Ikonion Journal of Mathematics 3(2) (2021) 1-8

Ikonion Journal of Mathematics

https://dergipark.org.tr/tr/pub/ikjm

Research Article

Open Access

https://doi.org/10.54286/ikjm.923786

ISSN: 2687-6531

An Investigation on The Behaviour of Unbounded Operators in Γ-Hilbert Space

Sahin Injamamul Islam[1], Nirmal Sarkar[2], Ashoke Das[3]

Keywords:

Γ-Hilbert space, Closed operator, Densely defined operator, Self-adjoint of densely defined operator, Symmetric of densely defined operator.

Abstract — In this paper, we investigate about the behavior of unbounded operators in Γ-Hilbert Space. Here we discussed about the adjoint, self-adjoint, symmetric and other related properties of densely defined operator. We proof some related theorems and corollaries and will show the characterizations of these operators in Γ-Hilbert space.

Subject Classification (2020): 46CXX, 46C05, 46C07,46C15,46C99,47L06.

1. Introduction

Γ-Hilbert space plays an important role in generalization of general linear quadratic control problems in an abstract space [1] which was motivated from the work of L.Debnath and Pitor Mikusinski [8] but there not enough literature found to study about the unbounded operators in Γ-Hilbert space. The definition of Γ-Hilbert space was introduced by Bhattacharya D.K. and T.E. Aman in their paper "Γ-Hilbert space and linear quadratic control problem" in 2003 [9]. Further development was made in 2017 by A.Ghosh, A.Das and T.E. Aman in their research paper [1]. In [6] S.Islam and A.Das discussed about the properties of bounded operators in Γ-Hilbert Space. Boundedness of an operator is a great tool to elaborate Γ-Hilbert Space. We often deal with operators which are not bounded. In this paper, we will briefly discuss the concept, methods and theory of unbounded operators in Γ-Hilbert Space. In this paper, after consulting the main author, we have made some changes in the main definition of Γ-Hilbert space [9].

First, we recall the definitions of Γ-Hilbert Space.

Definition 1.1. Let E be the linear space over the field F and Γ be a semi group with respect to addition. A mapping $\langle .,.,. \rangle : E \times \Gamma \times E \to$ F ($\mathbb{R}$ or $\mathbb{C}$) is called a Γ-Inner product on (E, Γ) if

- (i) $\langle .,.,. \rangle$ is linear in first variable and additive in second variable.
- (ii) $\langle u, \gamma, v \rangle = \langle v, \gamma, u \rangle \ \forall$ u, v ∈ E and γ ∈ Γ.
- (iii) $\langle u, \gamma, u \rangle > 0 \ \forall\, u \neq 0$.
- (iv) $\langle u, \gamma, u \rangle = 0$ if at least one of u, γ is zero.

[1] sahincool92@gmail.com (Corresponding Author); [2] nrmlsrkr@gmail.com; [3] ashoke.avik@gmail.com

[1,2,3] Department of Mathematics, Raiganj University, West Bengal,India

Article History: Received: 20.04.2021 — Accepted: 09.09.2021 — Published: 10.10.2021

$[(E,\Gamma),\langle .,.,.\rangle]$ is called a Γ-inner product space over F.

A complete Γ-inner product space is called Γ-Hilbert space.

Using the Γ-inner product, we may define three types of norm in a Γ-Hilbert space, namely (i) γ-norm (ii) Γ_{inf}- norm and (iii) Γ-norm.

Definition 1.2. Now if we write $\|u\|_\gamma{}^2 = \langle u,\gamma,u\rangle$, for $u \in H$ and $\gamma \in \Gamma$ then $\|u\|_\gamma{}^2$ satisfy all the conditions of norm.

Definition 1.3. If we define $\|\mathrm{u}\|_{\Gamma_{\text{inf}}} = \inf\{\|u\|_\gamma : \gamma \in \Gamma\}$. Clearly Γ_{inf}-norm satisfy all the conditions of the norm for $u \in H$.

Definition 1.4. If we write $\|u\|_\Gamma = \{\|u\|_\gamma : \gamma \in \Gamma\}$ then this norm is called the Γ-norm of the Γ-Hilbert space.

Definition 1.5. Let L be a non-empty subset of a Γ-Hilbert space H_Γ . Two elements x and y are said to be γ-orthogonal if their inner product $\langle x,\gamma,y\rangle = 0$. In symbol, we write $x \perp_\gamma y$.

2. Basic Concepts

In this section, we briefly discuss about the definition of densely defined operator and the adjoint, self-adjoint , symmetric etc of that operator. Also, related examples and theorem are mentioned in this part.

2.1. Extension of operators

Let S and T be two operators in a vector space E. $\mathrm{D_S}$ and $\mathrm{D_T}$ are the domains of S and T respectively. If

$$\mathrm{D_S} \subset \mathrm{D_T} \quad \text{and} \quad \mathrm{Sx} = \mathrm{Tx} \qquad \text{for every } \mathrm{x} \in D_S$$

then T is called an extension of S and we write $\mathrm{S} \subset \mathrm{T}$.

2.2. Densely defined operator

An operator T defined a linear map T from a subspace of H_Γ to H_Γ is called an operator in H_Γ and the subspace denoted by $\mathrm{D_T}$, is called the domain of T. Now an operator T is defined in a normed space E is called densely defined if its domain $\mathrm{D_T}$ is a dense subset of E , that is $\mathrm{cl}\ \mathrm{D_T} = \mathrm{E}$.

Example 2.2.1. The differential operator $\frac{d}{dx}$ is densely defined in $L^2(\mathbb{R})$, because the subspace of differentiable functions is dense in $L(\mathbb{R})^2$.

Theorem 2.2.2. Let T be a densely defined operator in a Γ-Hilbert space H_Γ and let E be the set of all $y \in \mathrm{H}_\Gamma$ for which $\langle Tx,\gamma,x\rangle$ where $\gamma \in \Gamma$ is a continuous functional on $\mathrm{D_T}$. There exists a unique operator S defined on E such that

$$\langle Tx,\gamma,x\rangle = \langle x,\gamma,Sy\rangle \text{ for all } x \in \mathrm{D_T} \text{ and } \mathrm{y} \in \mathrm{E}\,.$$

Proof: For any $\mathrm{y} \in \mathrm{E}$, consider the functional $f_y(x) = \langle Tx,\gamma,x\rangle$ where $\gamma \in \Gamma$. Being continuous on a dense subspace of H_Γ , has a unique extension to a continuous functional $\tilde{f}_y$ on H_Γ.

By Riesz representation theorem, there exists a unique $Z_y \in \mathrm{H}_\Gamma$ such that $\tilde{f}_y(x) = \langle x, \gamma, Z_y \rangle \;\forall\, x \in \mathrm{H}_\Gamma$. Now if we define $S(y) = Z_y$, then we will have

$$\langle Tx, \gamma, x \rangle = f_y(x) = \tilde{f}_y(x)$$
$$= \langle x, \gamma, Z_y \rangle$$
$$= \langle x, \gamma, Sy \rangle \text{ for all } x \in \mathrm{D_T} \text{ , y} \in \mathrm{E} \text{ and } \gamma \in \Gamma .$$

Also the linearity of S is obvious.

2.3. Adjoint of densely defined operator

Let T be an operator which is densely defined in a Γ-Hilbert space H_Γ. The adjoint T^* of T is the operator defined on the set of all $y \in \mathrm{H}_\Gamma$ for which $\langle Tx, \gamma, x \rangle$ where $\gamma \in \Gamma$ is a continuous function on $\mathrm{D_T}$ and such that

$$\langle Tx, \gamma, x \rangle = \langle x, \gamma, T^*y \rangle \text{ for all } x \in \mathrm{D_T} \text{ and } y \in D_{T^*}$$

Example 2.3.1. Let ${C^1}_0(\mathbb{R})$ denote the space of all continuously differentiable functions on $\mathbb{R}$. This is also a dense subspace of $L^2(\mathbb{R})$. Now consider the differentiable operator D which defined on ${C^1}_0(\mathbb{R})$. Since

$$\langle Dx, \gamma, y \rangle = \int_{-\infty}^{\infty} \left(\frac{d}{dt} x(t)\right) \gamma\, \overline{y(t)}\; dt$$
$$= -\int_{-\infty}^{\infty} x(t) \left(\frac{d}{dt}\, \overline{y(t)}\right) \gamma\, dt \qquad \text{for all } \gamma \in \Gamma .$$

$\therefore \langle Dx, \gamma, y \rangle$ is a continuous functional on ${C^1}_0(\mathbb{R})$.

Moreover,

$$\langle Dx, \gamma, y \rangle = -\int_{-\infty}^{\infty} x(t) \left(\frac{d}{dt}\, \overline{y(t)}\right) \gamma\, dt .$$
$$= \int_{-\infty}^{\infty} x(t)\; \overline{\left(-\frac{d}{dt} (y(t))\right)}\, \gamma\, dt .$$

Here it is not correct to write $D^* = -D$, since the domain of D^* is not ${C^1}_0(\mathbb{R})$.

2.4. Self –adjoint of densely defined operator

Let T be a densely defined operator in a Γ-Hilbert space H_Γ. Then T is called self-adjoint if $T = T^*$.

Note. $T = T^*$ implies that $D_{T^*} = D_T$ and $T(x) = T^*(x)$ for all $x \in D_T$. If T is a densely defined operator in H_Γ which is bounded then T has a unique extension to a bounded operator in H_Γ. Then the domain of T as well as its adjoint T^*, is the whole space H_Γ . If T is unbounded operators ,then T has an adjoint T^* such that $T(x) = T^*(x)$ whenever $x \in D_T \cap D_{T^*}$, but $D_{T^*} \neq D_T$ and thus T is not self-adjoint.

2.5. Symmetric Operator

We now consider a special kind of operator in Γ-Hilbert space . An operator T which is densely defined in Γ-Hilbert space H_Γ is called symmetric if for all $x, y \in D_T$, we have

$$\langle Tx, \gamma, y \rangle = \langle x, \gamma, Ty \rangle \text{ for all } \gamma \in \Gamma .$$

It is clear that if T is symmetric , then $\langle T(x), \gamma, x \rangle \in \mathbb{R}$ for every $x \in D_T$ and $\gamma \in \Gamma$. Also, it follows that a densely defined operator T is symmetric if and only if T^* extends T. If T is symmetric and $\mathrm{D_T} = \mathrm{H}_\Gamma$, then T is in fact a bounded operator on H_Γ. This leads as follows,

Let $E = \{\mathrm{T(x)} : \mathrm{x} \in \mathrm{H}_\Gamma , \|\mathrm{x}\|_\gamma \leq 1\}$. Then for a fixed $y \in \mathrm{H}_\Gamma$ and $\gamma \in \Gamma$, we have

$$|\langle T(x), \gamma, y \rangle| = |\langle x, \gamma, T(y) \rangle|$$
$$\leq \|x\|\, \|\gamma\| \|T(y)\|$$
$$\leq \|T(y)\| \text{ for all } x \in \mathrm{H}_\Gamma \text{ with } \|x\|, \|\gamma\| \leq 1 .$$

Also clearly every self-adjoint operator is symmetric.

Example 2.5.1. Suppose we consider an operator $A = \frac{id}{dt}$ with the domain $D_A = \{ f \in L^2([a,b]) : f'$ is continuous and $f(a) = f(b) = 0 \}$.
Now, since for all $\gamma \in \Gamma$, we have

$$\begin{aligned} \langle Af, \gamma, g \rangle &= \int_a^b if'(t)\,\gamma\,\overline{g(t)}\,dt \\ &= \int_a^b f(t)\,\gamma\,\overline{ig'(t)}\,dt \\ &= \langle f, \gamma, Ag \rangle \end{aligned}$$

$$\therefore \ \langle Af, \gamma, g \rangle = \langle f, \gamma, Ag \rangle$$

for all $f, g \in D_A$, A is symmetric.

$\langle Af, \gamma, g \rangle$ is a continuous functional on D_A for any function g continuously differentiable , no need to satisfying $g(a) = g(b)$.

Consequently , $D_{A^*} \neq D_A$ and A is not self-adjoint.

2.6. Closed Operator

A linear operator $T : E_1 \to E_2$ is said to be closed when the graph $G(T) = \{\langle x, \gamma, Tx \rangle : x \in D_T \text{ and } \gamma \in \Gamma\}$ is a closed subspace of $E_1 \times E_2$ that is

$$x_n \in D_T, x_n \to x \text{ and } Tx_n \to y$$

implies $x \in D_T$ and $Tx = y$.

3. Main Results

Theorem 3.1. Let A and B be densely defined operators in a Γ-Hilbert space H_Γ.

(a) If $A \subset B$, then $B^* \subset A^*$.

(b) If D_{B^*} is dense in H_Γ, then $B \subset B^{**}$.

Proof. (a) Let us consider $y \in D_{B^*}$ and $\gamma \in \Gamma$. Then as a function of x, $\langle Bx, \gamma, y \rangle$ is a continuous functional on D_B. Also $\langle Bx, \gamma, y \rangle$ is a continuous functional on D_A since $D_A \subset D_B$.

Now, $Bx = Ax$ for $x \in D_A$, so $\langle Ax, \gamma, y \rangle$ is a continuous functional on D_A. This proves that $y \in D_{A^*}$. Then the equality $A^*y = B^*y$ for $y \in D_{B^*}$ follows from the uniqueness of the adjoint operator.

(b) Let $x \in D_B$. Then for every $y \in D_{B^*}$ and $\gamma \in \Gamma$,we have

$$\langle Bx, \gamma, y \rangle = \langle x, \gamma, B^*y \rangle$$

It can be rewrite as

$$\langle B^*y, \gamma, x \rangle = \langle y, \gamma, Bx \rangle .$$

Since D_{B^*} is dense in H_Γ, B^{**} exists and we have

$\langle B^*y, \gamma, x \rangle = \langle y, \gamma, B^{**}x \rangle$ for all $y \in D_{B^*}$, $x \in D_{B^{**}}$ and $\gamma \in \Gamma$.

Now, by the proof of (a), we can show that $D_B \subset D_{B^{**}}$ and $B(x) = B^{**}(x)$ for any $x \in D_B$. Thus $B \subset B^{**}$.

Theorem 3.2. If T is a one-to-one operator in a Γ-Hilbert space and both T and its inverse T^{-1} are densely defined , then T^* is also one- to-one and $(T^*)^{-1} = (T^{-1})^*$.

Proof. Let $y \in D_{T^*}$. Then for every $x \in D_{T^{-1}}$ and $\gamma \in \Gamma$, we have $T^{-1}x \in D_T$ and hence

$$\begin{aligned}\langle T^{-1}x, \gamma, T^*x\rangle &= \langle TT^{-1}x, \gamma, y\rangle \\ &= \langle x, \gamma, y\rangle .\end{aligned}$$

This follows that $T^*y \in D_{(T^{-1})^*}$.

And also,

$$(T^{-1})^* \, T^*y = (T\,T^{-1})^* \, y = y \tag{3.1}$$

Now we take an arbitrary $y \in D_{(T^{-1})^*}$ .Then for each $x \in D_T$ and $\gamma \in \Gamma$, we have

$$Tx \in D_{T^{-1}} .$$

Hence

$$\langle Tx, \gamma, (T^{-1})^*y\rangle = \langle T^{-1}Tx, \gamma, y\rangle = \langle x, y\rangle \tag{3.2}$$

This shows that $(T^{-1})^*y \in D_{T^*}$. And $T^*(T^{-1})^*y = (T^{-1}T)^*y = y$. Now, from (3.1) and (3.2) it follows that $(T^*)^{-1} = (T^{-1})^*$.

Theorem 3.3. If A, B and AB are densely defined operators in H_Γ , then $B^* \, A^* = (AB)^*$.

Proof. Let $x \in D_{AB}$ and $y \in D_{B^*A^*}$. Since $x \in D_B$ and $A^*y \in D_{B^*}$, it follows that

$$\langle Bx, \gamma, A^*y\rangle = \langle x, \gamma, B^*A^*y\rangle \text{ for all } \gamma \in \Gamma .$$

On the other side, since $Bx \in D_A$ and $y \in D_{A^*}$, we have

$$\langle ABx, \gamma, y\rangle = \langle Bx, \gamma, A^*y\rangle \text{ for all } \gamma \in \Gamma .$$

Hence

$$\langle ABx, \gamma, y\rangle = \langle x, \gamma, B^*A^*y\rangle .$$

Since this holds for all $x \in D_{AB}$,we have $y \in D_{(AB)^*}$ and $(B^* \, A^*)y = (AB)^*y$. This implies, $B^* \, A^* = (AB)^*$.

Theorem 3.4. A densely defined operator T in a Γ-Hilbert space H_Γ is symmetric if and only if $T = T^*$.

Proof: Let us suppose $T = T^*$. Since for all $x \in D_T$ and $y \in D_{T^*}$ we have

$$\langle Tx, \gamma, y\rangle = \langle x, \gamma, T^*y\rangle \text{ where } \gamma \in \Gamma \tag{3.3}$$

Again we have

$$\langle Tx, \gamma, y\rangle = \langle x, \gamma, Ty\rangle \text{ for all } x, y \in D_T \tag{3.4}$$

Thus, T is symmetric. If T is symmetric then combining (3.3) and (3.4) we can conclude $T = T^*$.

Corollary 3.5. If T is a densely defined symmetric operator, then T^* is the maximal symmetric extension of T.

Proof. Let S be a symmetric operator in a Γ-Hilbert space H_Γ such that $T \subset S$. Then by the Theorem 3.3, we have

$$S^* \subset T^* .$$

Hence, $T \subset S \subset S^* \subset T^*$.

Theorem 3.6. If T is closed and invertible, then T^{-1} is closed.

Proof. Let us suppose that graph of T that is $G(T)$ is closed and $G(T) = \{(x, \gamma, Tx) : x \in D_T \text{ and } \gamma \in \Gamma\}$. Then obviously

$$G(T^{-1}) = \{(Tx, \gamma, x) : x \in D_T \text{ and } \gamma \in \Gamma\} \text{ is closed.}$$

Theorem 3.7. If T is densely defined operator, then T^* is closed.

Proof: If $y_n \in D_{A^*}$, $y_n \to y$ and $A^* y_n \to z$, then for any $x \in D_A$ & $\gamma \in \Gamma$ we have

$$\begin{aligned}\langle Ax, \gamma, y\rangle &= \lim_{n\to\infty} \langle Ax, \gamma, y_n\rangle \\ &= \lim_{n\to\infty} \langle x, \gamma, A^* y_n\rangle \\ &= \langle x, \gamma, z\rangle\end{aligned}$$

Hence, $y \in D_{A^*}$ and $A^* y = z$.

Note. If the given operator A is not closed then is it possible to extend A to a closed operator? Answer to that problem is to use the closure of $G(A)$ in $H_\Gamma \times H_\Gamma$ to define an operator. If closure of $G(A)$ defines an operator, then extension of A is closed.

Theorem 3.8. Every symmetric and densely defined operator in Γ-Hilbert space has a closed symmetric extension.

Proof. Let A be a densely defined, symmetric operator in a Γ-Hilbert space H_Γ. At first, we will show that condition $x_n \in D_A$, $x_n \to 0$, as $Ax_n \to y$ which implies that $y = 0$, is satisfied.

Let $x_n \to 0$ and $Ax_n \to y$. Since A is symmetric then for all $\gamma \in \Gamma$ we have

$$\begin{aligned}\langle y, \gamma, z\rangle &= \lim_{n\to 0} \langle Ax_n, \gamma, z\rangle \\ &= \lim_{n\to 0} \langle x_n, \gamma, Az\rangle \\ &= 0, \quad \text{for any } z \in D_A .\end{aligned}$$

This implies $y = 0$, as D_A is dense in H_Γ.

Now we have that there exists a closed operator B such that $G(B) = \text{Cl}G(A)$ and hence $A \subset B$. We have to prove that B is symmetric. If $x, y \in D_B$, then there exists $x_n, y_n \in D_A$ such that

$$x_n \to x \quad , \quad Ax_n \to Ax$$

and

$$y_n \to y \quad , \quad Bx_n \to Bx .$$

Since A is a symmetric operator , we have

$$\langle Ax_n, \gamma, y_n\rangle = \langle x_n, \gamma, Ay_n\rangle \text{ for all } \gamma \in \Gamma .$$

Then by letting $n \to \infty$, we have

$$\langle Bx, \gamma, y\rangle = \langle x, \gamma, By\rangle.$$

Hence B is symmetric.

Theorem 3.9. Let T be a closed densely defined operator in a Γ-Hilbert space H_Γ. Then

(a) For any $v, w \in H_\Gamma$, there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that $T(x) + y = v$ and $x - T^*(y) = w$.
(b) For any $w \in H_\Gamma$, there exist unique $x \in D_{T^*T}$ such that $x + T^*T(x) = w$.

Proof. (a) Consider the Γ-Hilbert space $H_{\Gamma_1} = H_\Gamma \times H_\Gamma$. Since T is closed, $G(T) = \{(x, \gamma, T(x)) : x \in D_T \text{ and } \gamma \in \Gamma\}$ is a closed subspace of H_{Γ_1}. Then by the projection theorem we have

$$H_{\Gamma_1} = G(T) + G(T)^{\perp_\gamma},$$

with

$$G(T) \cap G(T)^{\perp_\gamma} = \{0\}.$$

Now, $(u, y) \in G(T)^{\perp_\gamma}$ if and only if $\langle (x, Tx), \gamma, (u, y)\rangle = 0$ for all $x \in D_T$ and $\gamma \in \Gamma$. This implies, $\langle x, \gamma, u\rangle + \langle T(x), \gamma, y\rangle = 0$. That is $(u, y) \in G(T)^{\perp_\gamma}$ if and only if $\langle T(x), \gamma, y\rangle = \langle x, \gamma, -u\rangle$ for all $x \in D_T$. In other way,

$$(u, y) \in G(T)^{\perp_\gamma} \text{ if and only if } y \in D_{T^*} \text{ and } u = -T^*(y).$$

Since $(w, v) \in H_\Gamma \times H_\Gamma$, then there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that

$$(w, \gamma, v) = (x, \gamma, T(x)) + (-T^*(y), \gamma, y) \text{ for all } \gamma \in \Gamma.$$

That is, $w = x - T^*(y)$ and $v = T(x) + y$.

(b) Letting $v = 0$ in (a), then there exist unique $x \in D_T$ and $y \in D_{T^*}$ such that $T(x) + y = 0$ and $x - T^*(y) = w$. Thus $x - T^*(-T(x)) = 0$ implies, $x + T^*T(x) = w$, as desired.

References

[1] A. Ghosh, A. Das, T.E. Aman, *Representation theorem on Γ-Hilbert space*, International J. of Math Trends and Tech.,52(9), (2017) 608-615.

[2] B.K. Lahiri, *Elements of functional analysis*, The World Press Private Limited, (1982), Kolkata.

[3] B. V. Limaye, *Functional Analysis*, New age International (p) Limited, Publishers, Delhi, (2013), India.

[4] E. Kreyszig, *Introductory functional analysis with applications*, (1978), New York: Wiley.

[5] M.T. Garayev, M. Gürdal, S. Saltan, *Hardy type inequaltiy for reproducing kernel Hilbert space operators and related problems*, Positivity, 21(4), (2017), 1615-1623.

[6] S. I. Islam, *On Some bounded Operators and their characterizations in Γ-Hilbert Space,* Cumhuriyet Science Journal, 41(4), (2020), 854-861.

[7] J. B. Conway, *A course in functional analysis* (Vol. 96), (2019), Springer.

[8] L. Debnath, P. Mikusinski, *Introduction to Hilbert spaces with applications*, (2005), Academic press.

[9] T.E. Aman, D. K. Bhattacharya, *T-Hilbert space and linear quadratic control problem,* Revista de la Academia Canaria de Ciencias:= Folia Canariensis Academiae Scientiarum, 15(1), (2003), 107-114.