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ON THE MIN-PROJECTIVE MODULES

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Abstract. Let R be a commutative ring. An R-module M is called min-projective if $Ext^1_R(M, \frac{R}{I}) = 0$, for every simple ideal I. In this paper, we first give some results of min-projective R-modules on the some specific rings such as cotorsion rings, von Neumann regular rings and coherent rings. Then we investigate min-projective covers on universally min-projective rings. Finally, we deal with some characterizations of min-projective modules over a perfect ring.

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1. Introduction

Throughout this paper, R denotes a commutative ring. Let $\mathcal{C}$ be a class of R-modules and G be an element of $\mathcal{C}$. The homomorphism $\theta : G \longrightarrow K$ with $G \in \mathcal{C}$ is called a $\mathcal{C}$*-precover* of K if for any homomorphism $f : G^{'} \longrightarrow K$ with $G^{'} \in \mathcal{C}$, there exists a homomorphism $g : G^{'} \longrightarrow G$ such that $\theta o g = f$. Moreover, if the mapping g is automorphism of G when $G = G^{'}$ and $f = g$ then $\mathcal{C}$-precover θ is called the $\mathcal{C}$*-cover* of G. One can similarly define the dual of the $\mathcal{C}$*-preenvelope* and $\mathcal{C}$*-envelope*, see [6], for more details. An R-module C is said to be *cotorsion* if $Ext^1_R(F, C) = 0$, for all flat R-module F, see [15, Definition 3.1.1]. There are many papers which deal with to the concept of projective modules and injective modules and the related topics, see for instance [2,4,10,11]. An R-module N is called *finitely presented* if there exists the exact sequence $R^{(n)} \longrightarrow R^{(m)} \longrightarrow N \longrightarrow 0$ and an R-module M is called FP*-injective* if $Ext^1_R(N, M) = 0$ for any finitely presented R-module N. Also, an R-module N is called FP*-projective* if $Ext^1_R(N, M) = 0$ for any FP-injective R-module M. We refer the reader to [4] for more details about FP-projective and FP-injective modules. Note that for any R-module M, $\sigma_M : M \longrightarrow C(M)$, $\tau_M : M \longrightarrow FE(M)$ and $\xi_M : F(M) \longrightarrow M$ will denote, respectively, a cotorsion envelope, an FP-injective preenvelope and a flat cover for

M. An R-module M is called *min-projective* if $Ext^1_R(M, \frac{R}{I}) = 0$ for any simple ideal I. An R-module N is called *min-flat* if $Tor^R_1(N, \frac{R}{I}) = 0$ for any simple ideal I and if every R-module is min-projective or min-flat, then R is called a *universally min-projective* ring or a *universally min-flat* ring. Recall that a ring R is called *coherent* if every finitely generated ideal is finitely presented, see [15, Definition 1.1.4]. The *socle* of R, denoted by $Soc(R)$, is the direct sum of nonzero simple ideals of R. We use $Soc(R) \leq_e R$ to mean that $Soc(R)$ is an essential ideal of R and $r \in R$ is singular if $Ann_R(r) \leq_e R$. A ring R is called *von Neumann regular* if for each $r \in R$, there is $r' \in R$ with $rr'r = r$, see [12]. A ring R is said to be *perfect* when every R-module has a projective cover, see [15]. In this paper, some characterizations of min-projective modules on cotorsion rings, von Neumann regular rings, coherent rings, universally min-projective rings and perfect rings are given. For instance, it is shown that R is a cotorsion ring if and only if every flat R-module is min-projective; R is a von Neumann regular ring if and only if R is a coherent ring and every FP-projective R-module is min-projective; on universally min-flat rings with $Soc(R) \leq_e R$, R is a universally min-projective ring if and only if every min-flat R-module has an Ω-cover with the unique mapping property if only if R is a cotorsion ring with $Z(R) = 0$, where Ω is the class of min-projective R-modules and $Z(R)$ is the set of all singular elements. Also, we prove that R is a perfect ring if and only if every min-projective R-module is cotorsion if and only if every flat R-module is min-projective and every min-projective R-module has a cotorsion envelope with the unique mapping property if and only if for each R-homomorphism $f : M_1 \longrightarrow M_2$ with M_1 and M_2 min-projective , $\ker(f)$ is cotorsion.

2. Main Results

We start by the following definition.

Definition 2.1. Let R be a ring. An R-module M is called *min-projective* if $Ext^1_R(M, \frac{R}{I}) = 0$ for any simple ideal I.

It is well-known that if R is a cotorsion R-module, then R is a cotorsion ring. The following propositon shows that, on cotorsion rings, every flat module is a min-projective module.

Proposition 2.2. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a cotorsion ring;*
(2) *Every flat R-module is a min-projective.*

Proof. (1) $\Rightarrow$ (2) It is known from [5, Lemma 2.14], which I is a cotorsion ring. Now consider the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \frac{R}{I} \longrightarrow 0$. Then for every flat R-module M, we get the exact sequence $0 = Ext^1_R(M, R) \longrightarrow Ext^1_R(M, \frac{R}{I}) \longrightarrow Ext^2_R(M, I) = 0$. Hence $Ext^1_R(M, \frac{R}{I}) = 0$ and so M is min-projective R-module.
(2) $\Rightarrow$ (1) is clear. □

It is obvious that every projective module is a min-projective module. However, the following example shows that the converse is not true in general. Before this, we recall that a ring is said to be *hereditary* if all of its ideals are projective, see [12]. If R has no simple ideal, then the socle of R is defined to be zero and in this case it is clear that every R-module is a min-projective module. The following example shows that the definition of min-projective R-modules is a proper generalization of projective modules.

Example 2.3. *Let R be a ring.*

(a) *If R is a non-hereditary ring such that $Soc(R) = (0)$, then some of ideals of R are min-projective, while they are not projective R-modules. In particular, if $R \cong \frac{K[x_n : n \geq 1]}{(x_i x_j : i \geq 1 \text{ and } j \geq 1)}$, then $Soc(R) = (0)$ and so for every $n \geq 1$, the ideal (x_n) is a min-projective module but it is not projective.*

(b) *Let R be a reduced ring, that is R has no non-zero nilpotent element which is not decomposable (for example R can be an integral domain which is not a field). We show that R contains no simple ideal. By contrary, suppose that R contains a simple ideal, say Re. Then since R is reduced, we deduce that e is an idempotent element and so by Brauer's lemma (see* [8, 10.22]*), R is decomposable that is a contradiction. Hence every R-module is min-projective, because R contains no simple ideal. But not all R-modules are projective.*

(c) *Let $R \cong D_1 \times \cdots \times D_n$, where every D_i, $1 \leq i \leq n$, is an integral domain which is not field. Then every R-module is min-injective. But there are R-modules which are not projective.*

(d) *From Part (c), we conclude that any module over the ring $\mathbb{Z}$ of integers is min-projective. But not all $\mathbb{Z}$-modules are projective.*

In the following proposition, some properties of modules on cotorsion ring are studied.

It is trivial that min-projective modules are closed under extensions over any ring. So, we have the following proposition.

Proposition 2.4. *Let R be a cotorsion ring. Then*

(1) *Let $\alpha : N \longrightarrow M$ be a monomorphism. Then $coker(\alpha)$ is min-projective if and only if $coker(\sigma_M\alpha)$ is min-projective.*
(2) *Let N be a submodule of M. If M is min-projective and $\dfrac{M}{N}$ is flat, then N is also min-projective.*
(3) *Every cotorsion envelope of a min-projective R-module is min-projective.*

Proof. (1) and (3) are clear, by the fact which was mentioned before the proposition. Now, we prove (2). Let I be a simple ideal. Then the short exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow \dfrac{M}{N} \longrightarrow 0$ induces the exact sequence

$$0 = Ext^1_R(M, \frac{R}{I}) \longrightarrow Ext^1_R(N, \frac{R}{I}) \longrightarrow Ext^2_R(\frac{M}{N}, \frac{R}{I}) = 0.$$

The first equality follows by Proposition 2.2. Hence $Ext^1_R(N, \dfrac{R}{I}) = 0$ and so N is min-projective. □

Proposition 2.5. *A min-flat R-module M is min-projective if and only if the $\frac{R}{I}$-module $\frac{M}{MI}$ is min-projective, for every simple ideal I.*

Proof. This follows from the isomorphism $Ext^1_R(M, \dfrac{R}{I}) \simeq Ext^1_{\frac{R}{I}}(\dfrac{M}{MI}, \dfrac{R}{I})$, see [13, Lemma 5.1]. □

In the following proposition, we give some conditions under which the direct sum of a family of min-flat R-modules is min-projective. Before this, we recall that for any R-module M, the R-module $Hom_Z(M, \dfrac{Q}{Z})$ is denoted by M^+.

Proposition 2.6. *Let $\{\frac{M_i}{M_iI} : i \in I\}$ and $\{\frac{M_i^{++}}{M_i^{++}I} : i \in I\}$ be two indexed sets of min-projective $\frac{R}{I}$-modules, where I is a simple ideal. If every M_i is min-flat, then*

(1) *$\coprod_{i\in I} M_i$ is min-projective.*
(2) *$\coprod_{i\in I} M_i^{++}$ is min-projective.*

Proof. (1) By Proposition 2.5, we have $Ext^1_R(M_i, \dfrac{R}{I}) = 0$. Thus by [12, Theorem 7.13], $Ext^1_R(\coprod_{i\in I} M_i, \dfrac{R}{I}) \simeq \prod_{i\in I} Ext^1_R(M_i, \dfrac{R}{I}) = 0$ and so $\coprod_{i\in I} M_i$ is min-projective.
(2) This follows from [3, Lemma 3.2]. □

Remark 2.7. *Let R be a coherent ring. By* [6, Theorem 7.4.1]*, every R-module has a special FP-injective pre-envelope, i.e; there is an exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow L \longrightarrow 0$, where F is FP-injective and L is FP-projective and every R-module has a special FP-projective precover, i.e; there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$, where P is FP-projective and K is FP-injective.*

It is well-known that R is a von Neumann regular ring if and only if every R-module is flat, see [12, Theorem 4.9]. In the following theorem, we give a characterization of a von Neumann regular ring.

Recall that a $\mathcal{C}$-cover $\phi : M \longrightarrow N$ has the *unique mapping property* if for any homomorphism $f : A \longrightarrow N$ with $A \in \mathcal{C}$, there exists a unique $g : A \longrightarrow M$ such that $\phi g = f$. One can similarly define the dual of the $\mathcal{C}$-*envelope*, see [6].

Theorem 2.8. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a von Neumann regular ring;*
(2) *R is a coherent ring and every FP-projective R-module is min-projective.*

Proof. (1) ⇒ (2) By [2, Corollary 4.3], R is a coherent ring. Let N be an R-module. Then by Remark 2.7, there is an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$, where P is FP-projective and K is FP-injective. Therefore, for every FP-projective R-module M, we obtain the exact sequence

$$0 = Ext_R^1(M, P) \longrightarrow Ext_R^1(M, N) \longrightarrow Ext_R^2(M, K) = 0.$$

Thus $Ext_R^1(M, N) = 0$ and so every FP-projective R-module is min-projective.
(2) ⇒ (1) By [15, Theorem 2.3.1], every R-module has a FP-injective envelope and by Remark 2.7 and (2), its cokernel is FP-projective. Let M be a min-projective R-module. Then there exists a commutative diagram with the exact rows:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\tau_M} & FE(M) & \xrightarrow{\gamma} & L & \longrightarrow & 0 \\ & & & 0\searrow & \downarrow \tau_L\gamma & \swarrow \tau_L & & & \\ & & & & FE(L) & & & & \end{array}$$

Note that by [12, Proposition 7.24] and Remark 2.7, for every FP-injective R-module N, there exists a split exact sequence $0 \longrightarrow N \longrightarrow D \longrightarrow C \longrightarrow 0$, where D is an FP-injective R-module and C is an FP-projective R-module. So for every R-module K, $Ext_R^1(K, N) = 0$. Therefore by [5, Corollary 2.12] and [12, Theorem 3.56], every finitely presented R-module is projective. Hence every R-module is FP-injective. So the FP-injective envelopes τ_L and τ_M satisfy unique mapping property. Since $\tau_L \gamma \tau_M = 0 = 0\tau_M$, from (2), we have $\tau_L \gamma = 0$. Thus $L = im(\gamma) \subseteq ker(\tau_L) = 0$ and hence $L = 0$. Therefore, $M = FE(M)$ and so every min-projective R-module is FP-injective. Thus (1) follows from (2) and [2, Corollary 4.3] and the proof completes. □

Definition 2.9. Let R be a ring. A ring R is called *universally min-projective* if every R-module is min-projective.

Definition 2.10. Let R be a ring. A ring R is called *universally min-flat* when every R-module is min-flat.

Now, we present the following characterizations of the universally min-projective rings. From now and for simplicity, we denote the class of min-projective R-modules by Ω.

Theorem 2.11. *Let R be a ring. Then the following statements are equivalent:*

(1) *For any simple ideal I and any flat R-module M, the $\frac{R}{I}$-module $\frac{M}{MI}$ is min-projective;*
(2) *R is a cotorsion ring;*
Moreover, if R is a universally min-flat ring with $Soc(R) \leq_e R$, then the above conditions are equivalent to:
(3) *R is a cotorsion ring with $Z(R) = 0$;*
(4) *For every simple ideal I of R, $\frac{R}{I}$ is cotorsion and $J(R) = 0$;*
(5) *R is a universally min-projective ring;*
(6) *Every min-flat R-module has an Ω-cover with the unique mapping property;*
(7) *Every min-flat R-module is min-projective.*

Proof. (1) $\Rightarrow$ (2) We shall show that $Ext^1_R(M, R) = 0$, for every flat R-module M. Let I be a simple ideal of R. Then we have the short exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \dfrac{R}{I} \longrightarrow 0$. By [5, Lemma 2.14], $Ext^1_R(M, I) = 0$. Since every flat module is min-flat, Proposition 2.5 implies that M is min-projective. Hence $Ext^1_R(M, \dfrac{R}{I}) = 0$. Therefore, the above exact sequence induces the exact sequence

$$0 = Ext^1_R(M, I) \longrightarrow Ext^1_R(M, R) \longrightarrow Ext^1_R(M, \frac{R}{I}) = 0,$$

for every flat R-module M. Thus $Ext^1_R(M, R) = 0$, as desired.
(2) $\Rightarrow$ (1) This follows from Propositions 2.2 and 2.5.
(2) $\Rightarrow$ (3) R is a cotorsion ring by (2). Note that $\frac{R}{I}$ is flat and so by [5, Theorem 2.16] R is a PS ring (every simple ideal of R is projective). Therefore, $Soc(R)$ is projective and hence by [9, Exercise 12 (A), p.269], $Z(Soc(R)) = 0$. Therefore, by [1, Lemma 7.2], $Z(Soc(R)) = Z(R) \cap Soc(R) = 0$ and so $Z(R) = 0$.
(3) $\Rightarrow$ (4) Let I be a simple ideal of R and M be a flat R-module. Then we obtain the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \dfrac{R}{I} \longrightarrow 0$ which gives rise to the exactness of

$$\cdots \longrightarrow Ext^1_R(M, R) \longrightarrow Ext^1_R(M, \frac{R}{I}) \longrightarrow Ext^2_R(M, I) \longrightarrow \cdots$$

By (3), R is cotorsion. So, $Ext^1_R(M, R) = 0$. Also, from [5, Lemma 2.14], we conclude that $Ext^2_R(M, I) = 0$. Therefore, $Ext^1_R(M, \dfrac{R}{I}) = 0$ and so $\frac{R}{I}$ is cotorsion. Now, we claim that $J(R) = Ann_R(Soc(R)) = Z(R)$. Since the annihilator of any simple ideal is a maximal ideal, we deduce that $J(R) = Ann_R(Soc(R))$. It is clear that $Soc(R)^2 = 0$. Thus $Soc(R) \subseteq Ann(Soc(R))$. Now, since $Soc(R) \leq_e R$, we deduce that $Ann_R(Soc(R)) \leq_e R$ and so $Ann_R(Soc(R)) \subseteq Z(R)$. Since R is a cotorsion ring, we deduce that $\frac{R}{J(R)}$ is semisimple, by [7, Theorem 6]. Thus $Z(\frac{R}{J(R)}) = 0$ and hence $Z(R) \subseteq J(R)$. So, by (3), $J(R) = 0$.
(4) $\Rightarrow$ (5) Note that R is a von Neumann regular ring by [5, Theorem 2.16] and so by [5, Corollary 2.12], $\frac{R}{I}$ is injective. Hence (5) follows.
(5) $\Rightarrow$ (6) This is clear.
(6) $\Rightarrow$ (7) Let M be a min-flat R-module. Then there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & & & N' & & & & \\
 & & \phi \swarrow & & \alpha\phi\downarrow & & \searrow 0 & & \\
0 & \longrightarrow & K & \xrightarrow{\alpha} & N & \xrightarrow{\psi} & M & \longrightarrow & 0, \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where ψ and ϕ are Ω-cover with the unique mapping property. Since $\psi\alpha\phi = 0 = \psi$, we have $\alpha\phi = 0$ by (6). Therefore, $K = im(\phi) \subseteq ker(\alpha) = 0$ and so $K = 0$. Thus $M = N$ and hence every min-flat R-module is min-projective.
(7) $\Rightarrow$ (1) Let I be a simple ideal of R and M be a flat R-module. Then it is clear that M is a min-flat R-module. So, by (7), M is a min-projective R-module. Thus Proposition 2.5 implies that $\frac{M}{MI}$ is min-projective $\frac{R}{I}$-module and so we are done. □

Corollary 2.12. *Let R be a coherent ring. Then the following statements are equivalent:*

1. *R is a universally min-projective ring;*
2. *Every R-module has an Ω-cover with the unique mapping property;*
3. *For every simple ideal I, $\frac{R}{I}$ is cotorsion and every FP-projective R-module is min-projective.*

Proof. (1) $\Rightarrow$ (2) It is trivial.
(2) $\Rightarrow$ (3) By (2) and (7) of Theorem 2.11, $\frac{R}{I}$ is cotorsion. Let M be a FP-projective R-module, similar to proof (6) $\Rightarrow$ (7) of Theorem 2.11, R-module M is min-projective and so (3) follows.
(3) $\Rightarrow$ (1) This follows from Theorem 2.8 and [5, Corollary 2.12]. □

Corollary 2.13. *Let R be a coherent ring such that $\frac{R}{I}$ be an injective R-module, for every simple ideal of R. If $h : M \longrightarrow N$ is a homomorphism of min-projective R-modules, coker(h) is min-projective.*

Proof. By Theorem 2.8 and Corollary 2.12, M is FP-injective. So, the short exact sequence $0 \longrightarrow M \xrightarrow{h} N \longrightarrow \dfrac{N}{im(h)} \longrightarrow 0$ is pure. Therefore, for every R-module B, $Tor_1^R(B, \dfrac{N}{im(h)}) = 0$ and so by Proposition 2.2, $\frac{N}{im(h)}$ is min-projective. □

Corollary 2.14. *Let R be a coherent ring and $\frac{R}{I}$ be a cotorsion module, for every simple ideal I. Then R is a von Neumann regular ring if and only if R a is universally min-projective ring.*

Proof. This is a direct consequence of Theorem 2.8 and Corollary 2.12. □

From [14, Proposition 9.43], we know that R is a perfect ring if and only if every flat R-module is projective. In the following theorem, we give some other characterizations of perfect rings.

Theorem 2.15. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is a perfect ring;*
(2) *Every min-projective R-module is cotorsion;*
(3) *Every flat R-module is min-projective and every min-projective R-module has a cotorsion envelope with the unique mapping property;*
(4) *For each R-homomorphism $f : M_1 \longrightarrow M_2$ with M_1 and M_2 min-projective, $\ker(f)$ is cotorsion;*
(5) *For each min-projective R-module M, the functor $Hom_R(-, M)$ is exact with respect to each pure exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow L \longrightarrow 0$ in which P is projective. In addition, L and $\frac{R}{I}$-module $\frac{K}{KI}$ are min-projective, for every simple ideal I.*

Proof. (1) ⇒ (2) For every flat R-module F and every min-projective R-module M, $Ext_R^1(F, M) = 0$. So (2) follows.
(2) ⇒ (1) In the short exact sequence $0 \longrightarrow K \longrightarrow P \xrightarrow{\xi_L} L \longrightarrow 0$ with L flat, P projective and ξ_L projective cover of L, K is cotorsion. Thus (2) implies that P is cotorsion. So, for every flat R-module F, we obtain the exact sequence

$$0 = Ext_R^1(F, P) \longrightarrow Ext_R^1(F, L) \longrightarrow Ext_R^2(F, K) = 0.$$

Hence $Ext_R^1(F, L) = 0$ and so every flat R-module is cotorsion and by [15, Proposition 3.3.1], (1) follows.

(1) ⇒ (3) This is a direct consequence of [14, Proposition 9.43] and [5, Theorem 2.18].
(3) ⇒ (1) Let $C(M)$ be the cotorsion envelope of min-projective R-module M. There is the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\sigma_M} & C(M) & \xrightarrow{\gamma} & L & \longrightarrow & 0 \\ & & & 0\searrow & \downarrow \sigma_L\gamma & \swarrow \sigma_L & & & \\ & & & & C(L) & & & & \end{array}$$

Note that by [15, Theorem 3.4.2], L is flat and so σ_L exists. Hence $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$ and so $\sigma_L\gamma = 0$. Therefore, $L = im(\gamma) \subseteq ker(\sigma_L) = 0$ and so $M = C(M)$. Thus by (2), R is a perfect ring.
(1) ⇒ (4) This follows from the fact that every module is cotorsion over perfect rings.
(4) ⇒ (1) Let M be a min-projective R-module. From the above commutative diagram, we have $M = ker(\gamma) = ker(\sigma_L\gamma)$. Thus by (4), M is cotorsion. So, (1) follows from (2).
(1) ⇒ (5) For every R-module B, we obtain the exact sequence

$$0 \longrightarrow B \otimes_R K \longrightarrow B \otimes_R P \longrightarrow B \otimes_R L \longrightarrow 0.$$

Hence $Tor_1^R(L, B) = 0$ and so L is flat. Then for any min-projective R-module M, we have the following exact sequence

$$Hom_R(P, M) \longrightarrow Hom_R(K, M) \longrightarrow Ext_R^1(L, M) = 0.$$

By (2), M is cotorsion; therefore, the functor $Hom_R(-, M)$ is exact. By [14, Proposition 9.43], L is min-projective. Thus for every simple ideal I, we get the exact sequence

$$0 = Ext_R^1(P, \frac{R}{I}) \longrightarrow Ext_R^1(K, \frac{R}{I}) \longrightarrow Ext_R^2(L, \frac{R}{I}) = 0.$$

Hence $Ext_R^1(K, \frac{R}{I}) = 0$ and hence K is min-projective. Since P and L are flat, we deduce that K is flat and subsequently by Proposition 2.5, $\frac{K}{KI}$ is min-projective.
(5) ⇒ (1) For every min-projective R-module M and every flat R-module L, we have $Ext_R^1(L, M) = 0$. So, every min-projective R-module is cotorsion. Thus (1) follows from (2). □

Example 2.16. *Let $\mathbb{Z}$ be the ring of integer numbers. Then we show that the $\mathbb{Z}$-module $\mathbb{Z}$ is min-projective which is not cotorsion. Suppose to the contrary, $\mathbb{Z}$*

is cotorsion. Then by Proposition 2.2, every flat R-module is min-projective and so every flat R-module is cotorsion. Hence [15, Proposition 3.3.1] *implies that* $\mathbb{Z}$ *is a perfect ring and this contradicts this fact that* $\mathbb{Z}$ *is not a perfect ring, see* [12, Example 4.61].

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