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WHEN PRETORSION CLASSES COINCIDE WITH PRETORSION FREE CLASSES

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Abstract. This paper concerns with the study of pretorsion classes and pretorsion free classes considered as big lattices, ordered by class inclusion. We obtain structural results about these lattices and we apply them to characterize the rings for which these lattices coincide, as the Artinian principal ideal rings.

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1. Introduction

In this work R denotes an associative ring with unitary element 1 and R-mod denotes the category of left unital modules over the ring R.

We shall work with big lattices of classes of R-modules. Among the closure properties we will consider here are those of being closed under: monomorphisms ($\rightarrowtail$), homomorphic images ($\twoheadrightarrow$), direct sums ($\oplus$), direct products (Π), injective hulls ($E(\ )$), projective covers ($P(\ )$) and extensions (ext).

Also we consider classes closed under isomorphic copies of its elements.

If $\mathcal{P}$ is a set of some of the closure properties above, we denote by $\mathcal{L}_{\mathcal{P}}$ the (big) lattice of all classes of R-modules closed under the properties in $\mathcal{P}$. All these lattices have inclusion as their partial order and thus infima is given by intersection. The least element is $\{0\}$, denoted by $\mathbf{0}$, and the greatest element is R-mod, denoted by $\mathbf{1}$.

As examples we can mention the well known frame of hereditary torsion theories in R-mod (R-tors) which here coincides with $\mathcal{L}_{\{\twoheadrightarrow,\oplus,ext,\rightarrowtail\}}$. Another example is the boolean lattice of natural classes in R-mod (R-nat) which here will be denoted by $\mathcal{L}_{\{\rightarrowtail,\oplus,E(\ ),ext\}} = \mathcal{L}_{\{\rightarrowtail,\oplus,E(\ )\}}$ (see [9]).

We can also notice that some of these lattices have an underlying set of elements while other have a proper class of elements, in this last case we use the term big lattice instead of lattice.

In the proof of the main result we use the fact that $\mathcal{L}_{\{\rightarrowtail\}} = \mathcal{L}_{\{\twoheadrightarrow\}}$ is equivalent with R being an Artinian principal ideal ring (see [2, Theorem 38]).

Given $\mathcal{P}$ a set of closure properties and given a class $\mathcal{C}$ of R-modules we let $\xi_{\mathcal{P}}(\mathcal{C})$ denote the least class in $\mathcal{L}_{\mathcal{P}}$ containing $\mathcal{C}$ as a subclass. Respectively we let $\chi_{\mathcal{P}}(\mathcal{C})$ denote the largest class in $\mathcal{L}_{\mathcal{P}}$ in which $\mathcal{C}$ is a subclass.

For a preradical r in R-mod we let $\mathbb{T}_r = \{M \in R\text{-mod} \mid r(M) = M\}$ and $\mathbb{F}_r = \{M \in R\text{-mod} \mid r(M) = 0\}$. A hereditary torsion theory is an ordered pair $(\mathcal{T}, \mathcal{F})$ of classes or R-modules such that (i) $Hom(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$, (ii) $Hom(A, F) = 0$ for all $F \in \mathcal{F}$ implies that $A \in \mathcal{T}$, (iii) $Hom(T, A) = 0$ for all $T \in \mathcal{T}$ implies that $A \in \mathcal{F}$, (iv) $\mathcal{T}$ is closed under submodules. It is a known fact that for a hereditary torsion theory $\mathcal{T} \in \mathcal{L}_{\{\twoheadrightarrow,\oplus,ext,\rightarrowtail\}}$ and $\mathcal{F} \in \mathcal{L}_{\{\rightarrowtail,\Pi,ext,E()\}}$.

2. The lattices $\mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ and $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$

We begin by giving the description of the generated class in the big lattices $\mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ and $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$ respectively.

Remark 2.1. If X is a class of R-modules, it is easily seen that $\xi_{\{\rightarrowtail,\Pi\}}(X)$ is given by the class of the X-cogenerated R-modules and $\xi_{\{\twoheadrightarrow,\oplus\}}(X)$ is given by the class of the X-generated R-modules.

We denote by R-pr the big lattice of preradicals in R-mod.

It is well known that there is a one to one correspondence between the class of idempotent preradicals (R-idp) and $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$ and that there is a one to one correspondence between the class of radicals (R-rad) and $\mathfrak{L}_{\{\rightarrowtail,\Pi\}}$. In general R-idp and R-rad are not sublattices of R-pr.

Example 2.2. *Take* $R = \mathbb{Z}$.

(a) *Let d be the preradical that assigns to every abelian group its divisible part and s be the preradical wich assigns to every abelian group its socle. Then t and s are both idempotent but if $M = \mathbb{Z}_{p^\infty}$ then in $\mathbb{Z}$-idp we have that $(s \wedge d)(M) = \mathbb{Z}_p$ and $(s \wedge d)(s \wedge d)(M) = 0$. Thus $(s \wedge d)$ is not idempotent.*

(b) *Let t be the preradical that assigns to every abelian group its torsion subgroup and d as in (a). Then t and d are both radicals but if $M = \prod\limits_{p \in P} \mathbb{Z}_p$ with P the set of prime numbers, then in $\mathbb{Z}$-rad we have that $(t \vee d)(M) = \bigoplus\limits_{p \in P} \mathbb{Z}_p$ and $(t \vee d)\left(\frac{M}{(t \vee d)(M)}\right) \neq 0$. Hence $t \vee d$ is not a radical.*

Proposition 2.3. *If $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow\}}$, then R-rad is a complete sublattice of R-pr.*

Proof. Let $\{r_\alpha\}_{\alpha\in X}$ be a collection of radicals.

(1) We always have that $\bigwedge\limits_{\alpha\in X}\{r_\alpha\}$ is a radical. Indeed, if $M \in R$-mod, then

$$\left(\bigwedge_{\alpha\in X}\{r_\alpha\}\right)\left(\frac{M}{\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\right) = \bigcap_{\alpha\in X}\left(r_\alpha\left(\frac{M}{\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\right)\right).$$

Since $\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M) \subseteq r_\alpha(M)$ for each $\alpha \in X$, then by [8, Chapter VI. Lemma 1.1] we have that

$$r_\alpha\left(\frac{M}{\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\right) = \frac{r_\alpha(M)}{\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M)} \quad \forall\alpha\in X.$$

Then

$$\begin{aligned}\left(\bigwedge_{\alpha\in X}\{r_\alpha\}\right)\left(\frac{M}{\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\right) &= \bigcap_{\alpha\in X}\frac{r_\alpha(M)}{\left(\bigwedge\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\\ &= \bigcap_{\alpha\in X}\left(\frac{r_\alpha(M)}{\bigcap\limits_{\alpha\in X} r_\alpha(M)}\right) = 0.\end{aligned}$$

This proves that $\bigwedge\limits_{\alpha\in X}\{r_\alpha\}$ is a radical.

(2) Now if $M \in R$-mod, we have

$$\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)\left(\frac{M}{\left(\bigvee\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\right) = \sum_{\alpha\in X} r_\alpha\left(\frac{M}{\left(\bigvee\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}\right).$$

For each $\alpha\in X$, there exists an epimorphism

$$\frac{M}{r_\alpha(M)} \twoheadrightarrow \frac{M}{\left(\bigvee\limits_{\alpha\in X}\{r_\alpha\}\right)(M)}.$$

Since $\frac{M}{r_\alpha(M)} \in \mathbb{F}_{r_\alpha}$ and as $\mathbb{F}_{r_\alpha} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$, all of whose members we are assuming closed under homomorphic images, we have $\frac{M}{\left(\bigvee\limits_{\alpha\in X}\{r_\alpha\}\right)(M)} \in \mathbb{F}_{r_\alpha}$ for each $\alpha\in X$.

Thus

$$\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)\left(\frac{M}{\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)(M)}\right)=0$$

hence $\bigvee_{\alpha\in X}\{r_\alpha\}$ is a radical. $\square$

Proposition 2.4. *If $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}\subseteq\mathfrak{L}_{\{\rightarrowtail\}}$, then R-idp is a complete sublattice of R-pr.*

Proof. Let $\{r_\alpha\}_{\alpha\in X}$ be a collection of idempotent preradicals.

(1) We always have that $\bigvee_{\alpha\in X}\{r_\alpha\}$ is an idempotent preradical. Indeed, if $M\in R$-mod, then

$$\begin{aligned}\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)\left(\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)(M)\right)&=\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)\left(\sum_{\alpha\in X}r_\alpha(M)\right)\\&=\sum_{\alpha\in X}r_\alpha\left(\sum_{\alpha\in X}r_\alpha(M)\right).\end{aligned}$$

For each $\beta\in X$ it happens that

$$r_\beta\left(\sum_{\alpha\in X}r_\alpha(M)\right)\supseteq\sum_{\alpha\in X}r_\beta r_\alpha(M)$$

hence

$$\begin{aligned}\sum_{\alpha\in X}r_\alpha\left(\sum_{\alpha\in X}r_\alpha(M)\right)&\supseteq\sum_{\substack{\alpha\in X\\ \beta\in X}}r_\beta r_\alpha(M)\supseteq\sum_{\alpha\in X}r_\alpha r_\alpha(M)=\\&=\sum_{\alpha\in X}r_\alpha(M)=\left(\bigvee_{\alpha\in X}\{r_\alpha\}\right)(M).\end{aligned}$$

The reciprocal inclusion is clear.

(2) For infima, take $M\in R$-mod. Then

$$\begin{aligned}\left(\bigwedge_{\alpha\in X}\{r_\alpha\}\right)\left(\left(\bigwedge_{\alpha\in X}\{r_\alpha\}\right)(M)\right)&=\left(\bigwedge_{\alpha\in X}\{r_\alpha\}\right)\left(\bigcap_{\alpha\in X}r_\alpha(M)\right)\\&=\bigcap_{\alpha\in X}r_\alpha\left(\bigcap_{\alpha\in X}r_\alpha(M)\right).\end{aligned}$$

Since for each $\alpha\in X$, $r_\alpha(M)\in\mathbb{T}_{r_\alpha}\in\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$, all of whose members we are assuming closed under monomorphisms, we have

$$r_\beta\left(\bigcap_{\alpha\in X}r_\alpha(M)\right)=\bigcap_{\alpha\in X}r_\alpha(M),\ \text{for each }\beta\in X.$$

Thus

$$\left(\bigcap_{\beta\in X}r_\beta\right)\left(\bigcap_{\alpha\in X}r_\alpha(M)\right)=\bigcap_{\alpha\in X}r_\alpha(M)=\left(\bigwedge_{\alpha\in X}\{r_\alpha\}\right)(M)$$

hence $\bigwedge\limits_{\alpha \in X} \{r_\alpha\}$ is an idempotent preradical. $\square$

Given $\mathcal{C} \in \mathfrak{L}_{\mathcal{P}}$, a pseudocomplement for $\mathcal{C}$ in $\mathfrak{L}_{\mathcal{P}}$ is an element $\mathcal{D} \in \mathfrak{L}_{\mathcal{P}}$ such that $\mathcal{C} \cap \mathcal{D} = \mathbf{0}$ and which is maximal with respect to this property. If $\mathcal{D}$ is largest with this property, we will call $\mathcal{D}$ the strong pseudocomplement of $\mathcal{C}$ (S-pseudocomplement, for short).

If $\mathcal{C} \in \mathfrak{L}_{\mathcal{P}}$ we let $\mathcal{C}^{\perp_{\mathcal{P}}}$ denote to some pseudocomplement (S-pseudocomplement) of $\mathcal{C}$, if it exists. If each $\mathcal{C} \in \mathfrak{L}_{\mathcal{P}}$ has a pseudocomplement (S-pseudocomplement) we say that $\mathfrak{L}_{\mathcal{P}}$ is a pseudocomplemented (or S-pseudocomplemented) (big) lattice.

Remark 2.5. $\mathfrak{L}_{\{\rightarrowtail\}}$ and $\mathfrak{L}_{\{\twoheadrightarrow\}}$ are S-pseudocomplemented big lattices. Actually if $\mathcal{C} \in \mathfrak{L}_{\{\rightarrowtail\}}$ and $\mathcal{D} \in \mathfrak{L}_{\{\twoheadrightarrow\}}$, then:

$$\mathcal{C}^{\perp_{\{\rightarrowtail\}}} = \{M \in R\text{-mod} \mid M \text{ has no non zero submodules in } \mathcal{C}\}$$

and

$$\mathcal{D}^{\perp_{\{\twoheadrightarrow\}}} = \{M \in R\text{-mod} \mid M \text{ has no non zero homomorphic images in } \mathcal{D}\}.$$

Lemma 2.6. *If $\mathcal{C} \in \mathfrak{L}_{\{\rightarrowtail,\twoheadrightarrow\}}$ then:*
(1) $\mathcal{C}^{\perp_{\{\rightarrowtail\}}} \in \mathfrak{L}_{\{\Pi\}}$ *and*
(2) $\mathcal{C}^{\perp_{\{\twoheadrightarrow\}}} \in \mathfrak{L}_{\{\oplus\}}$.

Proof. (1) If $\{M_\alpha\}_{\alpha \in X}$ is a family of R-modules in $\mathcal{C}^{\perp_{\{\rightarrowtail\}}}$ and there exists

$$0 \neq C \overset{f}{\rightarrowtail} \prod_{\alpha \in X} \{M_\alpha\} \text{ with } C \in \mathcal{C}$$

then there exists $\beta \in X$ such that $C \overset{f}{\rightarrowtail} \prod\limits_{\alpha \in X} \{M_\alpha\} \overset{\rho_\beta}{\twoheadrightarrow} M_\beta$ is not zero. Thus $0 \neq \rho_\beta \circ f(C) \leq M_\beta$, and as $\mathcal{C}$ is closed under homomorphic images, then $\rho_\beta \circ f(C) \in \mathcal{C}$, a contradiction. So $\prod\limits_{\alpha \in X} \{M_\alpha\} \in \mathcal{C}$.

(2) It is similar to (1). $\square$

Proposition 2.7. (1) *If $\mathcal{C} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ and $\mathcal{C}$ is closed under homomorphic images then $\mathcal{C}$ has an S-pseudocomplement in $\mathfrak{L}_{\{\rightarrowtail,\Pi\}}$. Moreover*

$$\mathcal{C}^{\perp_{\{\rightarrowtail,\Pi\}}} = \mathcal{C}^{\perp_{\{\rightarrowtail\}}}$$

and $\mathcal{C}^{\perp_{\{\rightarrowtail,\Pi\}}} \in \mathfrak{L}_{\{\rightarrowtail,\Pi,E(),ext\}}$.

(2) *If $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$ and $\mathcal{C}$ is closed under monomorphisms, then $\mathcal{C}$ has an S-pseudocomplement in $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$. Moreover*

$$\mathcal{C}^{\perp_{\{\twoheadrightarrow,\oplus\}}} = \mathcal{C}^{\perp_{\{\twoheadrightarrow\}}}$$

and $\mathcal{C}^{\perp_{\{\twoheadrightarrow,\oplus\}}} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus,P(\ ),ext\}}$.

Proof. (1) By Lemma 2.6, $\mathcal{C}^{\perp_{\{\rightarrowtail\}}}$ is also closed under products, and so $\mathcal{C}^{\perp_{\{\rightarrowtail\}}} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ and $\mathcal{C}^{\perp_{\{\rightarrowtail\}}} \cap \mathcal{C} = \mathbf{0}$. Now, let $\mathcal{D} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ be such that $\mathcal{D} \cap \mathcal{C} = \mathbf{0}$. Since $\mathcal{D} \in \mathfrak{L}_{\{\rightarrowtail\}}$, then $\mathcal{D} \subseteq \mathcal{C}^{\perp_{\{\rightarrowtail\}}}$.

(2) It is similar to (1). □

Lemma 2.8. If $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}} \subseteq \mathfrak{L}_{\{\rightarrowtail\}}$ then every finitely generated (or finitely cogenerated) projective R-module P is injective.

Proof. By Remark 2.1, we have that

$$\xi_{\{\twoheadrightarrow,\oplus\}}\left(E\left(P\right)\right) = \left\{M \in R\text{-mod} \mid \exists\ E\left(P\right)^{(X)} \twoheadrightarrow M \text{ for some set } X\right\}.$$

By the present hypothesis $\xi_{\{\twoheadrightarrow,\oplus\}}\left(E\left(P\right)\right)$ is closed under submodules, thus $P \in \xi_{\{\twoheadrightarrow,\oplus\}}\left(E\left(P\right)\right)$. So P is $E\left(P\right)$-generated.

Since P is finitely generated (or finitely cogenerated) and projective, then P is a direct summand of a finite direct sum of copies of $E\left(P\right)$. Thus P is injective. □

Corollary 2.9. *If $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}} \subseteq \mathfrak{L}_{\{\rightarrowtail\}}$ then R is left self-injective.*

Remark 2.10. If $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow\}}$ then R is isomorphic to a finite direct product of right perfect left local rings.

Proof. If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory, then $\mathcal{F} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$, so that $\mathcal{F}$ is closed under homomorphic images and we conclude using [6]. □

Remark 2.11. Notice that in Remark 2.10, such a ring is Morita equivalent to a finite product of local right and left perfect rings. (see also [5, Theorem VI.2.4])

Lemma 2.12. *If $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow\}}$ then R is a finitely cogenerated injective cogenerator. In particular every finitely cogenerated projective module is injective.*

Proof. By Remark 2.1, we have that

$$\xi_{\{\rightarrowtail,\Pi\}}\left(R\right) = \left\{M \in R\text{-mod} \mid \exists\ M \rightarrowtail R^X \text{ for some set } X\right\}.$$

Since $R \in \xi_{\{\rightarrowtail,\Pi\}}\left(R\right)$ and since $\xi_{\{\rightarrowtail,\Pi\}}\left(R\right)$ is closed under homomorphic images and direct sums, then as each R-module is a quotient of a free R-module,

$$\xi_{\{\rightarrowtail,\Pi\}}\left(R\right) = R\text{-mod}.$$

In particular for a simple R-module S, there exists a set X and a monomorphism

$$E\left(S\right) \overset{\varphi}{\rightarrowtail} R^X.$$

Then there exists an $i \in X$ such that the composition

$$S \overset{i}{\hookrightarrow} E\left(S\right) \overset{\varphi}{\rightarrowtail} R^X \overset{\rho_i}{\twoheadrightarrow} R$$

is a monomorphism, $\rho_i : R^X \twoheadrightarrow R$ being a projection. Since $S \subseteq_{es} E(S)$, then $\rho_i\varphi : E(S) \rightarrowtail R$ is a monomorphism. Thus $E(S)$ is isomorphic to a direct summand of R.

By Remarks 2.10 and 2.11 we can assume that R is local.

If $R = E(S) \oplus J$ and $J \neq 0$, then R would have at least two maximal left ideals contradicting that R is local. Hence $R = E(S)$ and R is self-injective and finitely cogenerated. □

Now we will prove the main Theorem of this work:

Theorem 2.13. *The following assertions are equivalent for a ring R:*
(1) *R is an Artinian principal ideal ring.*
(2) $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$.
(3) $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}} \subseteq \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$.

Proof. $(1 \Rightarrow 2)$ By [2, Theorem 38] (1) is equivalent to $\mathfrak{L}_{\{\twoheadrightarrow\}} = \mathfrak{L}_{\{\rightarrowtail\}}$. Now, if $\mathcal{C} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$, then $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow\}}$ and since $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\oplus\}}$, then $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$. Hence $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$.

$(1 \Rightarrow 3)$ If $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$, by [2, Theorem 38], $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus,\rightarrowtail\}}$. On the other hand we have that R is a left Artinian ring, so by [4] $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus,\rightarrowtail,\Pi\}}$ and then $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}} \subseteq \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$.

$(3 \Rightarrow 1)$ By Corollary 2.9, R is left self-injective.

Let $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$, by (3) $\mathcal{C} \in \mathfrak{L}_{\{\twoheadrightarrow,\oplus,\rightarrowtail,\Pi\}}$. Then, by [4], R is left Artinian. Thus R is a QF-ring.

Noting that the condition $\mathfrak{L}_{\{\twoheadrightarrow,\oplus\}} \subseteq \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ holds when we take a quotient $\frac{R}{I}$ of R, we get that each factor $\frac{R}{I}$ is QF-ring too. Then by [7, Proposition 25.4.6B] R is an Artinian principal ideal ring.

$(2 \Rightarrow 1)$ By Lemma 2.12 we have that R is a left self-injective ring and we can suppose that R is a local right and left perfect ring and finitely cogenerated. Again notice that the condition $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow,\oplus\}}$ holds also with respect to a quotient $\frac{R}{I}$. Then for each two sided ideal I of R we have that $\frac{R}{I}$ is finitely cogenerated too.

Let J be the Jacobson radical of R. We claim J is nilpotent. Indeed, if $I = \bigcap\limits_{n\in\mathbb{N}} J^n$, then I is a two sided ideal of R. As $\frac{R}{I}$ is finitely cogenerated and $\bigcap\limits_{n\in\mathbb{N}} \left(\frac{J^n}{I}\right) = 0$, there exists $m \in \mathbb{N}$ such that $J^m = I$. Then J^m is an idempotent two sided ideal of R. By [8, Chapter VIII, Corollary 6.4] there exists an idempotent e such that $J^m = ReR$. Since $R = E(S)$ then R is indecomposable, thus its only idempotents are 0 and 1. Since $J^m \neq R$, then $e = 0$ and thus $J^m = 0$.

Now, we will see that R is left Artinian by induction on m, the nilpotency index of J. Since R is right perfect, for $m = 1$ we have that $R \cong \frac{R}{J}$ which is semisimple Artinian. If $m > 1$ then we consider the exact sequence

$$0 \to J^{m-1} \to R \to \frac{R}{J^{m-1}} \to 0$$

Since $\mathrm{Rad}\left(\frac{R}{J^{m-1}}\right) = \frac{J}{J^{m-1}}$ whose nilpotency index is at most $m-1$, then $\frac{R}{J^{m-1}}$ is left Artinian by induction hypothesis. On the other hand, J^{m-1} is an $\frac{R}{J}$-module and since $\frac{R}{J}$ is semisimple Artinian, we note that J^{m-1} is semisimple. Finally as $J^{m-1} \leq R$ and R is finitely cogenerated, then J^{m-1} is finitely cogenerated and semisimple. Hence J^{m-1} and R are left Artinian.

Since R is left self-injective, R is a QF-ring. Note that the same applies for a quotient $\frac{R}{I}$ of R. By [7, Proposition 25.4.6B], R is an Artinian principal ideal ring. $\square$

Proposition 2.14. *If* $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow\}}$, *then*

$$skel(\mathfrak{L}_{\{\rightarrowtail,\Pi\}}) = R - nat = \mathfrak{L}_{\{\rightarrowtail,\Pi,\twoheadrightarrow,E(),ext\}}.$$

Proof. Since $\mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\twoheadrightarrow\}}$, we have by Remark 2.10 that R is a finite direct product of right perfect left local rings. By [9, Propositions 2.4 and 2.5], every natural class is closed under direct products and quotients. Thus R-nat $= \mathfrak{L}_{\{\rightarrowtail,\Pi,\twoheadrightarrow,E(),ext\}}$.

If $\mathcal{C} \in \mathfrak{L}_{\{\rightarrowtail,\Pi\}}$, then by hypothesis and Proposition 2.7 we have that

$$\mathcal{C}^{\perp_{\{\rightarrowtail,\Pi\}}} = \mathcal{C}^{\perp_{\{\rightarrowtail\}}} = \{M \mid M \text{ has no non zero submodules in } \mathcal{C}\}.$$

Hence $\mathfrak{L}_{\{\rightarrowtail,\Pi\}}$ is S-pseudocomplemented.

Now, since R-nat $\subseteq \mathfrak{L}_{\{\rightarrowtail,\Pi\}} \subseteq \mathfrak{L}_{\{\rightarrowtail\}}$ and $skel(\mathfrak{L}_{\{\rightarrowtail\}}) = R$-nat (see [1, Theorem 12]) we can apply [3, Theorem 1.4] to obtain $skel(\mathfrak{L}_{\{\rightarrowtail,\Pi\}}) = R$-nat. $\square$

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