PAPER DETAILS

TITLE: Reconstruction of the nonlocal Sturm-Liouville operator with boundary conditions depending

on the parameter

AUTHORS: Ibrahim Adalar, Ahmet Sinan Özkan

PAGES: 314-320

ORIGINAL PDF URL: https://dergipark.org.tr/tr/download/article-file/2927927



RESEARCH ARTICLE

Reconstruction of the nonlocal Sturm-Liouville operator with boundary conditions depending on the parameter

Ibrahim Adalar^{*1}, A.Sinan Ozkan²

¹Zara Veysel Dursun Colleges of Applied Sciences, Sivas Cumhuriyet University Zara/Sivas, Turkey ²Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University 58140 Sivas, Turkey

Abstract

In the present paper, we consider the Sturm–Liouville equation with nonlocal boundary conditions depending polynomially on the parameter. We obtain a result and give an algorithm for the reconstruction of the coefficients of the problem using asymptotics of the nodal points.

Mathematics Subject Classification (2020). 34L05, 34B24, 34B10, 34A55

Keywords. nodal points, Sturm-Liouville equation, nonlocal boundary conditions, inverse problem

1. Introduction

One of the solution methods for the inverse problems of the Sturm-Liouville operators is to use the zeros of the eigenfunctions. These zeros are also called nodal points. Trying to reconstruct the coefficients of the operator from the asymptotic formula of the nodal points is known as inverse nodal problem. This problem for Sturm-Liouville operator was first investigated by McLaughlin in [20]. She succeeded in giving a uniqueness theorem for this type of inverse problems with Dirichlet boundary conditions. Some further numerical calculations for reconstruction of potential are given in [12]. In 1997, Yang [35] obtained a definite algorithm for the solution of inverse nodal problems with separated boundary conditions. Later, similar results for various boundary conditions were obtained in (see [1,4-7,10,11,14-19,21,28,30-32,36-38] and references therein). On the other hand, it can be said that the inverse nodal problem for nonlocal boundary conditions is a relatively new topic. Indeed, there exist only a few studies with these boundary conditions [8, 13, 26,27,33,34].

Nonlocal boundary conditions first appeared in Bitsadze and Samarskii's paper which includes some results on elliptic equations [3]. These conditions, which cannot be measured exactly at the boundary, have various applications in fields such as biology and physics (see [9,23]). Various spectral results for differential operators with boundary conditions of this type are obtained in [2,22,24,29].

^{*}Corresponding Author.

Email addresses: iadalar@cumhuriyet.edu.tr (I. Adalar), sozkan@cumhuriyet.edu.tr (A. S. Ozkan) Received: 31.01.2023; Accepted: 19.04.2023

In this study, we deal with the following boundary value problem $L = L(q, \alpha_i, \beta_i)$

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in (0, 1)$$
(1.1)

$$B_1(y) = a(\lambda)y'(0) + b(\lambda)y(0) - \gamma_0(\lambda)y(\xi_0) = 0, \qquad (1.2)$$

$$B_2(y) = c(\lambda)y'(1) + d(\lambda)y(1) - \gamma_1(\lambda)y(\xi_1) = 0, \qquad (1.3)$$

where $\xi_i \in \mathbb{Q} \cap (0, 1)$ for i = 0, 1 and λ is the spectral parameter. q(x) is a real valued continuously differentiable function; $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ and $d(\lambda)$ are monic polynomials such that

$$a(\lambda) = \sum_{i=0}^{\kappa} a_i \lambda^i, \ b(\lambda) = \sum_{i=0}^{\kappa} b_i \lambda^i,$$
$$c(\lambda) = \sum_{i=0}^{r} c_i \lambda^i, \ d(\lambda) = \sum_{i=0}^{r} d_i \lambda^i.$$

Also,

$$\gamma_0(\lambda) = \sum_{i=0}^k \alpha_i \lambda^i, \ \gamma_1(\lambda) = \sum_{i=0}^r \beta_i \lambda^i.$$

The main purpose of the present paper is to solve inverse nodal problem for L. We obtain q(x) which is the potential of operator L from asymptotics of the nodal points and give an algorithm for the reconstruction of coefficients α_k and β_r . Consequently, our main result is a kind of generalization of the first result in [25] in which the polynomials in the boundary conditions are constant. Moreover, it covers a wide class of nonlocal boundary conditions.

2. Main results

In this section, we will first give the asymptotics of the nodal points. Then with the help of this result we will obtain a uniqueness theorem and an algorithm. Let the eigenvalues of the problem (1.1)-(1.3) be $\{\lambda_n\}_{n\geq 0}$ and the eigenfunctions corresponding to these eigenvalues be $\{\varphi(x, \lambda_n)\}_{n\geq 0}$. The following theorems are the main results in this article.

Theorem 2.1. $\varphi(x, \lambda_n)$ has n - k - r nodal points in (0, 1) for sufficiently large n, namely x_n^j , j = 0, 1, 2, ..., n - k - r - 1, and the following asymptotic formula is valid

$$\begin{aligned} x_n^j &= \frac{j+1/2}{(n-k-r)} - (-1)^{n-m-r} \frac{\left[\beta_r \cos((n-k-r)\pi\xi_1) - \alpha_k \cos((n-k-r)\pi(1-\xi_0))\right]}{(n-k-r)^2 \pi^2} \frac{(j+1/2)}{(n-k-r)} + \\ &+ \frac{\left(Q(x_n^j) - 1\right)}{(n-k-r)^2 \pi^2} + \frac{\alpha_k \cos((n-k-r)\pi\xi_0)}{(n-k-r)^2 \pi^2} + o\left(\frac{1}{n^2}\right). \end{aligned}$$

where $Q(x) = \frac{1}{2} \int_0^x q(s) ds$.

Proof. Let $C(x,\lambda)$ and $S(x,\lambda)$ be the solutions of (1.1) under the initial conditions

$$S(0,\lambda) = 0, S'(0,\lambda) = 1$$

 $C(0,\lambda) = 1, C'(0,\lambda) = 0$

respectively. From [13] and [39], the functions $C(x, \lambda)$ and $S(x, \lambda)$ satisfy the following asymptotic relations for $|\lambda| \to \infty$,

$$C(x,\lambda) = \cos\sqrt{\lambda}x + \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}Q(x) + \frac{\cos\sqrt{\lambda}x}{\lambda}q_1(x) + O\left(\frac{1}{\lambda^{3/2}}\exp|\tau|x\right),$$
$$S(x,\lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{\cos\sqrt{\lambda}x}{\lambda}Q(x) + O\left(\frac{1}{\lambda^{3/2}}\exp|\tau|x\right),$$

where $q_1(x) = \frac{q(x) - q(0)}{4} - \frac{1}{8} \left(\int_0^x q(s) ds \right)^2$ and $\tau = \left| \text{Im} \sqrt{\lambda} \right|$.

The characteristic function of L is

$$\Delta(\lambda) = \det \begin{pmatrix} B_1(C) & B_1(S) \\ B_2(C) & B_2(S) \end{pmatrix}.$$
(2.1)

Since $\Delta(\lambda)$ is an entire function, L has a discrete spectrum. Also, the eigenvalues of L are the zeros of the function $\Delta(\lambda)$. From (2.1), we have that

$$\Delta(\lambda) = (b(\lambda) - \gamma_0(\lambda)C(\xi_0)) (c(\lambda)S'(1) + d(\lambda)S(1) - \gamma_1(\lambda)S(\xi_1))$$

- $(a(\lambda) - \gamma_0(\lambda)S(\xi_0)) (c(\lambda)C'(1) + d(\lambda)C(1) - \gamma_1(\lambda)C(\xi_1)).$ (2.2)

Using the asymptotics of $C(x, \lambda)$ and $S(x, \lambda)$ in (2.2), we find the following asymptotic expression for $\Delta(\lambda)$ as $\lambda \to \infty$:

$$\begin{aligned} \Delta(\lambda) &= a(\lambda)c(\lambda)\sqrt{\lambda}\sin\sqrt{\lambda} - c(\lambda)\gamma_0(\lambda)\sin(\sqrt{\lambda}\xi_0)\sin\sqrt{\lambda} - a(\lambda)d(\lambda)\cos\sqrt{\lambda} \\ &+ b(\lambda)c(\lambda)\cos\sqrt{\lambda} - c(\lambda)\gamma_0(\lambda)\cos(\sqrt{\lambda}\xi_0)\cos\sqrt{\lambda} + a(\lambda)\gamma_1(\lambda)\cos(\sqrt{\lambda}\xi_1) \\ &+ d(\lambda)\gamma_0(\lambda)\sin(\sqrt{\lambda}\xi_0)\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}} - a(\lambda)\gamma_1(\lambda)\sin(\sqrt{\lambda}\xi_0)\frac{\cos(\sqrt{\lambda}\xi_1)}{\sqrt{\lambda}} \\ &+ b(\lambda)d(\lambda)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - d(\lambda)\gamma_0(\lambda)\cos(\sqrt{\lambda}\xi_0)\frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} - b(\lambda)\gamma_1(\lambda)\frac{\sin(\sqrt{\lambda}\xi_1)}{\sqrt{\lambda}} \\ &+ \gamma_0(\lambda)\gamma_1(\lambda)\frac{\cos(\sqrt{\lambda}\xi_0)\sin(\sqrt{\lambda}\xi_1)}{\sqrt{\lambda}} + O(\frac{\lambda^{k+r}}{\sqrt{\lambda}}\exp|\tau|) \end{aligned}$$

and so

$$\Delta(\lambda) = \lambda^{k+r} \left[\sqrt{\lambda} \sin \sqrt{\lambda} + \beta_r \cos(\sqrt{\lambda}\xi_1) - \alpha_k \cos \sqrt{\lambda}(1-\xi_0) + o(\exp|\tau|) \right].$$
(2.3)

Let $G_n(\varepsilon) = \left\{ \sqrt{\lambda} : \left| \sqrt{\lambda} - (n-k-r)\pi \right| < \varepsilon \right\}$ for n = 1, 2, It follows from (2.3) that there exist some $M(\varepsilon) > 0$ such that $|\Delta(\lambda)| \ge M(\varepsilon) \left| \sqrt{\lambda} \right| \exp |\tau|$ for sufficiently large $\left| \sqrt{\lambda} \right|$ in $G_n(\varepsilon)$. For sufficiently large n, we can see that λ_n must be real number.

It can be shown using classical methods in [25] that the sequence $\{\lambda_n\}_{n\geq 0}$ satisfies the following asymptotic expression for $n \to \infty$:

$$\sqrt{\lambda_n} = (n-k-r)\pi
-(-1)^{n-k-r} \frac{[\beta_r \cos((n-k-r)\pi\xi_1) - \alpha_k \cos((n-k-r)\pi(1-\xi_0))]}{(n-k-r)\pi} + o(\frac{1}{n}).$$
(2.4)

Let $\varphi(x,\lambda)$ be the solution of equation (1.1), satisfying the initial conditions $\varphi(0,\lambda) = a(\lambda) - \gamma_0(\lambda)S(\xi_0), \ \varphi'(0,\lambda) = \gamma_0(\lambda)C(\xi_0) - b(\lambda).$ Thus, we have that

Thus, we have that

$$\varphi(x,\lambda_n) = C(x,\lambda_n)U(S(x,\lambda_n)) - S(x,\lambda_n)U(C(x,\lambda_n)).$$
(2.5)

From (2.4) and (2.5), we can see easily the following asymptotic formula

$$\varphi(x,\lambda_n) = \lambda_n^k \left(\cos\sqrt{\lambda_n}x + \frac{\sin\sqrt{\lambda_n}x}{\sqrt{\lambda_n}} \left(Q(x) - 1\right) + \frac{\alpha_k}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n} \left(x - \xi_0\right) + o\left(\frac{1}{\sqrt{\lambda_n}}\right) \right), \quad (2.6)$$

is valid for sufficiently large n. We get from (2.6) that $\varphi(x, \lambda_n)$ has exactly n - k - r zeros in (0, 1).

Use the asymptotic formula (2.6) to get

$$0 = \varphi(x_n^j, \lambda_n)$$
$$= \lambda_n^m \left(\cos \sqrt{\lambda_n} x_n^j + \frac{\sin \sqrt{\lambda_n} x_n^j}{\sqrt{\lambda_n}} \left(Q(x_n^j) - 1 \right) + \frac{\alpha_k}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} \left(x_n^j - \xi_0 \right) + o\left(\frac{1}{\sqrt{\lambda_n}} \right) \right)$$

and so

$$\tan\left(\sqrt{\lambda_n}x_n^j - \frac{\pi}{2}\right) = \frac{\left(Q(x_n^j) - 1\right)}{\sqrt{\lambda_n}} + \frac{\alpha_k}{\sqrt{\lambda_n}}\frac{\sin\sqrt{\lambda_n}\left(x_n^j - \xi_0\right)}{\sin\sqrt{\lambda_n}x_n^j} + o\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

This yields

$$x_n^j = \frac{(j+1/2)\pi}{\sqrt{\lambda_n}} + \frac{(Q(x_n^j)-1)}{\lambda_n} + \frac{\alpha_k}{\lambda_n} \frac{\sin\sqrt{\lambda_n} \left(x_n^j - \xi_0\right)}{\sin\sqrt{\lambda_n} x_n^j} + o\left(\frac{1}{\lambda_n}\right).$$
(2.7)

Using $\sqrt{\lambda_n} x_n^j = (j + 1/2) \pi + O(\frac{1}{n}), n \to \infty$ we can show

$$\frac{\sin\sqrt{\lambda_n}\left(x_n^j - \xi_0\right)}{\lambda_n \sin\sqrt{\lambda_n} x_n^j} = \frac{\cos\left((n-k-r)\pi\xi_0\right)}{(n-k-r)^2\pi^2} + o\left(\frac{1}{n^2}\right).$$
(2.8)

In addition, we obtain

$$\frac{1}{\sqrt{\lambda_n}} = \frac{1}{(n-k-r)\pi}$$
(2.9)

$$\times \left(1 + (-1)^{n-k-r} \frac{\left[\beta_r \cos\left((n-k-r)\pi\xi_1\right) - \alpha_k \cos\left((n-k-r)\pi(1-\xi_0)\right)\right]}{(n-k-r)^2 \pi^2} + o\left(\frac{1}{n^3}\right)\right)$$

$$\frac{1}{\lambda_n} = \frac{1}{(n-k-r)^2 \pi^2} + o\left(\frac{1}{n^3}\right)$$
(2.10)

using by (2.4).

Substituting (2.8), (2.9) and (2.10) in (2.7), it is concluded that

$$\begin{split} x_n^j &= \frac{j+1/2}{(n-k-r)} \\ &-(-1)^{n-k-r} \frac{(-1)^{n-k-r} \left[\beta_r \cos\left((n-k-r)\pi\xi_1\right) - \alpha_k \cos\left((n-k-r)\pi(1-\xi_0)\right)\right]}{(n-k-r)^2 \pi^2} \\ &\times \frac{(j+1/2)}{(n-k-r)} \\ &+ \frac{(Q(x_n^j)-1)}{(n-k-r)^2 \pi^2} + \frac{\alpha_k \cos\left((n-k-r)\pi\xi_0\right)}{(n-k-r)^2 \pi^2} + o\left(\frac{1}{n^2}\right). \end{split}$$

Let X_0 be a subsequence of the numbers x_n^j that is dense on (0, 1). According to above result, the existence of such a set is obvious.

Consider the problem $\tilde{L} = L\left(\tilde{q}, \tilde{\alpha}_i, \tilde{\beta}_i\right)$ under the same assumptions with L. Without loss of generality, we assume that $\int_0^1 (q(x) - \tilde{q}(x)) dx = 0$.

Theorem 2.2. If $X_0 = \widetilde{X}_0$ then $\alpha_k = \widetilde{\alpha}_k$, $\beta_r = \widetilde{\beta}_r$ and $q(x) = \widetilde{q}(x)$ a.e. in (0,1). Thus, the coefficients α_k , β_r and the potential q(x) are uniquely determined by X_0 .

Proof. Put $\xi_0 = \frac{p_0}{r_0}$ and $\xi_1 = \frac{p_1}{r_1}$, where $p_i, r_i \in \mathbb{Z}$ for i = 0, 1. For each fixed $x \in [0, 1]$, there exists a sequence (x_n^j) converges to x. For $n_s = 2sr_0r_1 + k + r$, $s \in \mathbb{Z}$, the

subsequence (x_n^j) converges also to x. Therefore we get from the asymptotic in Theorem 2.1 the following limit is finite and given equality holds:

$$\lim_{s \to \infty} (n - k - r)^2 \pi^2 \left(x_n^j - \frac{j + 1/2}{(n - k - r)} \right) = g(x) = (\alpha_k - \beta_r) x + Q(x) - 1 + \alpha_k,$$
(2.11)

Direct calculations in (2.11) yield

$$q(x) = 2\left(g'(x) - g(1) + g(0)\right) \tag{2.12}$$

$$\alpha_k = g(0) + 1,$$

 $\beta_r = 2g(0) - g(1) + 1$
(2.13)

Since $X_0 = \widetilde{X}_0$ then $g(x) = \widetilde{g}(x)$ and so $q(x) = \widetilde{q}(x)$, a.e. in (0, 1).

3. Algorithm

Let $X_0, \xi_i = \frac{p_i}{r_i}$ for i = 0, 1 be given. Then $q(x), \alpha_k$ and β_r can be reconstructed by the following algorithm:

- i) Denote $n_s = 2sr_0r_1 + k + r, s \in \mathbb{Z};$
- ii) Find q(x) by (2.12);
- iii) Find α_k and β_r by the formulas (2.13).

Example 3.1. We consider the following nonlocal boundary value problem

$$\ell y := -y'' + q(x)y = \lambda y, \quad x \in \Omega = (0, 1)$$
$$a(\lambda)y'(0) + b(\lambda)y(0) - \gamma_0(\lambda)y(\frac{2}{3}) = 0,$$
$$c(\lambda)y'(1) + d(\lambda)y(1) - \gamma_1(\lambda)y(\frac{5}{6}) = 0,$$

where $q(x) \in C^1[0,1]$; $a(\lambda)$, $b(\lambda)$, $c(\lambda)$, $d(\lambda)$, $\gamma_0(\lambda)$ and $\gamma_1(\lambda)$ are unknown coefficients of the problem. For sufficiently large n, let the nodal points provide the following asymptotic

$$\begin{aligned} x_n^j &= \frac{j+1/2}{(n-k-r)} + \\ &-(-1)^{n-k-r} \frac{\left[4\cos\left((n-k-r)\frac{5\pi}{6}\right) - 3\cos\left((n-k-r)\frac{\pi}{3}\right)\right]}{(n-k-r)^{2\pi^2}} \frac{(j+1/2)}{(n-k-r)} + \\ &+ \frac{3\left(\cos\left((n-k-r)\frac{2\pi}{3}\right) - 1/3\right)}{(n-k-r)^{2\pi^2}} - \frac{(j+1/2)}{6(n-k-r)^{3\pi^2}} + \frac{(j+1/2)^3}{6(n-k-r)^{5\pi^2}} + o\left(\frac{1}{n^2}\right). \end{aligned}$$

According to these data, we can calculate q(x), α_k and β_r . Let $n_s = 36s + k + r$, $s \in \mathbb{Z}$. One can calculate that

One can calculate that $\lim_{s \to \infty} (n - k - r)^2 \pi^2 \left(x_n^j - \frac{j + 1/2}{(n - k - r)} \right) = g(x) = -x + 2 - \frac{x}{6} + \frac{x^3}{6}.$ By the formulas (2.12) and (2.13) ;

$$\alpha_k = 3$$

$$\beta_r = 4$$

$$q(x) = x^2 - \frac{1}{3}$$

References

- S. Akbarpoor, H. Koyunbakan and A.Dabbaghian, Solving inverse nodal problem with spectral parameter in boundary conditions, Inv. Prob. Sci. Eng. 27 (12), 1790–1801, 2019.
- [2] S. Albeverio, R.O. Hryniv and L.P. Nizhnik, *Inverse spectral problems for non-local Sturm-Liouville operators*, Inverse Probl. 23 (2), 523, 2007.
- [3] A.V. Bitsadze and A.A. Samarskii, Some elementary generalizations of linear elliptic boundary value problems, Dokl. Akad. Nauk SSSR 185, 739–740, 1969.
- [4] S.A. Buterin and C.T. Shieh, *Inverse nodal problem for differential pencils*, Appl. Math. Lett. 22, 1240–1247, 2009.
- [5] Y.H. Cheng, C.K. Law and J. Tsay, *Remarks on a new inverse nodal problem*, J. Math. Anal. Appl. 248, 145–155, 2000.
- [6] Y.H. Cheng, Reconstruction of the Sturm-Liouville operator and a p-star graph with nodal data, Rocky Mt. J. Math. 42 (5), 1431–1446, 2012.
- [7] S. Currie and B.A. Watson, Inverse nodal problems for Sturm-Liouville equations on graphs, Inverse Problems, 23, 2029–2040, 2007.
- [8] Y. Çakmak and B. Keskin, Inverse nodal problem for the quadratic pencil of the Sturm-Liouville equations with parameter-dependent nonlocal boundary condition, Turkish J. Math. 47 (1), Article 26. 2023.
- [9] N. Gordeziani, On some non-local problems of the theory of elasticity, Bulletin of TICMI, 4, 43-46, 2000.
- [10] N. J. Guliyev, Inverse square singularities and eigenparameter-dependent boundary conditions are two sides of the same coin, Q. J. Math. haad004, 2023.
- [11] Y. Guo and G. Wei, *Inverse problems: dense nodal subset on an interior subinterval*, J. Differ. Equ. 255 (7), 2002–2017, 2013.
- [12] O.H. Hald and L.R. McLaughlin, Solutions of inverse nodal problems, Inv. Probl. 5, 307–347, 1989.
- [13] Y.T. Hu, C.F. Yang and X.C. Xu, Inverse nodal problems for the Sturm-Liouville operator with nonlocal integral conditions, J. Inv. Ill-Posed Probl. 25 (6), 799–806, 2017.
- [14] B. Keskin and A.S. Ozkan, Inverse nodal problems for impulsive Sturm-Liouville equation with boundary conditions depending on the parameter, Adv. Anal. 2 (3), 151-156, 2017.
- [15] H. Koyunbakan and S. Mosazadeh, Inverse nodal problem for discontinuous Sturm-Liouville operator by new Prüfer Substitutions, Math. Sci., 1–8, 2021.
- [16] H. Koyunbakan and E. Yilmaz, Reconstruction of the potential function and its derivatives for the diffusion operator, Z. Naturforschung A, 63 (3-4), 127–130, 2008.
- [17] C.K. Law and C.F. Yang, Reconstructing the potential function and its derivatives using nodal data, Inverse Probl. 14 299–312, 1998.
- [18] L.I. Mammadova and I.M. Nabiev, Uniqueness of recovery of the Sturm-Liouville operator with a spectral parameter quadratically entering the boundary condition, Vestn. Tomsk. Gos. Univ. Mat. Mech. (79), 14–24, 2022.
- [19] L.I. Mammadova, I.M. Nabiev and Ch. H. Rzayeva, Uniqueness of the solution of the inverse problem for differential operator with semiseparated boundary conditions, Baku Math. J. 1 (1), 47–52, 2022.
- [20] J.R. McLaughlin, Inverse spectral theory using nodal points as data- a uniqueness result, J. Differ. Equ. 73, 354–362, 1988.
- [21] S. Mosazadeh, The uniqueness theorem for inverse nodal problems with a chemical potential, Iran. J. Math. Chem. 8 (4), 403–411, 2017.
- [22] I.M. Nabiev, Reconstruction of the Differential Operator with Spectral Parameter in the Boundary Condition, Mediterr. J. Math. 19, 124, 2022.

- [23] A.M. Nakhushev, Equations of Mathematical Biology, Moscow: Vysshaya Shkola, 1995. (in Russian)
- [24] L. Nizhnik, Inverse nonlocal Sturm-Liouville problem, Inverse Probl. 26 (12), 125006, 2010.
- [25] A.S. Ozkan and İ. Adalar, Inverse nodal problems for Sturm-Liouville equation with nonlocal boundary conditions, J. Math. Anal. Appl. 520 (1), 126907, 2023.
- [26] A.S. Ozkan and I. Adalar, Inverse nodal problem for Dirac operator with integral type nonlocal boundary conditions, Math. Meth. Appl. Sci. 46 (1), 986–993, 2023.
- [27] A.S. Ozkan and B. Keskin, Inverse nodal problems for Sturm-Liouville equation with eigenparameter-dependent boundary and jump conditions, Inv. Probl. Sci. Eng. 23 (8), 1306–1312, 2015.
- [28] C.T. Shieh and V.A. Yurko, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, J. Math. Anal. Appl. 347 266–272, 2008.
- [29] F. Sun, K. Li and J. Cai, Bounds on the non-real eigenvalues of nonlocal indefinite Sturm-Liouville problems with coupled boundary conditions, Complex Anal. Oper. Theory, 16 (30), 2022.
- [30] Y.P. Wang and C.T. Shieh, Inverse problems for Sturm-Liouville operators on a compact equilateral graph by partial nodal data, Math. Models Meth. Appl. Sci. 44 (1), 693-704, 2021.
- [31] Y.P. Wang, E. Yılmaz and S. Akbarpoor, The numerical solution of inverse nodal problem for integro-differential operator by Legendre wavelet method, Int. J. Comput. Math. 100 (1), 219–232, 2023.
- [32] Y.P. Wang and V.A. Yurko, On the inverse nodal problems for discontinuous Sturm-Liouville operators, J. Differ. Equ. 260 (5), 4086–4109, 2016.
- [33] X.J. Xu and C.F. Yang, Inverse nodal problem for nonlocal differential operators, Tamkang J. Math. 50 (3), 337–347, 2019.
- [34] C.F. Yang, Inverse nodal problem for a class of nonlocal Sturm-Liouville operator, Math. Model. Anal. 15 (3), 383–392, 2010.
- [35] X.F. Yang, A solution of the nodal problem, Inverse Probl. 13 203–213, 1997.
- [36] X.F. Yang, A new inverse nodal problem, J. Differ. Equ. 169, 633–653, 2001.
- [37] C.F. Yang and X.P. Yang, Inverse nodal problems for the Sturm-Liouville equation with polynomially dependent on the eigenparameter, Inv. Probl. Sci. Eng. 19 (7), 951-961, 2011.
- [38] E. Yılmaz and H. Koyunbakan, Reconstruction of potential function and its derivatives for Sturm-Liouville problem with eigenvalues in boundary condition, Inv. Prob. Sci. Eng. 18 (7), 935-944, 2010.
- [39] V.A. Yurko, Inverse Spectral Problems for Differential Operators and Their Applications, Gordon and Breach, Amsterdam, 2000.