

## PAPER DETAILS

TITLE: LIE CENTRAL TRIPLE RACKS

AUTHORS: Guy Roger Biyogmam

PAGES: 58-65

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/232736>

## LIE CENTRAL TRIPLE RACKS

Guy Roger Biyogmam

Received: 18 February 2014; Revised 2 September 2014

Communicated by A. Çiğdem Özcan

**ABSTRACT.** This paper introduces Lie  $\mathfrak{c}$ -triple racks. These triple systems generalize both left and right racks to ternary algebras, and locally differentiates to  $gb$ -triple systems [3].

**Mathematics Subject Classification (2010):** 05B07, 18E10, 17A99

**Keywords:**  $gb$ -Triple systems, Lie triple systems, Lie racks

### 1. Introduction

The generalization of Lie algebras to algebras such as Lie triple systems, Jordan triple systems [8] and 3-Lie algebras [7] suggests a natural generalization of Leibniz algebras (non commutative Lie algebras) [11] to ternary algebras. One generalization is provided by Leibniz 3-algebras [5] for which the characteristic identity expresses the adjoint action as a derivation of the algebra. A second generalization of Leibniz algebras to ternary algebras is provided by Leibniz triple systems [4]. They are defined in such a way that Lie triple systems are a particular case. Recently, the author introduced  $gb$ -triple systems [3], another generalization of Leibniz algebras in which the ternary operation  $T : \mathfrak{g}^{\otimes 3} \rightarrow \mathfrak{g}$  expresses the map  $T_{a,b}(x) = T(a, x, b)$  as a derivation of  $\mathfrak{g}$  for all  $a, b \in \mathfrak{g}$ .

The local integration problem of these algebras generated from Lie's third theorem, which states that every finite-dimensional Lie algebra over the real numbers is associated with a Lie group. Partial solutions to this problem for Leibniz algebras (dubbed by Loday as the Coquecigrue problem) have been provided by several authors (see M. Kinyon [10], S. Covez [6]). The author extended Kinyon's results to Leibniz 3-algebras using Lie 3-racks [2]. In this paper we open the problem of integration of  $gb$ -triple systems. We follow Kinyon's approach [10] to open a path to a solution by defining an algebraic structure that locally differentiates to a  $gb$ -triple systems. We refer to these algebras as Lie  $\mathfrak{c}$ -triple racks. They appear to generalize both left and right Lie racks to ternary algebras; a particularity not supported by Lie 3-racks.

For the remainder of this paper, we assume that  $\mathfrak{K}$  is a field of characteristic different to 2.

## 2. $\mathfrak{c}$ -Triple racks

In this section we define  $\mathfrak{c}$ -triple racks and provide some examples. We also provide functorial connections with the category of groups and the category of racks.

Recall that a *gb-triple system* [3] is a  $\mathfrak{K}$ -vector space  $\mathfrak{g}$  equipped with a trilinear operation  $[-, -, -]_{\mathfrak{g}} : \mathfrak{g}^{\otimes 3} \longrightarrow \mathfrak{g}$  satisfying the identity

$$[A, B, [X, C, Y]_{\mathfrak{g}}]_{\mathfrak{g}} = [X, [A, B, C]_{\mathfrak{g}}, Y]_{\mathfrak{g}} - [[X, A, Y]_{\mathfrak{g}}, B, C]_{\mathfrak{g}} - [A, [X, B, Y]_{\mathfrak{g}}, C]_{\mathfrak{g}} \quad (1)$$

**Definition 2.1.** A  *$\mathfrak{c}$ -triple rack*  $(R, [-, -, -]_R)$  is a set  $R$  together with a ternary operation  $[-, -, -]_R : R \times R \times R \longrightarrow R$  satisfying the following conditions:

- (1)  $[x, [a, b, c]_R, y]_R = [[x, a, y]_R, [x, b, y]_R, [x, c, y]_R]_R$  ( $\mathfrak{c}$ -distributive property),
- (2) Given  $a, c, d \in R$ , there exists a unique  $x \in R$  such that  $[a, x, c]_R = d$ .

**Definition 2.2.** A  $\mathfrak{c}$ -triple rack  $(R, [-, -, -]_R)$  is said to be *pointed* if there is a distinguished element  $1 \in R$  satisfying

$$[1, y, 1]_R = y \text{ and } [a, 1, b]_R = 1 \text{ for all } a, b \in R.$$

A  $\mathfrak{c}$ -triple rack is said to be a *weak  $\mathfrak{c}$ -triple quandle* if it satisfies

$$[x, x, x]_R = x \text{ for all } x \in R.$$

A  $\mathfrak{c}$ -triple rack is a  *$\mathfrak{c}$ -triple quandle* if it satisfies

$$[a, y, b]_R = y \text{ if } a = y \text{ or } b = y.$$

Note that these generalize the notions of racks and quandles [9] to ternary operations. It is also clear that  $\mathfrak{c}$ -triple quandles are weak  $\mathfrak{c}$ -triple quandles but the converse is not true. See Example 2.4.

**Definition 2.3.** Let  $R$  and  $R'$  be two  $\mathfrak{c}$ -triple racks. A function  $\alpha : R \longrightarrow R'$  is said to be a *homomorphism of  $\mathfrak{c}$ -triple racks* if

$$\alpha([x, y, z]_R) = [\alpha(x), \alpha(y), \alpha(z)]_{R'} \text{ for all } x, y, z \in R.$$

This provides a category  $cpRACK$  of pointed  $\mathfrak{c}$ -triple racks and pointed  $\mathfrak{c}$ -triple rack homomorphisms.

**Example 2.4.** Let  $\Gamma := \mathbf{Z}[t^{\pm 1}, s]/(2s^2 + ts - s)$ . Any  $\Gamma$ -module  $M$  together with the ternary operation  $[-, -, -]_M$  defined by

$$[a, b, c]_M = sa + tb + sc$$

is a  $\mathfrak{c}$ -triple rack. Indeed,

$$\begin{aligned} [[x, a, y]_R, [x, b, y]_R, [x, c, y]_R]_R &= s(sx + ta + sy) + t(sx + tb + sy) + s(sx + tc + sy) \\ &= (2s^2 + st)(x + y) + ts(a + c) + t^2b \\ &= [x, [a, b, c]_R, y]_R \text{ since } 2s^2 + st = s. \end{aligned}$$

Therefore the  $\mathfrak{c}$ -distributive property is satisfied. For the second axiom, given  $a, c, d \in R$  one checks that  $x := t^{-1}(d - s(a + c))$  uniquely satisfies  $[a, x, c]_M = d$ . Note that  $M$  is a weak  $\mathfrak{c}$ -triple quandle that is not a  $\mathfrak{c}$ -triple quandle.

**Example 2.5.** Let  $G$  be a group with identity 1, and define on  $G$  the operation  $[-, -, -]_G$  by

$$[a, b, c]_G = acbc^{-1}a^{-1}.$$

Then  $(G, [-, -, -], 1)$  is a pointed weak  $\mathfrak{c}$ -triple quandle. Indeed, we have on one hand

$$[x, [a, b, c]_G, y]_G = xy[a, b, c]_Gy^{-1}x^{-1} = xyacb^{-1}c^{-1}a^{-1}y^{-1}x^{-1}.$$

On the other hand,

$$\begin{aligned} [[x, a, y]_G, [x, b, y]_G, [x, c, y]_G]_G &= [xyay^{-1}x^{-1}, xyby^{-1}x^{-1}, xycy^{-1}x^{-1}]_G \\ &= xyay^{-1}x^{-1}xyby^{-1}x^{-1}xyby^{-1}x^{-1}xyc^{-1}y^{-1}x^{-1}xya^{-1}y^{-1}x^{-1} \\ &= xyacb^{-1}c^{-1}a^{-1}y^{-1}x^{-1} \text{ by cancellation.} \end{aligned}$$

Therefore the  $\mathfrak{c}$ -distributive property is satisfied. For the second axiom, given  $a, c, d \in G$  one checks that  $x := c^{-1}a^{-1}dac$  uniquely satisfies  $[a, x, c]_M = d$ . Finally, it is clear that  $[x, x, x]_R = x$  for all  $x \in R$ .

As a consequence, we have the following:

**Proposition 2.6.** There is a faithful functor  $\mathfrak{F}$  from the category of groups to the category of pointed  $\mathfrak{c}$ -triple racks.

**Proof.** Define  $\mathfrak{F}$  by  $\mathfrak{F}(G) = (G, [-, -, -], 1)$  as in Example 2.5. Its left adjoint  $\mathfrak{F}'$  is defined as follows: Given a pointed  $\mathfrak{c}$ -triple rack  $R$ , consider the quotient group

$$G_R = \langle R \rangle / I$$

where  $\langle R \rangle$  is the free group on  $R$  and  $I$  is the normal subgroup of  $\langle R \rangle$  generated by the set  $\{(a^{-1}c^{-1}b^{-1}ca)([a, b, c]_R) : \text{with } a, b, c \in R\}$ . Indeed, given a morphism of  $\mathfrak{c}$ -triple racks  $\alpha : R \rightarrow \mathfrak{F}(G)$ , there is a unique morphism of groups  $\beta : \langle R \rangle \rightarrow G$  such that  $\alpha = \beta|_R$  by the universal property of free groups. So

$$\beta((a^{-1}c^{-1}b^{-1}ca)([a, b, c]_R)) = \alpha((a^{-1}c^{-1}b^{-1}ca)([a, b, c]_R)) = 1 \text{ for all } a, b, c \in R.$$

Now by the universal property of quotient groups, there is a unique morphism of groups  $\alpha_* : \mathfrak{F}'(R) \longrightarrow G$  such that the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{F}'(R) & \xrightarrow{\alpha_*} & G \\ \uparrow & & \uparrow id \\ R & \xrightarrow{\alpha} & \mathfrak{F}(G) \end{array}$$

□

**Example 2.7.** Let  $(R, \circ, 1)$  be a pointed rack. Then define on  $R$  the ternary operation by

$$[a, b, c]_R = a \circ (c \circ b).$$

It is easy to show that  $(R, [-, -, -], 1)$  is a pointed  $\mathfrak{c}$ -triple rack.

As a consequence we have the following:

**Proposition 2.8.** There is a faithful functor  $\mathfrak{H}$  from the category of pointed racks to the category of  $\mathfrak{c}$ -triple racks.

**Proof.** Define  $\mathfrak{H}$  by  $\mathfrak{H}((R, \circ, 1)) = (R, [-, -, -], 1)$  as in Example 2.7. Now, given a pointed  $\mathfrak{c}$ -triple rack  $(R, [-, -, -], 1)$ , one easily checks that the set  $R^{\times(3)}$  together with the binary operation

$$(a, b, c) \circ (x, y, z) = ([a, x, c]_R, [a, y, c]_R, [a, z, c]_R)$$

is a rack pointed at  $(1, 1, 1)$ . We then define the left adjoint  $\mathfrak{H}'$  of  $\mathfrak{H}$  by

$$\mathfrak{H}'((R, [-, -, -], 1)) = (R^{\times(3)}, \circ, (1, 1, 1)).$$

□

Let us observe that in the proof of Proposition 2.8 the set  $R^{\times(3)}$  is a quandle if  $R$  is a  $\mathfrak{c}$ -triple quandle.

### 3. From Lie $\mathfrak{c}$ -triple racks to gb-triple systems

In this section we define the notion of Lie  $\mathfrak{c}$ -triple racks. We show that the tangent functor  $T_1$  locally (at a specific point) maps Lie  $\mathfrak{c}$ -triple racks to gb-triple system.

**Definition 3.1.** A pointed  $\mathfrak{c}$ -triple rack  $(R, [-, -, -]_R, 1)$  is called a *Lie  $\mathfrak{c}$ -triple rack* if the underlying set  $R$  is a differentiable manifold and the ternary operation  $[-, -, -]_R : R \times R \times R \longrightarrow R$  is a smooth mapping.

Note that this definition appears to extend both left and right Lie racks [1] to ternary operations.

**Example 3.2.** Let  $G$  be a Lie group endowed with the operation

$$[a, b, c]_G = acbc^{-1}a^{-1}.$$

It follows by Example 2.5 that  $G$  is a Lie  $\mathfrak{c}$ -triple rack.

**Example 3.3.** Let  $(H, \{-, -, -\})$  be a group endowed with an antisymmetric ternary operation and  $V$  an  $H$ -module. Define the ternary operation  $[-, -, -]_R$  on  $R := V \times H$  by

$$[(a, A), (b, B), (c, C)]_R := (\{A, B, C\}b, ACBC^{-1}A^{-1}),$$

where  $a, b, c \in V$  and  $A, B, C \in H$ . Then  $(R, [-, -, -]_R, (0, 1))$  is a Lie  $\mathfrak{c}$ -triple rack.

For a pointed  $\mathfrak{c}$ -triple rack  $R$ , consider

$$\text{Aut}(R) := \{\xi : R \rightarrow R, \xi \text{ smooth bijection} : \xi([a, b, c]_R) = [\xi(a), \xi(b), \xi(c)]_R\}$$

and let  $\phi : R \times R \times R \rightarrow R$  be the mapping given by  $\phi(a, b, c) = [a, b, c]_R$ . We have as a consequence of the second axiom of Definition 2.1 that the map  $D : R \times R \rightarrow \text{Aut}(R)$ ,  $(a, c) \mapsto D(a, c) = \phi_{(a, c)}$  where  $\phi_{(a, c)}(x) = [a, x, c]_R$  for all  $x \in R$ , is well-defined differentiable map. Let  $D_* : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  be the induced map of tangent spaces, where  $\mathfrak{g} := T_1 R$  is the tangent space of  $R$  at the point 1 and  $\mathfrak{gl}(\mathfrak{g})$  is the Lie algebra associated to  $GL(\mathfrak{g})$ . Define a trilinear bracket on  $\mathfrak{g}$  by

$$[X, Y, Z]_{\mathfrak{g}} = D_*(X, Z)(Y).$$

**Proposition 3.4.** Let  $(R, [-, -, -]_R, 1)$  be a pointed  $\mathfrak{c}$ -triple rack. Then  $D(a, c) \in \text{Aut}(R)$  for all  $a, c \in R$ .

**Proof.**

$$\begin{aligned} D(a, c)([x, y, z]_R) &= \phi(a, c)([x, y, z]_R) \\ &= [a, [x, y, z]_R, c]_R \\ &= [[a, x, c]_R, [a, y, c]_R, [a, z, c]_R]_R \text{ by Definition 2.1} \\ &= [\phi_{(a, c)}(x), \phi_{(a, c)}(y), \phi_{(a, c)}(z)]_R \\ &= [D(a, c)(x), D(a, c)(y), D(a, c)(z)]_R. \end{aligned} \quad \square$$

**Remark 3.5.** Note that for all  $a, c \in R$ ,  $R$  acts on itself (considered as a differentiable manifold) via the maps  $\phi_{(a, c)}$  by Proposition 3.4. Also,  $\phi_{(a, c)}(1) = [a, 1, c]_R = 1$ . So the tangent functor  $T_1$  applied to  $\phi_{(a, c)} : R \rightarrow R$  yields a linear map  $\phi_{(a, c)*} : T_1 R \rightarrow T_1 R$ . Since  $\phi(a, c) \in \text{Aut}(R)$  by Proposition 3.4, it follows that  $\phi_{(a, c)*} \in GL(T_1 R)$ . Now let  $X \in T_1 R$  and denote by  $X_1 := \phi_{(a, c)*}(X)$  the vector field extension of  $X$ . Then  $X_1$  is generated by a one-parameter family of

diffeomorphisms  $\gamma_X : \mathbb{R} \rightarrow R$  with initial point  $\gamma_X(0) = 1$  and initial tangent vector  $d\gamma_X(0) = X$ . The corresponding exponential map (see [12, Chapter 9]) denoted  $\exp_1 : T_1(R) \rightarrow R$  is then defined by  $\exp_1(X) = \gamma_X(1)$ .

**Theorem 3.6.** *Let  $(R, [-, -, -]_R, 1)$  be a Lie  $\mathfrak{c}$ -triple rack and  $\mathfrak{g} := T_1 R$ . Then for all  $a, c \in R$ , the tangent mapping  $\phi_{(a,c)*} = T_1(\phi_{(a,c)})$  is an automorphism of  $\mathfrak{g}$ .*

**Proof.** Let  $X, Y, Z \in \mathfrak{g}$  and let  $x, y, z$  be respectively the images of  $X, Y$  and  $Z$  by the exponential map  $\exp_1$  (see Remark 3.5). By the  $\mathfrak{c}$ -distributive property of  $\mathfrak{c}$ -triple racks, we have

$$\phi_{(a,c)}(\phi_{(x,z)}(y)) = \phi_{(\phi_{(a,c)}(x), \phi_{(a,c)}(z))}(\phi_{(a,c)}(y))$$

which when successively differentiated at  $1 \in R$  with respect to the parameter  $\gamma_Y$  then  $\gamma_Z$  then  $\gamma_X$  yields

$$\phi_{(a,c)*}([X, Y, Z]_{\mathfrak{g}}) = [\phi_{(a,c)*}(X), \phi_{(a,c)*}(Y), \phi_{(a,c)*}(Z)]_{\mathfrak{g}} \quad (2). \quad \square$$

**Theorem 3.7.** *Let  $R$  be a Lie  $\mathfrak{c}$ -triple rack and  $A, C \in \mathfrak{g} := T_1 R$ . Let  $a, c$  be respectively the images of  $A$  and  $C$  by the exponential map  $\exp_1$ . Then the mapping  $D_{(A,C)*} : \mathfrak{g} \rightarrow gl(\mathfrak{g})$  is a derivation of  $\mathfrak{g}$ . Moreover,  $D_{(A,C)*}$  is exactly  $T_1(\Phi)$ , where  $\Phi$  is the mapping  $\Phi : R \times R \rightarrow GL(\mathfrak{g})$  defined by  $\Phi(a, c) = \phi_{(a,c)*}$ .*

**Proof.** From the proof of Theorem 3.6,  $\phi_{(a,c)*} \in GL(\mathfrak{g})$ . In addition, we have  $\phi_{(1,1)*} = I$ , where  $I$  is the identity of  $GL(\mathfrak{g})$ . Now differentiating  $\Phi$  at  $(1, 1)$  gives a map  $T_{(1,1)}(\Phi) : T_1(R \times R) \rightarrow gl(\mathfrak{g})$ . Also differentiating the identity (2) above at  $(1, 1)$  with respect to  $\gamma_{(A,C)}$  yields

$$\begin{aligned} D_{(A,C)*}(D_{(X,Z)*}(Y)) &= [A, [X, Y, Z]_{\mathfrak{g}}, C]_{\mathfrak{g}} \\ &= [[A, X, C]_{\mathfrak{g}}, Y, Z]_{\mathfrak{g}} + [X, [A, Y, C]_{\mathfrak{g}}, Z]_{\mathfrak{g}} + [X, Y, [A, Z, C]_{\mathfrak{g}}]_{\mathfrak{g}} \\ &= D_{(D_{(A,C)*}(X), Z)*}(Y) + D_{(X, Z)*}(D_{(A,C)*}(Y)) \\ &\quad + D_{(X, D_{(A,C)*}(Z))*}(Y). \end{aligned}$$

Hence  $D_{(A,C)*}$  is a derivation of  $\mathfrak{g}$  and the map  $T_1(\Phi)$  is exactly  $D_{(A,C)*}$ .  $\square$

From the calculations performed in the proofs of Theorem 3.6 and Theorem 3.7, we deduce that the ternary operation  $[-, -, -]_{\mathfrak{g}}$  satisfies the identity (1). We then have the following result:

**Corollary 3.8.** *Let  $R$  be a Lie  $\mathfrak{c}$ -triple rack and  $\mathfrak{g} := T_1 R$ . Then there exists a trilinear map  $[-, -, -]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $(\mathfrak{g}, [-, -, -]_{\mathfrak{g}})$  is a gb-triple system.*

**Remark 3.9.** Let  $G$  be the Lie  $\mathfrak{c}$ -triple rack of Example 3.2 and  $l$  the Lie algebra associated to the underlying group  $G$ . Then the bracket of the  $gb$ -triple system  $\mathfrak{g} = T_1(R)$  can be written in terms of the bracket of the Lie algebra  $l$  as

$$[X, Y, Z]_{\mathfrak{g}} = [Y, [X, Z]_l]_l.$$

To check that  $[-, -, -]_{\mathfrak{g}}$  satisfies the identity (1), let  $X, Y, A, B, C \in \mathfrak{g}$ ; we have on one hand

$$\begin{aligned} [X, Y, [A, B, C]_{\mathfrak{g}}]_{\mathfrak{g}} + [[A, X, C]_{\mathfrak{g}}, Y, B]_{\mathfrak{g}} &= [Y, [X, [A, B, C]_{\mathfrak{g}}]_l]_l + [Y, [[A, X, C]_{\mathfrak{g}}, B]_l]_l \\ &= [Y, [X, [B, [A, C]_l]_l]_l + [[X, [A, C]_l]_l, B]_l]_l \\ &= [Y, [[X, B]_l, [A, C]_l]_l]_l. \end{aligned}$$

On the other hand,

$$\begin{aligned} [A, [X, Y, B]_{\mathfrak{g}}, C]_{\mathfrak{g}} - [X, [A, Y, C]_{\mathfrak{g}}, B]_{\mathfrak{g}} &= [[X, Y, B]_{\mathfrak{g}}, [A, C]_l]_l - [[A, Y, C]_{\mathfrak{g}}, [X, B]_l]_l \\ &= [[Y, [X, B]_l]_l, [A, C]_l]_l - [[Y, [A, C]_l]_l, [X, B]_l]_l. \end{aligned}$$

The equality holds by the Jacoby identity.

**Acknowledgment.** The author would like to thank the referee for very helpful comments and suggestions.

## References

- [1] N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math., 178(2) (2003), 177-243.
- [2] G. R. Biyogmam, *Lie  $n$ -rack*, C. R. Math. Acad. Sci. Paris, 349(17-18) (2011), 957-960.
- [3] G. R. Biyogmam, *Introduction to  $gb$ -triple systems*, ISRN Algebra 2014, Art. ID 738154, 5 pp.
- [4] M. R. Bremner and J. Sánchez-Ortega, *Leibniz triple systems*, Commun. Contemp. Math., 16(1) (2014), 1350051, 19 pp.
- [5] J. M. Casas, J.-L. Loday and T. Pirashvili, *Leibniz  $n$ -algebras*, Forum Math., 14(2) (2002), 189-207.
- [6] S. Covez, *The local integration of Leibniz algebras*, Ann. Inst. Fourier (Grenoble), 63(1) (2013), 1-35.
- [7] V. T. Filippov,  *$n$ -Lie algebras*, Sibirsk. Mat. Zh., 26(6) (1985), 126-140.
- [8] N. Jacobson, *Lie and Jordan triple systems*, Amer. J. Math., 71 (1949), 149-170.
- [9] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, 23(1) (1982), 37-65.



- [10] M. K. Kinyon, *Leibniz algebras, Lie racks and digroups*, J. Lie Theory, 17(1) (2007), 99-114.
- [11] J.-L. Loday, *Une version non commutative des algebres de Lie: les algebres de Leibniz*, Enseign. Math., 39 (1993), 269-293.
- [12] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. I, (2nd edition) Publish or Perish, Inc., Wilmington, Del., 1979.

**Guy Roger Biyogmam**

Department of Mathematics  
Southwestern Oklahoma State University  
100 Campus Drive  
Weatherford, OK 73096, USA  
e-mail: [guy.biyogmam@swosu.edu](mailto:guy.biyogmam@swosu.edu)