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BLOCK TRANSITIVE $2 - (v, 17, 1)$ DESIGNS AND REE GROUPS

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ABSTRACT. This article is a contribution to the study of the automorphism groups of $2 - (v, k, 1)$ designs. Let \mathcal{D} be $2 - (v, 17, 1)$ design, $G \leq \text{Aut}(\mathcal{D})$ be block transitive and point primitive. If G is unsolvable, then $\text{Soc}(G)$, the socle of G , is not ${}^2G_2(q)$.

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1. Introduction

A $2 - (v, k, 1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set \mathcal{P} of v points and a collection \mathcal{B} of k -subsets of \mathcal{P} , called blocks, such that any 2-subsets of \mathcal{P} is contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq \text{Aut}(\mathcal{D})$ be a group of automorphisms of a $2 - (v, k, 1)$ design \mathcal{D} . Then G is said to be block transitive on \mathcal{D} if G is transitive on \mathcal{B} and is said to be point transitive (point primitive on \mathcal{D} if G is transitive (primitive) on \mathcal{P}). A flag of \mathcal{D} is a pair consisting of a point and a block through that point. Then G is flag transitive on \mathcal{D} if G is transitive on the set of flags.

The classification of block transitive $2 - (v, 3, 1)$ designs was completed about thirty years ago (see [2]). In [3], Camina and Siemons classified $2 - (v, 4, 1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2 - (v, 4, 1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [7]). Tong and Li [11] classified $2 - (v, 5, 1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [4] classified $2 - (v, 5, 1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [9] classified $2 - (v, k, 1)$ (where $k = 6, 7, 8, 9, 10$) designs with a block transitive, solvable group of automorphisms. In [5], Han and Ma classified $2 - (v, 11, 1)$ designs with a block transitive classical Simple groups of automorphisms.

This article is a contribution to the study of the automorphism groups of $2 - (v, k, 1)$ designs. Let \mathcal{D} be $2 - (v, 17, 1)$ design, $G \leq \text{Aut}(\mathcal{D})$ be block transitive and point primitive. We prove that following theorem.

Main Theorem. Let \mathcal{D} be $2 - (v, 17, 1)$ design, $G \leq \text{Aut}(\mathcal{D})$ be block transitive and point primitive. If G is unsolvable, then $\text{Soc}(G) \not\cong {}^2G_2(q)$.

2. Preliminary Results

Let \mathcal{D} be a $2 - (v, k, 1)$ design defined on the point set \mathcal{P} and suppose that G is an automorphism group of \mathcal{D} that acts transitively on blocks. For a $2 - (v, k, 1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G . Also, G^B denotes the permutation group induced by the action of G_B on the points of B , and so $G^B \cong G_B/G_{(B)}$.

The Ree groups ${}^2G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [10] as subgroups of $GL(7, q)$. Let $GF(q)$ be finite field of q elements, where $q = 3^{2n+1}$ for some positive integer $n \geq 1$. Set $t = 3^{n+1}$ so that $t^2 = 3q$. We give the following information about subgroups of ${}^2G_2(q)$. For each l dividing $2n + 1$, ${}^2G_2(3^l)$ denotes the subgroup of ${}^2G_2(q)$ consisting of all matrices in ${}^2G_2(q)$ with entries in subfield of 3^l . We use the symbols Q and K to note a Sylow 3-subgroup and a cyclic subgroup of order $q - 1$ of ${}^2G_2(q)$, respectively.

Lemma 2.1. ([6]) *Let $T \leq {}^2G_2(q)$ and T be maximal in ${}^2G_2(q)$. Then either T is conjugate to $P_6(l) = {}^2G_2(3^l)$ for some divisor l of $2n + 1$, or T is conjugate to one of the subgroups P_i in Table 1.*

Table 1: Group conjugate to T

Group	Structure	Remarks
P_1	$Q : K$	The normaliser of Q in ${}^2G_2(q)$
P_2	$(Z_2^2 \times D_{(q+1)/2}) : Z_3$	The normaliser of a fours-group
P_3	$Z_2 \times PSL(2, q)$	An involution centraliser
P_4	$Z_{q+t+1} : Z_6$	The normaliser of Z_{q+t+1}
P_5	$Z_{q-t+1} : Z_6$	The normaliser of Z_{q-t+1}

Lemma 2.2. ([8]) *Let $T = {}^2G_2(q)$ be an exceptional simple group of Lie type over $GF(q)$, and let G be a group with $T \trianglelefteq G \leq \text{Aut}(T)$. Suppose that M is a maximal subgroup of G not containing T , then one of the following holds:*

- (1) $|M| < q^3|G : T|$;
- (2) $T \cap M$ is a parabolic subgroup of T .

Lemma 2.3. ([5]) *Let G and $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a group and a design, and $G \leq \text{Aut}(\mathcal{D})$ be block transitive, point-primitive but not flag-transitive. Let $\text{Soc}(G) = T$. Then*

$$|T| \leq \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G : T|,$$

where $\alpha \in \mathcal{P}$, λ is the length of the longest suborbit of G on \mathcal{P} .

3. Proof of the Main Theorem

Proposition 3.1. *Let \mathcal{D} be $2 - (v, 17, 1)$ design, G be block transitive, point primitive but not flag transitive, then $v = 272b_2 + 1$.*

Proof. Let $b_1 = (b, v)$, $b_2 = (b, v - 1)$, $k_1 = (k, v)$, $k_2 = (k, v - 1)$. Obviously, $k = k_1 k_2$. Since $k = 17$, we get $k_1 = 1$. Otherwise, $k \mid v$, by [8], G is flag transitive, a contradiction. Thus $v = k(k - 1)b_2 + 1 = 272b_2 + 1$. \square

Proposition 3.2. *Let \mathcal{D} be $2 - (v, 17, 1)$ design, G be block transitive, point primitive but not flag transitive and $|T|$ be even. If G be unsolvable, then $|T| \leq 137|T_\alpha|^2|G : T|$.*

Proof. Let $B = \{1, 2, \dots, 17\} \in \mathcal{B}$. Since G is unsolvable, then the structure of G^B , the rank and subdegree of G do not occur:

Type of G^B	Rank of G	Subdegree of G
$\langle 1 \rangle$	273	$\overbrace{1, b_2, \dots, b_2}^{272}$

Otherwise, G^B is odd and G is also odd, a contradiction with $|T|$ be even. Thus $\lambda \geq 2b_2$. By Lemma 2.3 and Proposition 3.1,

$$\frac{|T|}{|T_\alpha|^2} \leq \frac{v}{\lambda} \cdot |G : T| \leq \frac{272b_2 + 1}{2b_2} \cdot |G : T| \leq 137|G : T|.$$

\square

Now we may prove our main theorem.

Suppose that $\text{Soc}(G) = {}^2G_2(q) = T$, then ${}^2G_2(q) \trianglelefteq G \leq \text{Aut}({}^2G_2(q))$. We have $G = T : \langle x \rangle$, where $x \in \text{Out}(T)$, the outer automorphisms group of T which may be generated by an automorphism of field. We may assume that x is an automorphism of field. Set $\circ(x) = m$, then $m \mid (2n + 1)$. Obviously, $|{}^2G_2(q)| = q^3(q^3 + 1)(q - 1)$. By [1] and $k = 17$, G is not flag transitive. Since G is point primitive, G_α ($\alpha \in \mathcal{P}$) is the maximal subgroup of G , T is block transitive in \mathcal{D} . Hence $M = G_\alpha$ satisfies one of the two cases in Lemma 2.2. We will rule out these cases one by one.

Case (1) $|M| < q^3|G : T|$.

By Proposition 3.2, we have an upper bound of $|T|$,

$$|T| < 137|T_\alpha|^2|G : T| < 137q^6|G : T| = 137q^6m.$$

We get

$$q - 1 < 137(2n + 1).$$

Let $2n + 1 = s \geq 3$, then $3^s < 138s$. Thus $s = 3, 5$.

If $s = 3$, then $|{}^2G_2(3^3)| = 3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37$. Since $v = 272b_2 + 1$ is odd, then $2^3 \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T . By Lemma 2.1, $T_\alpha \cong {}^2G_2(3)$, $(Z_2^2 \times D_{(q+1)/2}) : Z_3$ or $Z_2 \times PSL(2, q)$, where $q = 3^3$.

(i) $T_\alpha \cong {}^2G_2(3)$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7} - 1 = 6662330.$$

By Proposition 3.1, $156b_2 = 6662330$, a contradiction.

(ii) $T_\alpha \cong (Z_2^2 \times D_{(q+1)/2}) : Z_3$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3 \cdot 2^3 \cdot 7} - 1 = 59960978.$$

By Proposition 3.1, $156b_2 = 59960978$, a contradiction.

(iii) $T_\alpha \cong Z_2 \times PSL(2, q)$. We have

$$v - 1 = \frac{|T|}{|T_\alpha|} - 1 = \frac{3^9 \cdot 2^3 \cdot 7 \cdot 13 \cdot 19 \cdot 37}{3^3 \cdot 2^3 \cdot 7 \cdot 13} - 1 = 512486.$$

By Proposition 3.1, $156b_2 = 512486$, a contradiction.

If $s = 5$, then $|{}^2G_2(3^5)| = 3^{15} \cdot (3^{15} + 1) \cdot (3^5 - 1)$. Since $v = 272b_2 + 1$ is odd, then $2^3 \mid |T_\alpha|$. Clearly T_α is contained in some maximal subgroups of T . By Lemma 2.1,

$$T_\alpha \cong {}^2G_2(3), (Z_2^2 \times D_{(q+1)/2}) : Z_3 \text{ or } Z_2 \times PSL(2, q),$$

where $q = 3^5$. It is not difficult to exclude them by Proposition 3.1.

Case (2) $T \cap M$ is a parabolic subgroup of T .

By Lemma 2.1, the parabolic subgroup of ${}^2G_2(q)$ is conjugate to QK . Then the order of parabolic subgroup is $q^3(q - 1)$ and $v = q^3 + 1$. By Proposition 3.1, we have $q^3 = v - 1 = 272b_2$ and so $272 \mid q^3$, a contradiction.

This completes the proof the Main Theorem.

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