalities

3.1. Theorem. *If $f(z) = h(z) + \overline{g(z)}$ is of the form (1.1) and the condition*

$$(3.1) \quad \sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)|a_{p+k}| + (p+k-2)^n(p+k-\alpha)|b_{p+k}|] \leq (1-\alpha)|a_{-1}|$$

is satisfied then $f(z) \in MH_p^n(\alpha)$.

Proof. Set

$$p(z) = \begin{cases} -\frac{H^{n+1}f(z)}{H^n f(z)} + 2 - \alpha, & z \neq 0, \\ 1 - \alpha, & z = 0. \end{cases}$$

We must show that if the condition (3.1) is satisfied then $f(z) \in MH_p^n(\alpha)$. Therefore, it is sufficient to show that $p(z)$ is in the class PH which is the class of harmonic functions with positive real part, or equivalently prove that

$$(3.2) \quad |p(z) + 1| > |p(z) - 1|, \quad z \in U.$$

From (7) we obtain, for $z \in U^*$,

$$(3.3) \quad \frac{|-H^{n+1}f(z) + (2-\alpha+1)H^n f(z)|}{|H^n f(z)|} > \frac{|-H^{n+1}f(z) + (2-\alpha-1)H^n f(z)|}{|H^n f(z)|}.$$

Since,

$$\begin{aligned} & | -H^{n+1}f(z) + (2-\alpha+1)H^n f(z) | \\ &= \left| \frac{(\alpha-2)a_{-1}}{z} + \sum_{k=1}^{\infty} (p+k+2)^n(p+k+\alpha-1)a_{p+k} z^{p+k} \right. \\ & \quad \left. + (-1)^n \sum_{k=1}^{\infty} (p+k-2)^n(p+k+1-\alpha)\bar{b}_{p+k} \bar{z}^{p+k} \right| \\ (3.4) \quad & \geq \frac{(2-\alpha)|a_{-1}|}{|z|} - \sum_{k=1}^{\infty} (p+k+2)^n(p+k+\alpha-1)|a_{p+k}| |z|^{p+k} \\ & \quad - \sum_{k=1}^{\infty} (p+k-2)^n(p+k+1-\alpha)|\bar{b}_{p+k}| |\bar{z}|^{p+k} \\ & > |a_{-1}| + \sum_{k=1}^{\infty} (p+k+2)^n|a_{p+k}| - \sum_{k=1}^{\infty} (p+k-2)^n|b_{p+k}|, \end{aligned}$$

and

$$\begin{aligned}
 & \left| -H^{n+1}f(z) + (2 - \alpha - 1)H^n f(z) \right| \\
 &= \left| \frac{\alpha a_{-1}}{z} + \sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha+1) a_{p+k} z^{p+k} \right. \\
 &\quad \left. + (-1)^n \sum_{k=1}^{\infty} (p+k-2)^n (p+k-1-\alpha) \bar{b}_{p+k} \bar{z}^{p+k} \right| \\
 (3.5) \quad &\leq \frac{\alpha |a_{-1}|}{|z|} + \sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha+1) |a_{p+k}| |z|^{p+k} \\
 &\quad + \sum_{k=1}^{\infty} (p+k-2)^n (p+k-1-\alpha) |\bar{b}_{p+k}| |\bar{z}|^{p+k} \\
 &< |a_{-1}| + \sum_{k=1}^{\infty} (p+k+2)^n |a_{p+k}| - \sum_{k=1}^{\infty} (p+k-2)^n |b_{p+k}|,
 \end{aligned}$$

we see that inequality (3.1) holds in U^* by (3.4) and (3.5). The case $z = 0$ is trivial. Hence $p(z) \in PH$. So the proof of Theorem 3.1 is complete. $\square$

3.2. Corollary. *If we substitute $p = 0$, $n = 0$, $\alpha = 0$ and $a_{-1} = 1$ in the above theorem, then we have $\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \leq 1$ and $f \in MHS^*$, therefore f harmonic starlike in U^* [2, page 373, Theorem 2.1].*

3.3. Theorem. *Let the function $f_n(z)$ to be defined by (1.5). A necessary and sufficient condition for $f_n(z) \in \overline{MH}_p^n(\alpha)$ is that*

$$(3.6) \quad \sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha) a_{p+k} + (p+k-2)^n (p+k-\alpha) b_{p+k}] \leq (1-\alpha) a_{-1}.$$

The estimate (3.6) is sharp and the equality is attained for the function

$$f(z) = \frac{a_{-1}}{z} + \frac{(1-\alpha)a_{-1}}{(p+k+2)^n (p+k+\alpha)} z^{p+k} - \frac{(1-\alpha)a_{-1}}{(p+k-2)^n (p+k-\alpha)} \bar{z}^{p+k}.$$

Proof. In view of Theorem 3.1, it is sufficient to prove the "only if" part, since $\overline{MH}_p^n(\alpha) \subset MH_p^n(\alpha)$. Assume that $f_n(z) \in \overline{MH}_p^n(\alpha)$. Let z be a complex number. If $\Re(z) > 0$ then $\Re(1/z) > 0$. Thus from (1.4) we obtain

$$\Re \left\{ -\frac{H^{n+1}f(z)}{H^n f(z)} + (2 - \alpha) \right\} > 0$$

and

$$\begin{aligned}
 0 &< \Re \left\{ \frac{H^n f(z)}{-H^{n+1}f(z) + (2 - \alpha)H^n f(z)} \right\} \\
 &\leq \left| \frac{H^n f(z)}{-H^{n+1}f(z) + (2 - \alpha)H^n f(z)} \right| \\
 &= \left| \frac{a_{-1} + \sum_{k=1}^{\infty} (p+k+2)^n a_{p+k} z^{p+k+1} + |z|^2 \sum_{k=1}^{\infty} (p+k-2)^n b_{p+k} \bar{z}^{p+k-1}}{D} \right|,
 \end{aligned}$$

where

$$D = (1 - \alpha)a_{-1} - \sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha)a_{p+k} z^{p+k+1} \\ + |z|^2 \sum_{k=1}^{\infty} (p+k-2)^n (p+k-\alpha)b_{p+k} \bar{z}^{n+k-1}.$$

Hence

$$(3.7) \quad 0 \leq \frac{a_{-1} + \sum_{k=1}^{\infty} (p+k+2)^n a_{p+k} + \sum_{k=1}^{\infty} (p+k-2)^n b_{p+k}}{(1 - \alpha)a_{-1} - \sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha)a_{p+k} - \sum_{k=1}^{\infty} (p+k-2)^n (p+k-\alpha)b_{p+k}}.$$

From (3.7), we must have

$$\sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha)a_{p+k} + (p+k-2)^n (p+k-\alpha)b_{p+k}] \leq (1 - \alpha)a_{-1}.$$

Hence, the proof is completed. $\square$

3.4. Theorem. Let $0 \leq \alpha_1 \leq \alpha_2 < 1$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then $\overline{MH}_p^n(\alpha_2) \subset \overline{MH}_p^n(\alpha_1)$.

Proof. Let the function $f_n(z)$ defined by (1.5) be in the class $\overline{MH}_p^n(\alpha_2)$ and let $\alpha_1 = \alpha_2 - \delta$. Then, by Theorem 3.3, we have

$$\sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha_2)a_{p+k} + (-1)^n (p+k-2)^n (p+k-\alpha_2)b_{p+k} \leq (1 - \alpha_2)a_{-1}$$

and

$$\sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha_2)a_{p+k} + (-1)^n (p+k-2)^n (p+k-\alpha_2)b_{p+k} \\ \leq \frac{(1 - \alpha_2)a_{-1}}{p+1 - \alpha_2} \\ \leq a_{-1}.$$

Consequently,

$$\sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha_1)a_{p+k} + (-1)^n (p+k-2)^n (p+k-\alpha_1)b_{p+k} \\ = \sum_{k=1}^{\infty} (p+k+2)^n (p+k+\alpha_2)a_{p+k} + (-1)^n (p+k-2)^n (p+k-\alpha_2)b_{p+k} \\ + \delta \left(\sum_{k=1}^{\infty} (-1)^n (p+k-2)^n b_{p+k} + (p+k+2)^n a_{p+k} \right) \\ \leq (1 - \alpha_2)a_{-1}.$$

$\square$

3.5. Theorem. Let $0 \leq \alpha < 1$, $p \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then $\overline{MH}_p^{n+1}(\alpha) \subset \overline{MH}_p^n(\alpha)$.

Proof. The proof of Theorem 3.5 also follows from Theorem 3.3. $\square$

4. Distortion theorem

4.1. Theorem. *Let the function $f_n(z)$ be in the class $\overline{MH}_p^n(\alpha)$. Then, for $0 < |z| = r < 1$, we have*

$$(4.1) \quad \frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p} \leq |f_n(z)| \leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p}$$

where equalities hold for the functions

$$f(z) = \frac{a_{-1}}{z} \pm \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}z, \quad z = r$$

Proof. In view of Theorem 3.3, for $0 < |z| = r < 1$,

$$\begin{aligned} |f_n(z)| &= \left| \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k} z^{p+k}} \right| \\ &\leq \frac{a_{-1}}{r} + r^{1+p} \sum_{k=1}^{\infty} (a_{p+k} + b_{p+k}) \\ &\leq \frac{a_{-1}}{r} + \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \sum_{k=1}^{\infty} (p-1)^n(p+1-\alpha)(a_{p+k} + b_{p+k}) \\ &\leq \frac{a_{-1}}{r} + \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \\ &\quad \times \sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)a_{p+k} + (p+k-2)^n(p+k-\alpha)b_{p+k}] \\ &\leq \frac{a_{-1}}{r} + \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p} \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &= \left| \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k} z^{p+k}} \right| \\ &\geq \frac{a_{-1}}{r} - r^{1+p} \sum_{k=1}^{\infty} (a_{p+k} + b_{p+k}) \\ &\geq \frac{a_{-1}}{r} - \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \sum_{k=1}^{\infty} (p-1)^n(p+1-\alpha)(a_{p+k} + b_{p+k}) \\ &\geq \frac{a_{-1}}{r} - \frac{r^{1+p}}{(p-1)^n(p+1-\alpha)} \\ &\quad \times \sum_{k=1}^{\infty} [(p+k+2)^n(p+k+\alpha)a_{p+k} + (p+k-2)^n(p+k-\alpha)b_{p+k}] \\ &\geq \frac{a_{-1}}{r} - \frac{(1-\alpha)a_{-1}}{(p-1)^n(p+1-\alpha)}r^{1+p}. \end{aligned}$$

Thus, inequality (4.1) is obtained. $\square$

5. Closure theorem

Let the function $f_{n,j}(z)$ be defined by

$$(5.1) \quad f_{n,j}(z) = \frac{a_{-1,j}}{z} + \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k,j} z^{p+k}}, \quad j = 1, 2, \dots, m,$$

for $z \in U^*$.

Now we shall prove the following result for the closure of such a function in the class $\overline{MH}_p^n(\alpha)$.

5.1. Theorem. *Let the function defined by (5.1) be in the class $\overline{MH}_p^n(\alpha)$. Then the function $F_n(z)$ defined by*

$$F_n(z) = \frac{c_{-1}}{z} + \sum_{k=1}^{\infty} r_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} s_{p+k} z^{p+k}}$$

is a member of the class $\overline{MH}_p^n(\alpha)$, where

$$c_{-1} = \frac{1}{m} \sum_{j=1}^m a_{-1,j}, \quad r_{p+k} = \frac{1}{m} \sum_{j=1}^m a_{p+k,j} \quad \text{and} \quad s_{p+k} = \frac{1}{m} \sum_{j=1}^m b_{p+k,j}.$$

Proof. Since $f_{n,j}(z) \in \overline{MH}_p^n(\alpha)$, it follows from Theorem 3.3 that

$$\sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha) a_{p+k,j} + (p+k-2)^n (p+k-\alpha) b_{p+k,j}] \leq (1-\alpha) a_{-1,j}.$$

Hence,

$$\begin{aligned} & \sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha) r_{p+k} + (p+k-2)^n (p+k-\alpha) s_{p+k}] \\ &= \frac{1}{m} \sum_{j=1}^m \left[\sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha) a_{p+k,j} + (p+k-2)^n (p+k-\alpha) b_{p+k,j}] \right] \\ &\leq (1-\alpha) \frac{1}{m} \sum_{j=1}^m a_{-1,j} \\ &= (1-\alpha) c_{-1}, \end{aligned}$$

which implies that $F_n(z) \in \overline{MH}_p^n(\alpha)$. □

5.2. Theorem. *The class $\overline{MH}_p^n(\alpha)$ is a convex set.*

Proof. Let the function $f_{n,j}(z)$ ($j = 1, 2$) defined by (5.1) be in the class $\overline{MH}_p^n(\alpha)$. It is sufficient to prove that the function

$$H(z) = \lambda f_{n,1}(z) + (1-\lambda) f_{n,2}(z), \quad 0 \leq \lambda \leq 1,$$

is also in the class $\overline{MH}_p^n(\alpha)$. Since, for $0 \leq \lambda \leq 1$,

$$\begin{aligned} H(z) &= \frac{\lambda a_{-1,1} + (1-\lambda) a_{-1,2}}{z} + \sum_{k=1}^{\infty} [\lambda a_{p+k,1} + (1-\lambda) a_{p+k,2}] z^{p+k} \\ &\quad + (-1)^n \overline{\sum_{k=1}^{\infty} [\lambda b_{p+k,1} + (1-\lambda) b_{p+k,2}] z^{p+k}}, \end{aligned}$$

with the aid of Theorem 3.3, we have

$$\begin{aligned}
& \sum_{k=1}^{\infty} \{ (p+k+2)^n (p+k+\alpha) [\lambda a_{p+k,1} + (1-\lambda) a_{p+k,2}] \\
& \quad + (p+k-2)^n (p+k-\alpha) [\lambda b_{p+k,1} + (1-\lambda) b_{p+k,2}] \} \\
& = \lambda \sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha) a_{p+k,1} + (p+k-2)^n (p+k-\alpha) b_{p+k,1}] \\
& \quad + (1-\lambda) \sum_{k=1}^{\infty} [(p+k+2)^n (p+k+\alpha) a_{p+k,2} + (p+k-2)^n (p+k-\alpha) b_{p+k,2}] \\
& \leq (1-\alpha) \{ \lambda a_{-1,1} + (1-\lambda) a_{-1,2} \}.
\end{aligned}$$

Hence $H(z) \in \overline{MH}_p^n(\alpha)$. This completes the proof of Theorem 5.2. $\square$

5.3. Theorem. Let, for $p \geq 2$ and $z \in U^*$,

$$\begin{aligned}
h_p(z) &= g_p(z) = \frac{a_{-1}}{z}, \\
h_{p+k}(z) &= \frac{a_{-1}}{z} + \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)} z^{p+k}, \text{ and,} \\
g_{p+k}(z) &= \frac{a_{-1}}{z} + \frac{(-1)^n(1-\alpha)a_{-1}}{(p+k-2)^n(p+k-\alpha)} \bar{z}^{p+k}.
\end{aligned}$$

Then $f_n(z) \in \overline{MH}_p^n(\alpha)$ if and only if it can be expressed in the form

$$f_n(z) = \sum_{k=0}^{\infty} [x_{p+k} h_{p+k}(z) + y_{p+k} g_{p+k}(z)],$$

where

$$x_{p+k} \geq 0, y_{p+k} \geq 0, \text{ and } \sum_{k=0}^{\infty} (x_{p+k} + y_{p+k}) = 1.$$

Proof. Let $f_n(z) = \sum_{k=0}^{\infty} [x_{p+k} h_{p+k}(z) + y_{p+k} g_{p+k}(z)]$, with

$$x_{p+k} \geq 0, y_{p+k} \geq 0 \text{ and } \sum_{k=0}^{\infty} (x_{p+k} + y_{p+k}) = 1.$$

Then, we have

$$\begin{aligned}
f_n(z) &= \sum_{k=0}^{\infty} [x_{p+k} h_{p+k}(z) + y_{p+k} g_{p+k}(z)] \\
&= x_p h_p(z) + y_p g_p(z) + \sum_{k=1}^{\infty} (x_{p+k} + y_{p+k}) \frac{a_{-1}}{z} \\
&\quad + \sum_{k=1}^{\infty} \frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)} z^{p+k} + \sum_{k=1}^{\infty} \frac{(-1)^n(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)} \bar{z}^{p+k} \\
&= \sum_{k=0}^{\infty} (x_{p+k} + y_{p+k}) \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} \frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)} z^{p+k} \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^n(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)} \bar{z}^{p+k}.
\end{aligned}$$

Since

$$\begin{aligned} & \sum_{k=1}^{\infty} \left[(p+k+2)^n(p+k+\alpha) \frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)} \right. \\ & \quad \left. + (p+k-2)^n(p+k-\alpha) \frac{(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)} \right] \\ & = (1-\alpha)a_{-1} \sum_{k=1}^{\infty} (x_{p+k} + y_{p+k}) \leq (1-\alpha)a_{-1}, \end{aligned}$$

by Theorem 3.3, $f_n(z) \in \overline{MH}_p^n(\alpha)$.

Conversely, we suppose that $f_n(z) \in \overline{MH}_p^n(\alpha)$ and

$$a_{p+k} = \frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)}, \quad b_{p+k} = \frac{(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)}$$

for $k = 1, 2, \dots$. Hence, we obtain

$$\begin{aligned} f_n(z) &= h(z) + (-1)^n \overline{g(z)} = \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} + (-1)^n \overline{\sum_{k=1}^{\infty} b_{p+k} z^{p+k}} \\ &= \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} \frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)} z^{p+k} \\ & \quad + \sum_{k=1}^{\infty} \frac{(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)} \bar{z}^{p+k} \\ &= \sum_{k=0}^{\infty} (x_{p+k} + y_{p+k}) \frac{a_{-1}}{z} + \sum_{k=1}^{\infty} \frac{(1-\alpha)a_{-1}x_{p+k}}{(p+k+2)^n(p+k+\alpha)} z^{p+k} \\ & \quad + \sum_{k=1}^{\infty} \frac{(-1)^n(1-\alpha)a_{-1}y_{p+k}}{(p+k-2)^n(p+k-\alpha)} \bar{z}^{p+k} \\ &= \frac{a_{-1}}{z} x_p + \frac{a_{-1}}{z} y_p + \sum_{k=1}^{\infty} \left(\frac{a_{-1}}{z} + \frac{(1-\alpha)a_{-1}}{(p+k+2)^n(p+k+\alpha)} \right) x_{p+k} z^{p+k} \\ & \quad + \sum_{k=1}^{\infty} \left(\frac{a_{-1}}{z} + \frac{(-1)^n(1-\alpha)a_{-1}}{(p+k-2)^n(p+k-\alpha)} \right) y_{p+k} \bar{z}^{p+k} \\ &= \sum_{k=0}^{\infty} [x_{p+k} h_{p+k}(z) + y_{p+k} g_{p+k}(z)]. \end{aligned}$$

This completes the proof of Theorem 5.3. □

References

- [1] Aouf, M. K. and Hossen, H. M. *New criteria for meromorphic p -valent starlike functions*, Tsukuba J. Math. **17** (2), 181–186, 1993.
- [2] Bostancı, H., Yalcin, S. and Öztürk, M. *On meromorphically harmonic starlike functions with respect to symmetric conjugate points*, J. Math. Anal. and Appl. **328**, 370–379, 2007.
- [3] Cluine, J. and Sheil-Small, T. *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A1 Math. **9**, 3–25, 1984.
- [4] Darwish, H. E. *Meromorphic p -valent starlike functions with negative coefficients*, Indian J. Pure Appl. Math. **33** (7), 967–976, 2002.
- [5] Hengartner, W. and Schober, G. *Univalent harmonic functions*, Trans. Amer. Math. Soc. **299**, 1–31, 1987.

- [6] Jahangiri, J. M. *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Mariae Currie-Sklodowska, Sec. A, **52**, 57–66, 1998.
- [7] Jahangiri, J. M. *Harmonic meromorphic starlike functions*, Bull. Korean Math. Soc. **37** (2), 291–301, 2000.
- [8] Joshi, S. B. and Sangle, N. D. *Meromorphic starlike functions with negative and missing coefficients*, Far East J. Math. Sci. (FMJS) **26** (2), 289–301, 2007.
- [9] Murugusundaramoorthy, G. *Starlikeness of multivalent meromorphic harmonic functions*, Bull. Korean Math. Soc. **40** (4), 553–564, 2003.
- [10] Murugusundaramoorthy, G. *Harmonic meromorphic convex functions with missing coefficients*, J. Indones. Math. Soc. (MIHMI) **10** (1), 15–22, 2004.
- [11] Jahangiri, J. M. and Silverman, H. *Meromorphic univalent harmonic functions with negative coefficients*, Bull. Korean Math. Soc. **36**, 763–770, 1999.
- [12] Uraleggaddi, B. A. and Somanatha, C. *New criteria for meromorphic starlike univalent functions*, Bull. Austral. Math. Soc. **43**, 137–140, 1991.