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ON CONDITIONS FOR CONSTELLATIONS

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Abstract. A *constellation* is a set with a partially-defined binary operation and a unary operation satisfying certain conditions, which, loosely speaking, provides a 'one-sided' analogue of a category, where we have a notion of 'domain' but not of 'range'. Upon the introduction of an ordering, we may define so-called *inductive constellations*. These prove to be a significant tool in the study of an important class of semigroups, termed *left restriction semigroups*, which arise from the study of systems of partial transformations. In this paper, we study the defining conditions for (inductive) constellations and determine that certain of the original conditions from previous papers are redundant. Having weeded out this redundancy, we show, by the construction of suitable counterexamples, that the remaining conditions are independent.

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1. Introduction

The study of certain semigroups via related types of ordered categories is a well-established technique in semigroup theory; perhaps the most famous example is that of inverse semigroups, which may be studied from the viewpoint of so-called *inductive groupoids*: a special type of ordered category in which all arrows are invertible. The correspondence between inverse semigroups and inductive groupoids may be cast in technical category-theoretic terms (by saying that a certain category of inverse semigroups is isomorphic to a certain category of inductive groupoids) but, at a practical level, this correspondence simply means that we may always construct an inductive groupoid from any given inverse semigroup in a very straightforward manner, and vice versa. This has proved to be a particularly useful technique in the study of inverse semigroups, and forms the core of the book [12].

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Inverse semigroups arise from the consideration of systems of injective partial transformations of a set, but we may extend the ‘inductive groupoid’-type techniques to the study of semigroups which arise from systems of *arbitrary* partial transformations. In particular, so-called *two-sided restriction semigroups* may be connected with *inductive categories* (essentially, inductive groupoids in which we drop the insistence upon arrows being invertible) in a manner entirely analogous to that between inverse semigroups and inductive groupoids. For further details and references on two-sided restriction semigroups, and on their connection with inductive categories, see [9] and [10], respectively.

As their name suggests, two-sided restriction semigroups are not the only class of ‘restriction semigroups’ which may be considered: *left restriction semigroups* have seen much study — see, for example, [5,9].[1] Moreover, given the proven utility of ‘inductive groupoid/category’-type techniques in the study of inverse semigroups and two-sided restriction semigroups, it is desirable to have similar techniques at our disposal in the one-sided case of left restriction semigroups. In this instance, (inductive) categories are not the appropriate objects to consider, since these are inherently ‘two-sided’ (owing to the presence of both the ‘domain’ and ‘range’ operations). Instead, we must use what is essentially a ‘one-sided’ analogue of a category, where we have a notion of ‘domain’ but not of ‘range’. Such an object was introduced in [6] and is termed a *constellation*; this is a set together with a partially-defined binary operation $\cdot$ and a unary operation $^+$ which satisfy certain conditions — these conditions were chosen to be as ‘category-like’ as possible, to emphasize the fact that a constellation is ‘half a category’. The particular constellations which correspond to left restriction semigroups in the required manner are so-called *inductive constellations*: partially ordered constellations with certain additional properties. The ‘partial actions’ of inductive constellations have already been considered in [7], as a means of informing the study of those of left restriction semigroups [5].

We reiterate the fact that a constellation possesses an analogue of the ‘domain’ in a category, in the form of the $^+$ operation (see Definition 2.1), but that it has no analogue of ‘range’. Thus a constellation is essentially ‘half a category’. It is in fact possible to construct a full category from a left restriction semigroup, as Jackson and Stokes [11] have shown. However, we have reasons to prefer the ‘constellation approach’: see [6, p. 263 and §4]. Furthermore, see [6, pp. 262–263] for a brief

[1]With regard to the title of [5], we note that left restriction semigroups were formerly known as ‘weakly left E-ample semigroups’. Indeed, these semigroups have appeared in the literature under a range of different names — see [9] for more details.

discussion of some other ways in which a category may be constructed from a left restriction semigroup, and for comments on other attempts to generalise a category by modelling domain but not range.

An inductive constellation is defined by a rather lengthy list of conditions. First, we have four conditions defining the basic constellation (i.e., the underlying structure of the constellation, before any ordering is applied), followed by six conditions defining the ordering, and then one final condition for 'inductivity'. Moreover, in the special cases of *replete* and *right cancellative* constellations, further conditions are added. The purpose of this paper is to contribute to the development of the theory of constellations by studying the defining conditions and removing any redundancy therein. Once all such redundancy has been removed, we will show that the remaining conditions are independent by constructing suitable counterexamples.

We note that this will be the study of the conditions for constellations *as they stand*, i.e., the study of the conditions employed in [6,7]. As we will point out in the relevant places, the conditions for inductive constellations were designed, first of all to mimic the conditions used to define an inductive category, and secondly to echo the definition of a left restriction semigroup (in the formulation employed by the 'York school' — see [9]). We make no claim that the conditions for constellations used in [6,7] give the best possible representation of a constellation. They were developed in the first place as the most *practical* set of conditions — a set of conditions which best evokes those for both categories and left restriction semigroups and which may therefore be easily grasped by those researchers who are accustomed to handling inductive categories, inductive groupoids, inverse semigroups and left restriction semigroups in the manner of [12] or, more recently, [10]. Thus, other than removing redundancies, we make little effort to improve the conditions under consideration — we hope to make this the subject of future work.

With regard to the style in which we present the conditions, we retain the largely English formulation of [6,7] and eschew any form of logic of partial terms, such as that of Beeson [1]. We do this for reasons of accessibility (and for uniformity with previous work), though, again, such a reformulation may be made the subject of future work. [2]

[2] I am grateful to the referee for making the observation that the use of a logic of partial terms may improve the presentation of the conditions. It is worth noting that such a reformulation (and an improvement to the constellation conditions) has been considered by Robin Cockett in his unpublished *Notes on Constellations* (April 2010). Cockett's recasting of the conditions helps

We begin the paper with Section 2, where we consider unordered constellations. This section is divided into three subsections. In the first (2.1), we define the notion of a constellation and show that the four conditions defining this object are in fact independent. In the remaining two subsections, we consider the special cases of replete (2.2) and right cancellative (2.3) constellations; we show that some simplification of conditions is possible in the latter case, but not in the former.

In Section 3, we introduce an ordering onto the constellations and consider the defining conditions for an inductive constellation. We show that some slight simplification is possible, and then show that the remaining conditions are independent. We then go on to consider the special cases of inductive replete constellations and inductive right cancellative constellations. Throughout Sections 2 and 3, we provide examples of the objects under consideration. However, we omit the (in most cases, easy) verification that these examples do indeed have the required properties.

Although all of the examples in this paper were computer-generated, it is still useful to have techniques available for the manual verification of their various properties. We therefore conclude the paper with an appendix which sketches some such techniques.

Certain conventions will be adopted throughout this paper. To begin with, since we will be considering such examples in our independence proofs, we will use the term *quasi-constellation* to refer to an object which satisfies all but one of the required conditions for a constellation. The examples $C_{2.3(\mathrm{a})}$, $C_{2.3(\mathrm{b})}$, $C_{2.3(\mathrm{c})}$ and $C_{2.3(\mathrm{d})}$ in the proof of Theorem 2.3 are all quasi-constellations, for instance. In the ordered case, we will use the term *quasi-inductive* to refer to a constellation which satisfies all of the ordering/inductivity conditions but one. For both quasi-constellations and quasi-inductive constellations, the particular condition which is *not* satisfied in each instance will always be clear from the context.

We will also adopt some conventions for the presentation of the (counter)examples which follow. The elements of the (quasi-)constellations under consideration will be denoted by $a, b, c, \ldots$, depending on the number of elements. We therefore omit the labels from multiplication tables, with the understanding that the elements appear in alphabetical order. Thus, for example, the multiplication table for $C_{2.3(\mathrm{b})}$ is

	a	b
a	a	b
b	b	$\times$

to emphasize the fact that a constellation is 'half a category' and so he adopts a new name: *left category*.

but we will write it simply as

$$\begin{array}{cc} a & b \\ b & \times \end{array}$$

A '$\times$' will indicate an undefined product. Note that the product $a \cdot b$ is obtained from the a-row and the b-column.

Similar 'ordering' conventions will be applied to the descriptions of other features of (inductive) constellations. For example, again in the description of $C_{2.3(\mathrm{b})}$, we specify the unary operation $^+$ (see Definition 2.1) by

$$^+ : \ a, \ a,$$

which is our shorthand for

$$a^+ = a, \ b^+ = a.$$

Later on, when we come to ordered constellations, we will have reason to work with so-called *restrictions* and *corestrictions* (particular partially defined functions of two variables — see Definition 3.1); these will be denoted by $x|y$, where x, y are elements of the constellation satisfying certain conditions. In order to describe these functions, we introduce some more shorthand. For example, we find the following

$$\begin{array}{rcl} a|\bullet & : & a, \ b, \ c, \ \times \\ d|\bullet & : & d, \ d, \ d, \ d \end{array}$$

in (3.1), as the restrictions for $C_{2.10}$; this indicates that

$$\begin{array}{l} a|a = a, \ a|b = b, \ a|c = c, \ \text{but } a|d \text{ is undefined} \\ d|a = d|b = d|c = d|d = d. \end{array}$$

One final convention will be that if $\mathcal{X}$ denotes a set of conditions and (X) is a condition in $\mathcal{X}$, then

$$\mathcal{X}_{(\mathrm{X})} := \mathcal{X} \setminus \{(X)\}.$$

2. Unordered constellations

2.1. Constellations with no extra conditions. Let C be a set and let $\cdot$ be a partial binary operation on C, i.e., a binary operation which is not necessarily defined for every pair of elements from C. If, for $x, y \in C$, the product $x \cdot y$ *is* defined, we denote the fact by '$\exists x \cdot y$'. In expressions such as '$\exists x \cdot (y \cdot z)$', we implicitly assume that $\exists y \cdot z$. An *idempotent* in C (with respect to $\cdot$) is an element $e \in C$ such that $\exists e \cdot e$ and $e \cdot e = e$. The set of idempotents in C will be denoted by $E(C)$. An idempotent $e \in C$ is termed a *left identity* for an element $x \in C$ if

$\exists e \cdot x$ and $e \cdot x = x$. A unary operation $^+$ on C will be called *image idempotent* if its image is contained in $E(C)$; in this case, the set

$$E = \{x^+ : x \in C\} \subseteq E(C)$$

is the *distinguished subset* (of the unary operation $^+$). (Observe that this set-up is designed to mimic that of a left restriction semigroup, which, in the formulation of the 'York school' [9], is defined in terms of a distinguished subsemilattice E. The parallel between (inductive) constellations and left restriction semigroups then suggests that certain semigroup properties may have analogues for constellations — see [6, Lemma 3.6], for example.)

Definition 2.1. [6] Let C be a set and $\cdot$ be a partial binary operation on C. Suppose further that $^+$ is an image idempotent unary operation on C with distinguished subset E. We call $(C, \cdot, ^+)$ a *constellation (with distinguished subset E)* if the following conditions hold:

- (C1) if $\exists x \cdot (y \cdot z)$, then $\exists (x \cdot y) \cdot z$, in which case, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
- (C2) $\exists x \cdot (y \cdot z)$ if and only if $\exists x \cdot y$ and $\exists y \cdot z$;
- (C3) for each $x \in C$, x^+ is the unique left identity for x in E;
- (C4) if $\exists a \cdot g$, for $a \in C$ and $g \in E$, then $a \cdot g = a$.

Observe the similarity between this definition and that of a category found in [12, p. 78] or [10, Definition 3.1]. Conditions (C1)–(C3) are straightforward analogues (though with range deleted from (C3)). Condition (C4), on the other hand, does not appear in the definition of a category but it is easy to show that a category does indeed satisfy an analogue of this condition. Our mimicry of the definition of a category also explains the presence of the two conditions (C1) and (C2), which might just as well be combined into a single condition. We will retain (C1) and (C2) as separate conditions for most of the paper, for the reasons outlined in the introduction, though we hope to develop a new, combined condition in future work.[3]

We note that, strictly speaking, the object defined in Definition 2.1 should be termed a *left* constellation, since it is also possible to define its left-right dual. However, since we will be dealing only with constellations as defined in Definition 2.1, we will omit the 'left' (but see footnote 2 on page 3). In addition, we will usually discard the phrase 'with distinguished subset E', reinstating it only when it is needed to aid clarity. Finally, we note that since, in most cases, the operations $\cdot$ and

[3]Such a combined condition has in fact already been considered by Cockett — see footnote 2 on page 3.

$^+$ will be clear, we will refer to 'the constellation C', rather than the longer-winded 'the constellation $(C, \cdot, ^+)$'.

Example 2.2. *The following multiplication table defines a constellation $C_{2.2}$:*[4]

$$\begin{array}{cccc} a & c & c & a \\ \times & \times & \times & \times \\ c & a & a & c \\ \times & b & \times & d \end{array} \qquad ^+ : \ a, \ d, \ a, \ d$$

Let $\mathcal{C}$ denote the set of conditions given in Definition 2.1:

$$\mathcal{C} := \{\text{(C1), (C2), (C3), (C4)}\}.$$

Theorem 2.3. *$\mathcal{C}$ forms a set of independent defining conditions for a constellation.*

Proof. We present four counterexamples:

$\mathcal{C}_{(\mathrm{C1})} \nvdash$ (C1). Let $C_{2.3(\mathrm{a})}$ be the following quasi-constellation:

$$\begin{array}{ccc} c & a & a \\ b & c & b \\ a & b & c \end{array} \qquad ^+ : \ c, \ c, \ c$$

(C1) fails: $\exists a \cdot (b \cdot a)$ and $\exists (a \cdot b) \cdot a$ but $a \cdot (b \cdot a) \neq (a \cdot b) \cdot a$.

$\mathcal{C}_{(\mathrm{C2})} \nvdash$ (C2). Let $C_{2.3(\mathrm{b})}$ be the following quasi-constellation:

$$\begin{array}{cc} a & b \\ b & \times \end{array} \qquad ^+ : \ a, \ a$$

(C2) fails: $\exists b \cdot a$ and $\exists a \cdot b$, but $\nexists b \cdot (a \cdot b)$.

$\mathcal{C}_{(\mathrm{C3})} \nvdash$ (C3). Let $C_{2.3(\mathrm{c})}$ be the following quasi-constellation:

$$\begin{array}{cccc} \times & \times & \times & \times \\ c & b & c & \times \\ \times & \times & \times & \times \\ d & d & d & d \end{array} \qquad ^+ : \ b, \ b, \ b, \ d$$

(C3) fails: a^+ is not a left identity for a.

[4]Note that here, and also in subsequent theorems, there are smaller examples that we may choose. The examples presented here, however, have been selected because they may be reused later on.

$\mathcal{C}_{(\mathrm{C4})} \nvDash$ (C4). Let $C_{2.3(\mathrm{d})}$ be the following quasi-constellation:

$$\begin{array}{ccccc} a & \times & \times & \times & \times \\ c & \times & \times & \times & \times \\ c & \times & \times & \times & \times \\ d & b & c & d & \times \\ e & e & e & e & e \end{array} \qquad {}^{+}: \ a,\ d,\ d,\ d,\ e$$

(C4) fails: $\exists b \cdot a$ but $b \cdot a \neq b$. □

Notice that we may characterise E as the set of all idempotents e in C such that $e^{+} = e$. In general, we will not have $E = E(C)$.

2.2. Replete constellations.

Definition 2.4. [6] A constellation $(C, \cdot, ^{+})$ is called *replete* if

(C5) $E = E(C)$.

The significance of replete constellations comes from the fact that, under the correspondence of inductive constellations to left restriction semigroups, inductive replete constellations correspond to *full* left restriction semigroups: those left restriction semigroups whose distinguished subsemilattice E is in fact the whole subset of idempotents of the semigroup [6, Corollary 5.1].

The following demonstrates that not every constellation is replete:

Example 2.5. *The constellation $C_{2.2}$ is replete. On the other hand, let $C_{2.5}$ be the following constellation:*

$$\begin{array}{cc} a & a \\ a & b \end{array} \qquad {}^{+}: \ b,\ b$$

Observe that $C_{2.5}$ is not replete, since $E(C_{2.5}) = C_{2.5}$ but $E = \{b\}$.

Let $\mathcal{R}$ denote the set of conditions which define a replete constellation:

$$\mathcal{R} := \mathcal{C} \cup \{(\mathrm{C5})\}.$$

Theorem 2.6. *$\mathcal{R}$ forms a set of independent defining conditions for a replete constellation.*

Proof. We present five counterexamples:

$\mathcal{R}_{(\mathrm{C1})} \nvDash$ (C1). The quasi-constellation $C_{2.3(\mathrm{a})}$ is replete but does not satisfy (C1).

$\mathcal{R}_{(\mathrm{C2})} \nvDash$ (C2). The quasi-constellation $C_{2.3(\mathrm{b})}$ is replete but does not satisfy (C2).

$\mathcal{R}_{(\mathrm{C3})} \nvDash$ (C3). The quasi-constellation $C_{2.3(\mathrm{c})}$ is replete but does not satisfy (C3).

$\mathcal{R}_{(\mathrm{C4})} \nvdash$ (C4). The quasi-constellation $C_{2.3(\mathrm{d})}$ is replete but does not satisfy (C4).

$\mathcal{R}_{(\mathrm{C5})} \nvdash$ (C5). The constellation $C_{2.5}$ is not replete. □

2.3. Right cancellative constellations.

Definition 2.7. [6] A constellation $(C, \cdot, {}^+)$ is called *right cancellative* if it satisfies the following additional condition:

(C6) if $\exists x \cdot z$, $\exists y \cdot z$ and $x \cdot z = y \cdot z$, then $x = y$.

The significance of right cancellative constellations comes from the fact that, under the correspondence of inductive constellations to left restriction semigroups, inductive right cancellative constellations correspond to a special case of left restriction semigroups, termed *left ample semigroups* [6, Corollary 5.3]. These semigroups have been studied extensively by Fountain [3,4], amongst others — see [9].

Example 2.8. *The constellation $C_{2.2}$ is right cancellative.*

Recall that $C_{2.2}$ is also replete. Indeed, in general, we have the following:

Lemma 2.9. [8, Lemma 9.1.6] *Every right cancellative constellation is replete.*

Proof. We know that $E \subseteq E(C)$; we need to show the reverse inclusion. Suppose that $e \in E(C)$, i.e., $\exists e \cdot e$ and $e \cdot e = e$. But $\exists e^+ \cdot e$ and $e^+ \cdot e = e$, whence $e^+ = e$, by right cancellation. Thus $e \in E$, as required. □

Thus, since not every constellation is replete (Example 2.5), we conclude that not every constellation is right cancellative. The following example demonstrates further that not every replete constellation is right cancellative:

Example 2.10. *Let $C_{2.10}$ be the following constellation:*

$$\begin{array}{cccc} a & b & c & \times \\ b & a & c & \times \\ \times & \times & \times & \times \\ d & d & d & d \end{array} \qquad {}^+ : \ a,\ a,\ a,\ d$$

Then $C_{2.10}$ is replete but not right cancellative, since $\exists a \cdot c$ and $\exists b \cdot c$ with $a \cdot c = b \cdot c$.

A simplification to the conditions for a right cancellative constellation has already been noted in [8]:

Lemma 2.11. [8, Lemma 9.1.7] $\{$(C1), (C2), (C6)$\} \vdash$ (C4).

Proof. Let $a \in C$ and $g \in E$ be such that $\exists a \cdot g$. Then, since $\exists g \cdot g$, we deduce that $\exists a \cdot (g \cdot g)$, by (C2). Further, $\exists (a \cdot g) \cdot g$ and $(a \cdot g) \cdot g = a \cdot (g \cdot g)$, by (C1). Thus $(a \cdot g) \cdot g = a \cdot g$, since g is idempotent, and $a \cdot g = a$, by (C6). □

Moreover, a further simplification is possible if we introduce the following weakened version of condition (C3), where we drop uniqueness:

(C3′) for each $x \in C$, x^+ is a left identity for x.

(We note that the dropping of uniqueness from (C3) seems to be appropriate since uniqueness is not an algebraic property.)

Lemma 2.12. $\{(\text{C3}'), (\text{C6})\} \vdash (\text{C3})$.

Proof. Suppose that $x \in C$ has a second left identity y, i.e., $\exists y \cdot x$ and $y \cdot x = x$. But we know that $\exists x^+ \cdot x$ and $x^+ \cdot x = x = y \cdot x$. It therefore follows from right cancellation that $y = x^+$. □

Thus, conditions (C1), (C2), (C3′) and (C6) suffice to define a right cancellative constellation. Let us denote this set of conditions by $\mathcal{RC}$.

Theorem 2.13. *$\mathcal{RC}$ forms a set of independent defining conditions for a right cancellative constellation.*

Proof. We present four counterexamples:

$\mathcal{RC}_{(\text{C1})} \nvdash (\text{C1})$. The quasi-constellation $C_{2.3(\text{a})}$ is right cancellative but does not satisfy (C1).

$\mathcal{RC}_{(\text{C2})} \nvdash (\text{C2})$. The quasi-constellation $C_{2.3(\text{b})}$ is right cancellative but does not satisfy (C2).

$\mathcal{RC}_{(\text{C3}')} \nvdash (\text{C3}')$. The quasi-constellation $C_{2.3(\text{c})}$ is right cancellative but does not satisfy (C3′).

$\mathcal{RC}_{(\text{C6})} \nvdash (\text{C6})$. The constellation $C_{2.10}$ is not right cancellative. □

3. Ordered and inductive constellations

We now investigate the introduction of a partial order to the various types of constellations considered in the previous section.

3.1. Inductive constellations.

Definition 3.1. [6] Let $(C, \cdot, ^+)$ be a constellation and let C be partially ordered by $\leq$. We call $(C, \cdot, ^+, \leq)$ an *ordered constellation* if the following conditions hold:

(O1) if $a \leq c$, $b \leq d$, $\exists a \cdot b$ and $\exists c \cdot d$, then $a \cdot b \leq c \cdot d$;

(O2) if $a \leq b$, then $a^+ \leq b^+$;

(O3) for $e \in E$ and $a \in C$ such that $e \leq a^+$, there exists a *restriction* $e|a$ which is the unique element x with the properties $x \leq a$ and $x^+ = e$;

(O4) for all $e \in E$ and all $a \in C$, there exists a *corestriction* $a|e$ which is the maximum element x with the properties $x \leq a$ and $\exists x \cdot e$;

(O5) for $x, y \in C$ and $e \in E$, if $\exists x \cdot y$, then $((x \cdot y)|e)^+ = (x|(y|e)^+)^+$;

(O6) if $e, f \in E$, then, whenever the restriction $e|f$ is defined, it coincides with the corresponding corestriction.

Ordered replete constellations and ordered right cancellative constellations are simply replete constellations and right cancellative constellations, respectively, which possess a partial order satisfying (O1)–(O6).

As with the conditions given in Definition 2.1, the above conditions for the ordering of a constellation, as well as the inductive condition of Definition 3.3 below, are designed to mimic the definition of the ordering on an inductive groupoid given in [12, p. 108], which may also be applied more generally to an inductive category — see, for example, [10, Definition 3.2]. In particular, the presence of the restriction and corestriction operations is intended to emphasis the parallel between inductive constellations and inductive categories for those who are used to working with the latter. It also suggests constellation analogues of certain results for inductive categories and groupoids — compare [6, Lemma 3.4] with [12, Lemma 4.1.3], for example.

We will denote by ι the equality relation on a set.

Example 3.2. *If we endow the constellation $C_{2.10}$ with the following ordering*

$$\leq \; = \iota \cup \{(d,a),(d,b),(d,c)\}$$

$$\begin{array}{lcl} a|\bullet & : & a,\ b,\ c,\ \times \\ d|\bullet & : & d,\ d,\ d,\ d \end{array} \qquad \begin{array}{lcl} \bullet|a & : & a,\ b,\ d,\ d \\ \bullet|d & : & d,\ d,\ d,\ d \end{array} \tag{3.1}$$

then it becomes an ordered constellation.

As indicated in the introduction, it is so-called *inductive* constellations that we are most interested in, given their fundamental connection with left restriction semigroups. These are a special type of ordered constellation. Let $(C, \cdot, ^+, \leq)$ be an ordered constellation, and let $e, f \in E$; if the greatest lower bound of e and f exists (with respect to $\leq$), we denote it by $e \wedge f$.

Definition 3.3. Let $(C, \cdot, ^+, \leq)$ be an ordered constellation. We call $(C, \cdot, ^+, \leq)$ an *inductive constellation* if it satisfies the following additional condition:

(I) $e \wedge f$ exists in E for all $e, f \in E$ and is equal to $e|f$.

Inductive replete constellations and inductive right cancellative constellations are simply replete constellations and right cancellative constellations, respectively, which possess a partial order satisfying (O1)–(O6) and (I).

Example 3.4. *The constellation $C_{2.10}$ with the ordering of* (3.1) *is inductive with $a \wedge d = d$.*

In subsequent examples, we will not specify $\wedge$, since it can easily be read off from the description of the ordering. It is clear that if C is an ordered constellation with $|E| = 1$, then C is necessarily inductive. Observe also that if D is an ordered constellation with $|E| > 1$ and $\leq\, = \iota$, then D *cannot* be inductive, since $e, f \in E$ will not have any common lower bounds.

The following demonstrates that not every ordered constellation is inductive:

Example 3.5. *Let $C_{3.5}$ be the following constellation:*

$$\begin{array}{cc} a & a \\ b & b \end{array} \qquad {}^{+} :\ a,\ b$$

We endow $C_{3.5}$ with the following ordering:

$$\leq\, = \iota \cup \{(b, a)\} \qquad \begin{array}{ccl} a|\bullet & : & a,\ \times \\ b|\bullet & : & b,\ b \end{array} \qquad \begin{array}{ccl} \bullet|a & : & a,\ b \\ \bullet|b & : & a,\ b \end{array} \tag{3.2}$$

But $C_{3.5}$ is not inductive under this ordering, since $a \wedge b = b \neq a|b$.

In earlier papers [6,7], we have started by considering ordered constellations and then added (I) to study inductive constellations. In the present paper, however, we will not take the intermediate step of studying ordered constellations, but will move straight to the inductive case.

Lemma 3.6. $\{$(O3), (O4), (I)$\} \vdash$ (O6).

Proof. Suppose that $e \leq f$, for $e, f \in E$, so that the restriction $e|f$ is defined. Notice that e also satisfies the defining conditions of this restriction ($e \leq f$ and $e^+ = e$), so, by uniqueness of restrictions, we have that $e|f = e$, as a restriction.

We now consider $e|f$ as a corestriction. We have $e|f = e \wedge f = e$, using (I), together with the fact that $e \leq f$. We see then that the restriction and corestriction $e|f$ coincide. □

Thus, in the presence of (I), we may discard condition (O6).

Note that (O5) depends on (O4): we cannot make statements about the corestriction without first defining it. This is somewhat problematic when it comes to showing the independence of these conditions: for example, how can we verify (O5) in the case where we negate (O4)? To get around this problem, we replace (O5) by the following slightly reworded condition:

(O5′) for $x, y \in C$ and $e \in E$, if $\exists x \cdot y$ and if the appropriate corestrictions are defined, then $((x \cdot y)|e)^+ = (x|(y|e)^+)^+$.

It is clear that in the presence of (O4), (O5′) implies (O5). In fact, we must make a similar replacement in the case of condition (I):

(I′) $e \wedge f$ exists in E for all $e, f \in E$ and is equal to the corestriction $e|f$, whenever the latter is defined.

Lemma 3.7. {(C2), (C3′), (O4), (I)} $\vdash$ (C3).

Proof. Suppose that $\exists y \cdot x$ with $y \cdot x = x$ and $y \in E$. We will show that $y = x^+$. First of all, note that since $\exists y \cdot x$ and $y \cdot x = y \cdot (x^+ \cdot x)$, we have $\exists y \cdot x^+$, by (C2). This, together with the fact that $y \leq y$, gives us $y \leq y|x^+$, by maximality of corestrictions. But then

$$y \leq y|x^+ = y \wedge x^+ \leq y,$$

so that $y \wedge x^+ = y$. We conclude that $y \leq x^+$.

For the reverse inequality, we first note that since $y \cdot x = x$ and $\exists x^+ \cdot x$, we have $\exists x^+ \cdot (y \cdot x)$. It follows from (C2) that $\exists x^+ \cdot y$, and so $x^+ \leq x^+|y$, by maximality of the corestriction $x^+|y$. But then, as above,

$$x^+ \leq x^+|y = x^+ \wedge y \leq x^+,$$

whence $x^+ \leq y$ and so $x^+ = y$, as required. □

Let $\mathcal{O}$ denote the following set of conditions

$$\mathcal{O} = \{(O1), (O2), (O3), (O4), (O5'), (I')\}$$

and let $\mathcal{C}'$ denote the set $\mathcal{C}$ with (C3) replaced by (C3′). We put $\mathcal{I} = \mathcal{C}' \cup \mathcal{O}$; the conditions in $\mathcal{I}$ suffice to define an inductive constellation.

Theorem 3.8. *$\mathcal{I}$ forms a set of independent defining conditions for an inductive constellation.*

Proof. We present ten counterexamples:

$\mathcal{I}_{(C1)} \nvdash$ (C1). We endow the quasi-constellation $C_{2.3(a)}$ with the ordering

$$\leq = \iota \qquad c|\bullet \ : \ a,\ b,\ c \qquad \bullet\,|c \ : \ a,\ b,\ c \tag{3.3}$$

Then $C_{2.3(a)}$ is inductive but does not satisfy (C1).

$\mathcal{I}_{(\mathrm{C2})} \nvDash$ (C2). We endow the quasi-constellation $C_{2.3(\mathrm{b})}$ with the ordering

$$\leq\, = \iota \qquad a|\bullet :\ a,\ b \qquad \bullet\,|a :\ a,\ b \tag{3.4}$$

Then $C_{2.3(\mathrm{b})}$ is inductive but does not satisfy (C2).

$\mathcal{I}_{(\mathrm{C3}')} \nvDash$ (C3′). We endow the quasi-constellation $C_{2.3(\mathrm{c})}$ with the ordering:

$$\leq\, = \iota \cup \{(d,a),(d,b),(d,c)\}$$

$$\begin{array}{lcl} b|\bullet & : & a,\ b,\ c,\ \times \\ d|\bullet & : & d,\ d,\ d,\ d \end{array} \qquad \begin{array}{lcl} \bullet|b & : & d,\ b,\ d,\ d \\ \bullet|d & : & d,\ d,\ d,\ d \end{array} \tag{3.5}$$

Then $C_{2.3(\mathrm{c})}$ is inductive but does not satisfy (C3′).

$\mathcal{I}_{(\mathrm{C4})} \nvDash$ (C4). We endow the quasi-constellation $C_{2.3(\mathrm{d})}$ with the ordering:

$$\leq\, = \iota \cup \{(d,a),(e,a),(e,b),(e,c),(e,d)\}$$

$$\begin{array}{lcl} a|\bullet & : & a,\ \times,\ \times,\ \times,\ \times \\ d|\bullet & : & d,\ b,\ c,\ d,\ \times \\ e|\bullet & : & e,\ e,\ e,\ e,\ e \end{array} \qquad \begin{array}{lcl} \bullet|a & : & a,\ b,\ c,\ d,\ e \\ \bullet|d & : & d,\ e,\ e,\ d,\ e \\ \bullet|e & : & e,\ e,\ e,\ e,\ e \end{array} \tag{3.6}$$

Then $C_{2.3(\mathrm{d})}$ is inductive but does not satisfy (C4).

$\mathcal{I}_{(\mathrm{O1})} \nvDash$ (O1). We endow the constellation $C_{2.2}$ with the ordering:

$$\leq\, = \iota \cup \{(a,b),(a,d)\} \qquad \begin{array}{lcl} a|\bullet & : & a,\ a,\ c,\ a \\ d|\bullet & : & \times,\ b,\ \times,\ d \end{array} \qquad \begin{array}{lcl} \bullet|a & : & a,\ a,\ c,\ d \\ \bullet|d & : & a,\ a,\ c,\ d \end{array} \tag{3.7}$$

(O1) fails: $a \leq a$, $a \leq b$, $\exists a \cdot a$ and $\exists a \cdot b$, but $(a \cdot a, a \cdot b) \notin\, \leq$.

$\mathcal{I}_{(\mathrm{O2})} \nvDash$ (O2). Let $C_{3.8(\mathrm{a})}$ be the following constellation:

$$\begin{array}{ccccc} a & \times & \times & \times & e \\ b & \times & \times & \times & c \\ \times & b & c & \times & \times \\ d & d & d & d & d \\ \times & a & e & \times & \times \end{array} \qquad {}^{+} :\ a,\ c,\ c,\ d,\ a$$

We endow $C_{3.8(\mathrm{a})}$ with the ordering:

$$\leq\, = \iota \cup \{(a,b),(d,a),(d,b),(d,c),(d,e),(e,c)\}$$

$$\begin{array}{lcl} a|\bullet & : & a,\ \times,\ \times,\ \times,\ e \\ c|\bullet & : & \times,\ b,\ c,\ \times,\ \times \\ d|\bullet & : & d,\ d,\ d,\ d,\ d \end{array} \qquad \begin{array}{lcl} \bullet|a & : & a,\ b,\ d,\ d,\ d \\ \bullet|c & : & d,\ d,\ c,\ d,\ e \\ \bullet|d & : & d,\ d,\ d,\ d,\ d \end{array} \tag{3.8}$$

(O2) fails: $a \leq b$ but $(a^+, b^+) \notin\, \leq$.

$\mathcal{I}_{(\mathrm{O3})} \nvDash (\mathrm{O3})$. Let $C_{3.8(\mathrm{b})}$ be the following constellation:

$$\begin{array}{cc} \times & \times \\ a & b \end{array} \qquad {}^{+} : \ b, \ b$$

We endow $C_{3.8(\mathrm{b})}$ with the ordering

$$\leq \, = \iota \cup \{(b, a)\} \qquad \bullet\,|b \ : \ b, \ b \tag{3.9}$$

(O3) fails: there is no unique restriction $b|a$, since both a and b possess the required properties.

$\mathcal{I}_{(\mathrm{O4})} \nvDash (\mathrm{O4})$. Let $C_{3.8(\mathrm{c})}$ be the following constellation:

$$\begin{array}{cc} a & b \\ \times & \times \end{array} \qquad {}^{+} : \ a, \ a$$

We endow $C_{3.8(\mathrm{c})}$ with the following ordering:

$$\leq \, = \iota \qquad a|\bullet : \ a, \ b \tag{3.10}$$

(O4) fails: there is no element x with $x \leq b$ and $\exists x \cdot a$, i.e., no corestriction $b|a$.

$\mathcal{I}_{(\mathrm{O5}')} \nvDash (\mathrm{O5}')$. Let $C_{3.8(\mathrm{d})}$ be the following constellation:

$$\begin{array}{ccccc} a & e & a & \times & e \\ \times & c & b & \times & \times \\ \times & b & c & \times & \times \\ d & d & d & d & d \\ \times & a & e & \times & \times \end{array} \qquad + : \ a, \ c, \ c, \ d, \ a$$

We endow $C_{3.8(\mathrm{d})}$ with the following ordering:

$$\leq \, = \iota \cup \{(a, c), (d, a), (d, b), (d, c), (d, e), (e, b)\}$$

$$\begin{array}{lcl} a|\bullet & : & a, \ e, \ a, \ \times, \ e \\ c|\bullet & : & \times, \ b, \ c, \ \times, \ \times \\ d|\bullet & : & d, \ d, \ d, \ d, \ d \end{array} \qquad \begin{array}{lcl} \bullet|a & : & a, \ d, \ a, \ d, \ d \\ \bullet|c & : & a, \ b, \ c, \ d, \ e \\ \bullet|d & : & d, \ d, \ d, \ d, \ d \end{array} \tag{3.11}$$

(O5′) fails: $\exists b \cdot b$ but $(b \cdot b|a)^{+} \neq (b|(b|a)^{+})^{+}$.

$\mathcal{I}_{(\mathrm{I})} \nvDash (\mathrm{I})$. The constellation $C_{3.5}$ is ordered by (3.2) but is not inductive. □

3.2. Inductive replete constellations. Let $\mathcal{R}'$ denote the set $\mathcal{R}$ with (C3) replaced by (C3′), and put $\mathcal{IR} := \mathcal{R}' \cup \mathcal{O}$. Note that we may apply Lemma 3.7 to deduce (C3) from the conditions in $\mathcal{IR}$. Thus, the conditions in $\mathcal{IR}$ suffice to define an inductive replete constellation.

Theorem 3.9. *$\mathcal{IR}$ forms a set of independent defining conditions for an inductive replete constellation.*

Proof. We present eleven counterexamples:

$\mathcal{IR}_{(C1)} \nvDash$ (C1). The quasi-constellation $C_{2.3(a)}$ with the ordering of (3.3) is both replete and inductive, but does not satisfy (C1).

$\mathcal{IR}_{(C2)} \nvDash$ (C2). The quasi-constellation $C_{2.3(b)}$ with the ordering of (3.4) is both replete and inductive, but does not satisfy (C2).

$\mathcal{IR}_{(C3')} \nvDash$ (C3′). The quasi-constellation $C_{2.3(c)}$ with the ordering of (3.5) is both replete and inductive, but does not satisfy (C3′).

$\mathcal{IR}_{(C4)} \nvDash$ (C4). The quasi-constellation $C_{2.3(d)}$ with the ordering of (3.6) is both replete and inductive, but does not satisfy (C4).

$\mathcal{IR}_{(C5)} \nvDash$ (C5). We endow the constellation $C_{2.5}$ with the ordering

$$\leq = \iota \qquad b|\bullet \ : \ a, \ b \qquad \bullet\,|b \ : \ a, \ b$$

Then $C_{2.5}$ is inductive but not replete.

$\mathcal{IR}_{(O1)} \nvDash$ (O1). The constellation $C_{2.2}$ with the ordering of (3.7) is replete and quasi-inductive, but does not satisfy (O1).

$\mathcal{IR}_{(O2)} \nvDash$ (O2). The constellation $C_{3.8(a)}$ with the ordering of (3.8) is replete and quasi-inductive, but does not satisfy (O2).

$\mathcal{IR}_{(O3)} \nvDash$ (O3). The constellation $C_{3.8(b)}$ with the ordering of (3.9) is replete and quasi-inductive, but does not satisfy (O3).

$\mathcal{IR}_{(O4)} \nvDash$ (O4). The constellation $C_{3.8(c)}$ with the ordering (3.10) is replete and quasi-inductive, but does not satisfy (O4).

$\mathcal{IR}_{(O5')} \nvDash$ (O5′). The constellation $C_{3.8(d)}$ with the ordering of (3.11) is replete and quasi-inductive, but does not satisfy (O5′).

$\mathcal{IR}_{(I)} \nvDash$ (I). The constellation $C_{3.5}$ with the ordering of (3.2) is replete and quasi-inductive, but does not satisfy (I). □

3.3. Inductive right cancellative constellations. Recall that the set $\mathcal{RC}$ is a set of independent defining conditions for a right cancellative constellation. We put $\mathcal{IRC} := \mathcal{RC} \cup \mathcal{O}$, so that the conditions in $\mathcal{IRC}$ suffice to define an inductive right cancellative constellation.

Theorem 3.10. *$\mathcal{IRC}$ forms a set of independent defining conditions for an inductive right cancellative constellation.*

Proof. We present ten counterexamples:

$\mathcal{IRC}_{(C1)} \nvdash$ (C1). The quasi-constellation $C_{2.3(a)}$ with the ordering of (3.3) is both right cancellative and inductive, but does not satisfy (C1).

$\mathcal{IRC}_{(C2)} \nvdash$ (C2). The quasi-constellation $C_{2.3(b)}$ with the ordering of (3.4) is both right cancellative and inductive, but does not satisfy (C2).

$\mathcal{IRC}_{(C3')} \nvdash$ (C3′). The quasi-constellation $C_{2.3(c)}$ with the ordering of (3.5) is both right cancellative and inductive, but does not satisfy (C3′).

$\mathcal{IRC}_{(C6)} \nvdash$ (C6). The constellation $C_{2.10}$ with the ordering of (3.1) is inductive, but not right cancellative.

$\mathcal{IRC}_{(O1)} \nvdash$ (O1). The constellation $C_{2.2}$ with the ordering of (3.7) is right cancellative and quasi-inductive, but does not satisfy (O1).

$\mathcal{IRC}_{(O2)} \nvdash$ (O2). The constellation $C_{3.8(a)}$ with the ordering of (3.8) is right cancellative and quasi-inductive, but does not satisfy (O2).

$\mathcal{IRC}_{(O3)} \nvdash$ (O3). The constellation $C_{3.8(b)}$ with the ordering of (3.9) is right cancellative and quasi-inductive, but does not satisfy (O3).

$\mathcal{IRC}_{(O4)} \nvdash$ (O4). The constellation $C_{3.8(c)}$ with the ordering (3.10) is right cancellative and quasi-inductive, but does not satisfy (O4).

$\mathcal{IRC}_{(O5')} \nvdash$ (O5′). The constellation $C_{3.8(d)}$ with the ordering of (3.11) is right cancellative and quasi-inductive, but does not satisfy (O5′).

$\mathcal{IRC}_{(I)} \nvdash$ (I). The constellation $C_{3.5}$ with the ordering of (3.2) is right cancellative and quasi-inductive, but does not satisfy (I). □

Appendix A. Techniques for handling constellations

Although the counterexamples presented here are all computer-generated, it is still useful for us to have in hand some manual techniques for verifying the conditions for constellations. This appendix summarises some of these methods.

A.1. Unordered constellations.

A.1.1. *A generalised Light's associativity test.* In [2], Clifford and Preston presented a method, *Light's associativity test*, for verifying the associativity of a binary operation given by a small Cayley table. We can apply a generalised version of this test in the case of constellations. Suppose that we are given the following multiplication table for a finite constellation $C = \{a, b, c, \ldots\}$:

$\cdot$	a	b	c	$\cdots$
a	x_{11}	x_{12}	x_{13}	$\cdots$
b	x_{21}	x_{22}	x_{23}	$\cdots$
c	x_{31}	x_{32}	x_{33}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The entries x_{ij} are drawn from the set $C \cup \{\times\}$, where, as usual, the symbol '$\times$' will indicate an undefined product. For each $x \in C$, we define two new operations:

$$u * v = u \cdot (x \cdot v) \qquad \text{and} \qquad u \circ v = (u \cdot x) \cdot v.$$

The operation $\cdot$ is then associative if and only if the two operations $*$ and $\circ$ coincide for all $x \in C$. The $*$-table is obtained from the $\cdot$-table by replacing the y-column by the $x \cdot y$-column, for each $y \in C$ (if $\nexists x \cdot y$, we replace the y-column by a column of $\times$s). Similarly, the $\circ$-table is obtained from the $\cdot$-table by replacing the z-row by the $z \cdot x$-row, for each $x \in C$ (if $\nexists z \cdot x$, we replace the z-row by a row of $\times$s). In fact, we need not write down the $\circ$-table: we need only compare the z-row of the $*$-table with the $z \cdot x$-row of the $\cdot$-table. If these agree (modulo the comments which follow), then $*$ and $\circ$ coincide for that value of x.

The fact that our operation $\cdot$ is partial, together with the fact that the implication in (C1) goes only one way, means that there is an extra complication for us to watch out for when applying this generalised associativity test. When we compare the z-row of the $*$-table for x with the $z \cdot x$-row of the $\cdot$-table, we are comparing the products $z * v = z \cdot (x \cdot v)$ and $z \circ v = (z \cdot x) \cdot v$, for each $v \in C$. The existence of the product $z \cdot (x \cdot v)$ implies that of the product $(z \cdot x) \cdot v$, but not vice versa. This means that $(z \cdot x) \cdot v$ may be defined when $z \cdot (x \cdot v)$ is not, but not vice versa. Thus, we may have an element from C in a particular position in the $z \cdot x$-row of the $\cdot$-table, but only a '$\times$' in the corresponding position in the z-row of the $*$-table. The converse situation, however, may not occur: if an element of C appears in the $*$-table, then it must also appear in the appropriate position in the $\cdot$-table.

We illustrate this generalised associativity test with an example. Consider $C_{2.2}$:

$\cdot$	a	b	c	d
a	a	c	c	a
b	$\times$	$\times$	$\times$	$\times$
c	c	a	a	c
d	$\times$	b	$\times$	d

We draw the $*$-table for $x = a$:

$*$	a	b	c	d
a	a	c	c	a
b	$\times$	$\times$	$\times$	$\times$
c	c	a	a	c
d	$\times$	$\times$	$\times$	$\times$

Observe that the a-row of this second table coincides with the $a \cdot a = a$-row of the $\cdot$-table. Similarly, the c-row of the $*$-table agrees with the $c \cdot a = c$-row of the $\cdot$-table. The b- and d-rows of the $*$-table are rows of $\times$s because $\nexists b \cdot a$ and $\nexists d \cdot a$. Continuing in this way for $x = b, c, d$, we may verify condition (C1). In fact, we may also use the $*$-tables to verify (C2). The entry in the s-row and t-column of the $*$-table for $x = a$, for example, represents the value $s \cdot (a \cdot t)$. If a '$\times$' appears in this position, then the product $s \cdot (a \cdot t)$ is undefined; we should then be able to glance at the $\cdot$-table and see that one or both of $s \cdot a$ and $a \cdot t$ is undefined, thereby checking (C2).

We can also use $C_{2.2}$ to illustrate the phenomenon described two paragraphs ago by drawing the $*$-table for $x = b$:

$*$	a	b	c	d
a	$\times$	$\times$	$\times$	$\times$
b	$\times$	$\times$	$\times$	$\times$
c	$\times$	$\times$	$\times$	$\times$
d	$\times$	$\times$	$\times$	$\times$

The d-row of this $*$-table coincides with the $d \cdot b = b$-row of the $\cdot$-table, whilst the b-row of the $*$-table consists entirely of $\times$s because $\nexists b \cdot b$. The a- and c-rows, however, illustrate our earlier point: the a-row of the $*$-table should agree with the $a \cdot b = c$-row of the $\cdot$-table, but it instead contains only $\times$s. This is because, for example, $\exists (a \cdot b) \cdot a$, but $\nexists a \cdot (b \cdot a)$.

A.1.2. *Further comments.* When verifying the conditions for unordered constellations, we can often make our work easier by spotting certain tricks. For instance, observe that condition (C2) is an 'existence' condition and has nothing to say about the values of products. Thus, when checking the conditions for a constellation such as $C_{2.5}$, in which all products are defined, (C2) is immediate.

Conditions (C3) and (C4), which relate to the behaviour of the 'distinguished idempotents' in E are easily read off from the multiplication table, as is (C5) for replete constellations: we simply identify the idempotents of the constellation by reading down the leading diagonal of the multiplication table and then compare these with the elements appearing as images of the unary operation $^+$.

As in the case of multiplication tables for right cancellative semigroups, we can check condition (C6) for right cancellative constellations by examining the table and checking that no column contains any repetitions (other than $\times$).

A.2. Inductive constellations. The verification that the ordering on a constellation satisfies the required properties can be rather longer than that of conditions (C1)–(C4) (and perhaps (C5) and (C6)). We suggest some methods for tackling this.

A.2.1. *Partial ordering.* The way in which we have presented the ordering, in the form $\leq\, = \iota \cup \kappa$ ($\kappa \subseteq C \times C$ and $\iota \cap \kappa = \emptyset$), for each ordered constellation makes it very easy to check that this is indeed a partial order. First of all, it is clear that the ordering is reflexive, since we have explicitly stated that the equality relation ι is contained in it. To verify anti-symmetry, we simply examine κ and check that if $(x, y) \in \kappa$ (where, of course, $x \neq y$, because all pairs (x, x) are contained in ι), then $(y, x) \notin \kappa$. For transitivity, we again examine κ and ensure that if $(x, y), (y, z) \in \kappa$, then $(x, z) \in \kappa$. When checking transitivity, we may ignore ι, since it is clear that the transitivity property holds for $(x, x), (x, y) \in\, \leq$, or indeed, $(x, y), (y, y) \in\, \leq$.

A.2.2. *(O1).* Depending upon the particular ordering under consideration, the verification of (O1) can be rather lengthy. In order to reduce the effort that this takes, if not the time, we suggest the following systematic method. Suppose that $(C, \cdot, ^+, \leq)$ is an ordered constellation, with $\leq$ given in the form $\leq\, = \iota \cup \kappa$. We agree upon some order in which to list the relations in $\leq$; the order chosen is immaterial — it simply allows us to be systematic. Suppose that the list begins $s \leq t$, $u \leq v$, $w \leq x$, $y \leq z$, ... We then draw up a table of the following form:

Relation 1	**Relation 2**	**Reading left → right**	**Reading right → left**
$s \leq t$	$u \leq v$		
"	$w \leq x$		
"	$y \leq z$		
⋮	⋮		
$u \leq v$	$w \leq x$		
"	$y \leq z$		
⋮	⋮		

That is, we take each relation on our list and pair it up in turn with every other relation that appears *after it* on the list. Then, working row by row, we try to compare the two relations in the row, first reading from left to right, and then from right to left. Suppose that both $s \cdot u$ and $t \cdot v$ are defined. Then we must check that $s \cdot u \leq t \cdot v$; we put our verifications in the appropriate row of the table, under the column 'Reading left $\to$ right':

Relation 1	**Relation 2**	**Reading left $\to$ right**	**Reading right $\to$ left**
$s \leq t$	$u \leq v$	$s \cdot u \leq t \cdot v$	
"	$w \leq x$		
"	$y \leq z$		
$\vdots$	$\vdots$		

Suppose, however, that $\nexists u \cdot s$. Thus, when we take relation 2 first, the conditions of (O1) are not satisfied and we therefore have nothing to check. We note this in the table:

Relation 1	**Relation 2**	**Reading left $\to$ right**	**Reading right $\to$ left**
$s \leq t$	$u \leq v$	$s \cdot u \leq t \cdot v$	$\nexists u \cdot s$
"	$w \leq x$		
"	$y \leq z$		
$\vdots$	$\vdots$		

Continuing in this systematic way, we complete the table and thereby verify that (O1) holds.

We illustrate this method by considering it in the case of $C_{2.3(\mathrm{c})}$. We take the relations in $\leq$ in the following order:

$$a \leq a, \quad b \leq b, \quad c \leq c, \quad d \leq a, \quad d \leq b, \quad d \leq c, \quad d \leq d.$$

Then the first few rows of the table will be as follows:

Relation 1	**Relation 2**	**Reading left $\to$ right**	**Reading right $\to$ left**
$a \leq a$	$d \leq a$	$\nexists a \cdot d$	$\nexists a \cdot a$
"	$d \leq b$	"	$d \cdot a = d \leq b \cdot a = c$
"	$d \leq c$	"	$\nexists c \cdot a$
$b \leq b$	$d \leq a$	$\nexists b \cdot d$	$\nexists a \cdot b$
"	$d \leq b$	"	$d \cdot b = d \leq b \cdot b = b$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Thus, the relations $a \leq a$ and $d \leq a$ do not admit any comparison, in either order, since $\nexists a \cdot d$ and $\nexists a \cdot a$. Nor may we compare $a \leq a$ and $d \leq b$, in that order, since $\nexists a \cdot d$. However, we may compare them the other way around, and (O1) does indeed hold in this case, as $\exists d \cdot a$, $\exists b \cdot a$ and $d \cdot a \leq b \cdot a$. Notice that we have not tried to compare $a \leq a$ with $b \leq b$. In fact, $\nexists a \cdot b$, but even if it were defined, there would be no need to compare these relations. More generally, there is no need for us to compare relations of the form $x \leq x$ and $y \leq y$, since, if $\exists x \cdot y$, it follows immediately from reflexivity that $x \cdot y \leq x \cdot y$.

A.2.3. *(O2)–(O5) and (I).* Condition (O2) is particularly easy to check: we simply check that if $(x, y) \in \leq$, then $(x^+, y^+) \in \leq$. Note that we need only check those pairs in κ — (O2) follows for those in ι by reflexivity.

Unfortunately, in the case of (O3) and (O4), there is nothing to be done but to verify the properties of the given restrictions and corestrictions, one by one. However, we note a phenomenon that occurs several times in the examples given in this paper. In the description of the ordering for $C_{2.10}$ in (3.1), for instance, we find the corestriction $c|a = d$. To verify the maximality of this element, we observe that it is in fact the *only* element with the required properties ($c|a \leq c$ and $\exists(c|a) \cdot a$) and is therefore maximal by default.

As with (O1), the verification of condition (O5) can be somewhat laborious, but the work can be lessened by the spotting of some tricks. One example of this is in the case of $C_{3.5}$ (Example 3.5), where we observe that this is in fact a left zero semigroup, i.e., a semigroup with $x \cdot y = x$, for all x, y. Observing also that $x|e = x$, for all $x \in C_{3.5}$ and all $e \in E = C_{3.5}$, we have

$$(x \cdot y|e)^+ = (x|e)^+ = x^+ = (x|y^+)^+ = (x|(y|e)^+)^+,$$

and so there is no need to check each case individually. A similar, though less dramatic, simplification is possible in the case of $C_{2.3(\mathrm{c})}$ (with the ordering given in (3.5)), for example, where we observe that all products and all corestrictions involving d are equal to d. This takes care of all such cases in one go and leaves only the cases $\{x = b,\ y = a,\ e = b\}$, $\{x = y = e = b\}$ and $\{x = b,\ y = c,\ e = b\}$. In fact, the second of these is immediate: for any $e \in E$, we will always have $e \cdot e = e$, $e^+ = e$ and $e|e = e$, and so (O5) follows for this combination.

Condition (I) is particularly easy to verify. In the case of $C_{3.8(\mathrm{d})}$, for example, we have $E = \{a, c, d\}$ and all possible (unordered) pairs of elements from this set appear in the description of $\leq$, allowing us to read off that $a \wedge c = a$ and $a \wedge d = c \wedge d = d$. Indeed, the only example presented here which is in any way tricky, with regard to inductivity, is $C_{3.8(\mathrm{a})}$. Again, we have $E = \{a, c, d\}$, but this time neither (a, c) nor (c, a) appears in $\leq$, so we cannot simply read off $a \wedge c$. We must instead note that d is the only common lower bound for a and c, whence $a \wedge c = d$.

By the comments following Example 3.4, the inductivity of many of the examples in this paper is very easy to verify, simply because they have trivial distinguished subset — $C_{3.8(\mathrm{b})}$, for instance.

A.2.4. *Trivial ordering.* Note that in the case of $C_{2.3(\mathrm{a})}$ (in the proof of Theorem 3.8), for example, we have $\leq\, = \iota$. Indeed, sweeping simplifications take place in the case of trivial ordering. Observe, first of all, that (O1) and (O2) reduce to statements about $\cdot$ and $^+$ being well-defined, something that we assume anyway, and so these conditions become redundant. Further, a restriction $e|a$ is defined only if $e = a^+$, and we necessarily have $e|a = a$. Similarly, we must have the corestriction $a|e = a$. Condition (O5), however, requires a little more care. Before we deal with it, we first note the following lemma:

Lemma A.1. [6, Lemma 2.2] *In a constellation C, if $\exists a \cdot b$, then $(a \cdot b)^+ = a^+$.*

Then, if $\exists x \cdot y$, we have, using Lemma A.1,

$$(x \cdot y|e)^+ = (x \cdot y)^+ = x^+ = (x|y^+)^+ = (x|(y|e)^+.$$

Therefore, it would seem that (O5) is also immediate in the case of trivial ordering. However, we must be cautious: the proof of Lemma A.1 uses (C2) and so (O5) must be checked explicitly in any instance where we do not have (C2) — the case of $C_{2.3(\mathrm{b})}$ in the proof of Theorem 3.8, for example.

Finally, from our comments following Example 3.4, (I) can only hold in the case of trivial ordering if $|E| = 1$.

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