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INTEGRALS AND A MASCHKE-TYPE THEOREM FOR WEAK HOPF π-COALGEBRAS

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Abstract. Let π be a discrete group. We shall introduce the more general concept of an integral of a weak Doi-Hopf π-datum (H, A, C), where H is a weak Hopf π-coalgebra coacting on an algebra A and acting on a π-coalgebra $C = \{C_\alpha\}_{\alpha\in\pi}$. We prove that there exists a total integral $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$, then any representation of (H, A, C) is injective in a functorial way, as a corepresentation of C and vice versa. As the application of the existence of a total integral, we prove the Maschke-type Theorem for weak Doi-Hopf π-modules.

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1. Introduction

The category ${}^{C}\mathcal{U}(H)_A$ of Doi-Hopf modules over the bialgebra H was introduced in [9]. It is the category of the modules over the algebra A which are also comodules over the coalgebra C and satisfy certain compatibility condition involving H. The study of ${}^{C}\mathcal{U}(H)_A$ turned out to be very useful: it was shown in [3, 9] that many categories such as the module and comodule categories over bialgebras, the Hopf modules category [13], and the Yetter-Drinfeld category [11, 25] are special cases of ${}^{C}\mathcal{U}(H)_A$. Many results known for module categories over bialgebras or Hopf algebras were generalized to Hopf group-coalgebra [21].

Hopf group-algebras appeared in the work of Turaev [14] on homotopy quantum field theories as a generalization of ordinary Hopf algebras. Let us note that there exists a symmetric monoidal category, the so-called Turaev category, constructed by Caenepeel and De Lombaerde [4] the Hopf algebras which are the same as Hopf group-coalgebras. A purely algebraic study of Hopf group-coalgebras can be found in the references Virelizier ([19, 20]), Wang ([21-24]) and Zunino ([26-27]).

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As a generalization of ordinary Hopf group-coalgebras and weak Hopf algebras [1], weak Hopf group-coalgebras were studied in the work of Van Daele and Wang [18]. Weak Hopf group-coalgebras not only provide examples of weak multiplier Hopf algebras ([17, 18]) , but also provide an approach to construct a class of new braided crossed categories in the sense of Turaev. A purely algebraic study of weak Hopf group-coalgebras can be found in [17]. Many results in Hopf group-coalgebra had been generalized to weak Hopf group-coalgebra. For example, the Fundamental Theorem of weak Hopf group-comodules was given in [17]. In the sequel, the fundamental theorem had been generalized to weak relative Hopf group-comodules in [12].

In this paper, we generalize the definition of Doi-Hopf π-modules to the case when H is a weak Hopf group-coalgebra and develop the theory of weak Doi-Hopf π- modules.

This article is organized as follows:

In Section 2, we recall defintions and basic results related to separable functors and (weak) Hopf group-coalgebras and in Section 3, we recall some important adjoint functors and introduce some weak Doi-Hopf π-modules.

In Section 4, inspired by the idea adopted by Menini and Militaru ([10]) or Caenepeel et al.([5]), we introduce the notion of integral for weak Doi-Hopf π-datums and prove that if there exists $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ a total integral, then the natural transformation $\rho : F_A \circ 1_{\pi-C_{\mathcal{U}(H)_A}} \to F_A \circ G \circ F^C$ splits. Conversely, if the natural transformation $\rho : F_A \circ 1_{\pi-C_{\mathcal{U}(H)_A}} \to F_A \circ G \circ F^C$ splits, then there exists $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ a total integral(see Theorem 4.4). The application of the integral is also considered (see Theorem 4.7).

In Section 5, we prove the Maschke-type theorem for weak Doi-Hopf π-modules (see Theorem 5.2).

2. Preliminaries

Throughout this paper, we always let π be a discrete group with a neutral element e and k a field. If U and V are k-spaces, $T_{U,V} : U \otimes V \to V \otimes U$ will denote the flip map defined by $T_{U,V}(u \otimes v) \to v \otimes u$, for all $u \in U$ and $v \in V$.

2.1. Separable Functors. Separable functors were introduced in [8]: futher applications and properties have been discussed in [28]and [6]. Separable functors of the second kind were introduced in [7]. Let $F : \mathcal{C} \to \mathcal{D}$ and $\mathcal{H} : \mathcal{C} \to \mathcal{E}$ be covariant functors. We then have functors

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet), \mathrm{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)), \mathrm{Hom}_{\mathcal{E}}(\mathcal{H}(\bullet), \mathcal{H}(\bullet)) : \mathcal{C}^{op} \times \mathcal{C} \to \underline{Sets}$$

and natural transformations

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(\bullet,\bullet) \to \mathrm{Hom}_{\mathcal{D}}(F(\bullet),F(\bullet)), \mathfrak{H} : \mathrm{Hom}_{\mathcal{C}}(\bullet,\bullet) \to \mathrm{Hom}_{\mathcal{E}}(H(\bullet),H(\bullet))$$

given by

$$\mathcal{F}_{C,C'}(f) = F(f), \mathfrak{H}_{C,C'}(f) = \mathcal{H}(f)$$

for $f : C \to C^{'}$ in $\mathcal{C}$. The functor F is called $\mathcal{H}$-separable if there exsits a natural transformation

$$\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F(\bullet),F(\bullet)) \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{H}(\bullet),\mathcal{H}(\bullet))$$

such that $\mathcal{P} \circ \mathcal{F} = \mathfrak{H}$.

We recall two properties: The first is Maschke$^{'}$s Theorem for $\mathcal{H}$-separable functors(see [7, Prop. 2.4]): Let F be an $\mathcal{H}$-separable functor. If $f : C \to C^{'}$ in $\mathcal{C}$ is such that $F(f)$ has a left , right, or two-sided inverse in $\mathcal{D}$, then $\mathcal{H}(f)$ has a left, right, or two-sided inverse in $\mathcal{E}$. The second one is called Rafael $^{'}$s Theorem for $\mathcal{H}$-separability(see [7, Theorem 2.7]). If F has a right adjoint G, then F is $\mathcal{H}$-separable if and only if there exists a natural transformation $\xi : \mathcal{H}GF \to \mathcal{H}$ such that $\xi_C \circ \mathcal{H}(\eta_C) = I_{\mathcal{H}(C)}$ for any $C \in \mathcal{C}$.

2.2. The π-Coalgebras. Recall from Turaev ([14]) and Virelizier ([19]) that a π-coalgebra is a family of k-spaces $C = \{C_\alpha\}_{\alpha\in\pi}$ together with a family of k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : \Delta_{\alpha\beta} \to \Delta_\alpha \otimes \Delta_\beta\}_{\alpha,\beta\in\pi}$ (called a *comultiplication*) and a k-linear map $\varepsilon : C_e \to k$ (called a *counit*) such that Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}, \tag{2. 1}$$

$$(id_{C_\alpha} \otimes \varepsilon) \circ \Delta_{\alpha,e} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha}) \circ \Delta_{e,\alpha}, \tag{2. 2}$$

for any $\alpha, \beta, \gamma \in \pi$.

Remark 2.1. $(C_e, \Delta_{e,e}, \varepsilon)$ is an ordinary coalgebra in the sense of Sweedler. Following the Sweedler$^{'}$s notation for π-coalgebras, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, one write

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}. \tag{2. 3}$$

The coassociativity axiom (2.1) gives that, for any $\alpha, \beta, \gamma \in \pi$ and $c \in C_{\alpha\beta\gamma}$,

$$c_{(1,\alpha\beta)(1,\alpha)} \otimes c_{(1,\alpha\beta)(2,\beta)} \otimes c_{(2,\gamma)} = c_{(1,\alpha)} \otimes c_{(2,\beta\gamma)(1,\beta)} \otimes c_{(1,\beta\gamma)(2,\gamma)}, \tag{2. 4}$$

which is written as $c_{(1,\alpha)} \otimes c_{(2,\beta)} \otimes c_{(3,\gamma)}$. Inductively, we can define $c_{(1,\alpha_1)} \otimes c_{(2,\alpha_2)} \otimes \cdots c_{(n,\alpha_n)}$, for any $c \in C_{\alpha_1\alpha_2\cdots\alpha_n}$. The axiom (2.2) gives that, for any $\alpha \in \pi$ and $c \in C_\alpha$,

$$\varepsilon(c_{(1,e)})c_{(2,\alpha)} = c = c_{(1,\alpha)}\varepsilon(c_{(2,e)}). \tag{2. 5}$$

2.3. The π-C-Comodules. Let C be a π-coalgebra. A left π-C-comodule is a k-vector space V endowed with a family of k-linear maps $\rho^V = \{\rho^V_\alpha : V \to C_\alpha \otimes V\}_{\alpha\in\pi}$ such that for all $\alpha, \beta \in \pi$ and $v \in V$,

$$v_{<-1,\alpha>} \otimes v_{<0,0><-1,\beta>} \otimes v_{<0,0><0,0>} = v_{<-1,\alpha\beta>(1,\alpha)} \otimes v_{<-1,\alpha\beta>(2,\beta)} \otimes v_{<0,0>}, \tag{2. 6}$$

$$\varepsilon(v_{<-1,e>})v_{<0,0>} = v, \tag{2. 7}$$

where we use the the standard notation $\rho^V_\alpha(v) = v_{<-1,\alpha>} \otimes v_{<0,0>}$.

2.4. Weak Hopf π-Coalgebras. We recall from Van Daele and Wang ([18]) that a *weak semi-Hopf π-coalgebra* $H = \{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ is a family of algebras $\{H_\alpha,\ m_\alpha,\ 1_\alpha\}_{\ \alpha\in\pi}$ and at the same time a π-coalgebra $\{H_\alpha, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon\}_{\alpha,\beta\in\pi}$ such that

(i) The comultiplication $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ is a homomorphism of algebras (not necessary unit-preserving) such that

$$(\Delta_{\alpha,\beta} \otimes id_{H_\alpha})\Delta_{\alpha\beta,\gamma}(1_{\alpha\beta\gamma}) = (\Delta_{\alpha,\beta} \otimes 1_\gamma)(1_\alpha \otimes \Delta_{\beta,\gamma}(1_{\beta\gamma})), \tag{2. 8}$$

$$(\Delta_{\alpha,\beta} \otimes id_{H_\alpha})\Delta_{\alpha\beta,\gamma}(1_{\alpha\beta\gamma}) = (1_\alpha \otimes \Delta_{\beta,\gamma}(1_{\beta\gamma}))(\Delta_{\alpha,\beta}(1_{\alpha,\beta}) \otimes 1_\gamma), \tag{2. 9}$$

for all $\alpha, \beta, \gamma \in \pi$.

(ii) The counit $\varepsilon : H_e \to k$ is a k-linear map satisfying the identity

$$\varepsilon(gxh) = \varepsilon(gx_{(2,e)})\varepsilon(x_{(1,e)}h) = \varepsilon(gx_{(1,e)})\varepsilon(x_{(2,e)}h), \tag{2. 10}$$

for all $g, h, x \in H_e$.

A *weak Hopf π-coalgebra* is a weak semi-Hopf π-coalgebra $H = \{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ endowed with a family of k-linear maps $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}$ (called *antipode*) such that the following data hold:

$$m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h) = 1_{(1,\alpha)}\varepsilon(h1_{(2,e)}), \tag{2. 11}$$

$$m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h) = \varepsilon(1_{(1,e)}h)1_{(2,\alpha)}, \tag{2. 12}$$

$$S_\alpha(g_{(1,\alpha)})g_{(2,\alpha^{-1})}S_\alpha(g_{(3,\alpha)}) = S_\alpha(g), \tag{2. 13}$$

for all $h \in H_e$, $g \in H_\alpha$ and $\alpha \in \pi$.

Remark 2.2. $H = \{H_\alpha, m_\alpha, 1_\alpha, \Delta, \varepsilon\}_{\alpha\in\pi}$ is an ordinary weak Hopf algebra. The set of axioms of the above definition is not self-dual. A weak Hopf π-coalgebra H is said to be of finite type if, for all $\alpha \in \pi$, H_α is finite-dimensional as a k-vector space. Note that it does not mean that $\bigoplus_{\alpha\in\pi} H_\alpha$ is finite-dimensional (unless $H_\alpha = 0$ for all but a finite number of $\alpha \in \pi$). The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of the weak Hopf π-coalgebra H is said to be bijective if each S_α is bijective.

Let H be a weak hopf π-coalgebra. Define the family of linear maps $\varepsilon^t = \{\varepsilon^t_\alpha : H_e \to H_\alpha\}_{\alpha\in\pi}$ and $\varepsilon^s = \{\varepsilon^s_\alpha : H_e \to H_\alpha\}_{\alpha\in\pi}$ by the formulars

$$\varepsilon^s_\alpha(h) = m_\alpha(S_{\alpha^{-1}} \otimes id_{H_\alpha})\Delta_{\alpha^{-1},\alpha}(h) = 1_{(1,\alpha)}\varepsilon(h1_{(2,e)}), \qquad (2.\ 14)$$

$$\varepsilon^t_\alpha(h) = m_\alpha(id_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}(h) = \varepsilon(1_{(1,e)}h)1_{(2,\alpha)}, \qquad (2.\ 15)$$

for any $h \in H_e$ and $\alpha \in \pi$, where $\varepsilon^t, \varepsilon^s$ are called the π-target and π-source counital maps. Introduce the notations $H^t = \varepsilon^t(H) = \{\varepsilon^t_\alpha(H_e)\}_{\alpha\in\pi}$ and $H^s = \varepsilon^s(H) = \{\varepsilon^s_\alpha(H_e)\}_{\alpha\in\pi}$ for their images.

Let H be a weak hopf π-coalgebra. Then we have the following properties, we refer to [18] for full detail.

(W1) $\Delta_{\alpha,\beta}(1_{\alpha\beta}) \in H^s_\alpha \otimes H^t_\alpha$, for all $\alpha, \beta \in \pi$;
(W2) $\varepsilon^t_\alpha(gh) = \varepsilon^t_\alpha(g\varepsilon^t_e(h))$, $\varepsilon^t_\alpha(\varepsilon^t_e(g)h) = \varepsilon^t_\alpha(g)\varepsilon^t_\alpha(h)$, for all $g, h \in H_e$;
(W3) $\Delta_{\alpha,\beta}(H^t_{\alpha\beta}) \subseteq H_\alpha \otimes H^t_\beta$, $\Delta_{\alpha,\beta}(H^s_{\alpha\beta}) \subseteq H^s_\alpha \otimes H_\beta$;
(W4) $x_{(1,\alpha)} \otimes \varepsilon^t_\beta(x_{(2,e)}) = 1_{(1,\alpha)}x \otimes 1_{(2,\beta)}$, for all $\alpha, \beta \in \pi$ and $x \in H_e$;
(W5) $\varepsilon^s_\beta(x_{(1,e)}) \otimes x_{(2,\alpha)} = 1_{(1,\beta)} \otimes x1_{(2,\alpha)}$, for all $\alpha, \beta \in \pi$ and $x \in H_e$;
(W6) $\varepsilon^t_\alpha(h)\varepsilon^s_\alpha(g) = \varepsilon^s_\alpha(g)\varepsilon^t_\alpha(h)$, for all $g, h \in H_e$;
(W7) $S_\alpha(xy) = S_\alpha(y)S_\alpha(x)$, for all $\alpha \in \pi$ and $x, y \in H_\alpha$;
(W8) $S_\alpha(1_\alpha) = 1_{\alpha^{-1}}$, for all $\alpha \in \pi$;
(W9) $\Delta_{\beta^{-1},\alpha^{-1}} \circ S_{\alpha\beta} = T_{H_{\alpha^{-1}},H_{\beta^{-1}}} \circ (S_\alpha \otimes S_\beta) \circ \Delta_{\alpha,\beta}$, for all $\alpha, \beta \in \pi$;
(W10) $\varepsilon^t_\alpha \circ S_e = \varepsilon^t_\alpha \circ \varepsilon^s_e = S_{\alpha^{-1}} \circ \varepsilon^s_{\alpha^{-1}}$;
(W11) $x_{(1,\alpha)} \otimes \varepsilon^s_\beta(x_{(2,e)}) = x1_{(1,\alpha)} \otimes S_{\beta^{-1}}(1_{(2,\beta^{-1})})$, for all $\alpha, \beta \in \pi$ and $x \in H_\alpha$;
(W12) $\varepsilon^t_\beta(x_{(1,e)}) \otimes x_{(2,\alpha)} = S_{\beta^{-1}}(1_{(1,\beta^{-1})}) \otimes 1_{(2,\alpha)}x$, for all $\alpha, \beta \in \pi$ and $x \in H_\alpha$;
(W13) If H is of finite type, then the antipode S is bijective.

Definition 2.3. Let H be a weak Hopf π-coalgebra over the field k. A k-algebra A is called a *weak left π-H-comodule algebra* if there exists a family of maps $\rho^A = \{\rho^A_\alpha : A \to H_\alpha \otimes A\}$ such that

$$(id_{H_\alpha} \otimes \rho^A_\beta) \circ \rho^A_\alpha = (\Delta_{\alpha,\beta} \otimes id_A) \circ \rho^A_{\alpha\beta}, \qquad (2.\ 16)$$

$$(\varepsilon \circ id_A) \circ \rho^A_e = id_A, \qquad (2.\ 17)$$

$$\rho^A_\alpha(1_A) = (\varepsilon^s_\alpha \circ id_A) \circ \rho^A_e(1_A), \qquad (2.\ 18)$$

$$\rho^A_\alpha(ab) = \rho^A_\alpha(a)\rho^A_\alpha(b), \qquad (2.\ 19)$$

for all $\alpha, \beta \in \pi$ and $a, b \in A$. We use the standard notation $\rho^A_\alpha(a) = a_{<-1,\alpha>} \otimes a_{<0,0>}$.

Lemma 2.4. *Let A be a weak left π-H-comodule algebra. Then, for all $\alpha \in \pi$ and $a \in A$,*

$$1_{<-1,\alpha>} \otimes 1_{<0,0><-1,e>} \otimes 1_{<0,0><0,0>} = 1_{\alpha(1,\alpha)} \otimes 1_{\alpha(2,e)}1^{'}_{<-1,e>} \otimes 1^{'}_{<0,0>}, \tag{2. 20}$$

$$\varepsilon^{s}_{\alpha}(a_{<-1,e>}) \otimes a_{<0,0>} = 1_{<-1,\alpha>} \otimes a1_{<0,0>}, \tag{2. 21}$$

$$\varepsilon^{t}_{\alpha}(a_{<-1,e>}) \otimes a_{<0,0>} = S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a. \tag{2. 22}$$

Proof. The proof is straightforward. □

Definition 2.5. A π-coalgebra C is called a *weak right π-H-module coalgebra* if there exists a family of maps $\cdot : C_\alpha \otimes H_\alpha \to C_\alpha$ such that

$$(c \cdot h) \cdot h^{'} = c \cdot hh^{'}, \tag{2. 23}$$

$$c \cdot 1_\alpha = c, \tag{2. 24}$$

$$c \cdot \varepsilon^{t}_{\alpha}(g) = \varepsilon_C(c_{(1,e)} \cdot g)c_{(2,\alpha)}, \tag{2. 25}$$

$$\Delta_{\alpha,\beta}(c^{'} \cdot g^{'}) = c^{'}_{(1,\alpha)} \cdot g^{'}_{(1,\alpha)} \otimes c^{'}_{(2,\beta)} \cdot g^{'}_{(2,\beta)}, \tag{2. 26}$$

for all $c \in C_\alpha$, $c^{'} \in C_{\alpha\beta}$, $h, h^{'} \in H_\alpha$, $g^{'} \in H_{\alpha\beta}$, $g \in H_e$ and $\alpha, \beta \in \pi$.

Lemma 2.6. *E.q (2. 25) is equivalent to*

$$\varepsilon_C(c \cdot h) = \varepsilon_C(c \cdot \varepsilon^{t}_{e}(h)), \tag{2. 27}$$

for all $c \in C_e$ and $h \in H_e$.

Proof. Assume E.q (2.27) holds. For all $c \in C_\alpha$ and $g \in H_e$, we have

$$\begin{aligned}
\varepsilon_C(c_{(1,e)} \cdot g)c_{(2,\alpha)} &\overset{(2.27)}{=} \varepsilon_C(c_{(1,e)} \cdot \varepsilon^{t}_{e}(g))c_{(2,\alpha)} \\
&\overset{(2.26)}{=} \varepsilon_C(c_{(1,e)} \cdot 1_{(1,e)}\varepsilon^{t}_{e}(g))c_{(2,\alpha)} \cdot 1_{(2,\alpha)} \\
&\overset{(W3)}{=} \varepsilon_C(c_{(1,e)} \cdot \varepsilon^{t}_{\alpha}(g)_{(1,e)})c_{(2,\alpha)} \cdot \varepsilon^{t}_{\alpha}(g)_{(2,\alpha)} \\
&\overset{(2.26,2.5)}{=} c \cdot \varepsilon^{t}_{\alpha}(g).
\end{aligned}$$

Thus E.q (2. 25) holds.

Conversely, assume E.q (2. 25) holds. Taking $\alpha = e$, $c \in C_e$ and $g \in H_e$, we have

$$c \cdot \varepsilon^{t}_{e}(g) = \varepsilon_C(c_{(1,e)} \cdot g)c_{(2,e)}.$$

So

$$\varepsilon_C(c \cdot \varepsilon^{t}_{e}(g)) \overset{(2.25)}{=} \varepsilon_C(c_{(1,e)} \cdot g)\varepsilon_C(c_{(2,e)}) \overset{(2.5)}{=} \varepsilon_C(c \cdot g).$$

The proof of the Lemma is completed. □

3. Weak Doi-Hopf π-Module: Functors And Structure

Let H be a weak Hopf π-coalgebra. A *weak Doi-Hopf π-datum* is a triple (H, A, C), where A is a weak left π-H-comodule algebra and C a weak right π-H-module coalgebra.

A *weak Doi-Hopf π-module* M is a right A-module which is also a left π-C-comodule with the coaction structure $\rho^M = \{\rho^M_\lambda : M \to C_\lambda \otimes M\}_{\lambda\in\pi}$ such that the following compatible condition holds:

$$\rho^M_\alpha(m \cdot a) = m_{<-1,\alpha>} \cdot a_{<-1,\alpha>} \otimes m_{<0,0>} \cdot a_{<0,0>}, \tag{3. 1}$$

for all $\alpha \in \pi$ and $m \in M, a \in A$.

The set of weak Doi-Hopf π-modules together with both an A-module maps and a π-C-comodule maps will form a category of weak Doi-Hopf π-modules and will be denoted by ${}^{\pi-C}\mathcal{U}(H)_A$ (called a *weak Doi-Hopf π-modules category*).

Let $F^C :{}^{\pi-C} \mathcal{U}(H)_A \to \mathcal{U}_A$ be the forgetful functor which forgets the π-C-coaction and

$$G : \mathcal{U}_A \to{}^{\pi-C} \mathcal{U}(H)_A, \quad M \mapsto G(M) = \bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes M}$$

its right adjoint, where $\overline{C_\alpha \otimes M} = \{c \cdot 1_{<-1,\alpha>} \otimes m \cdot 1_{<0,0>} | m \in M,\ c \in C_\alpha\}$ and the weak Doi-Hopf π-module structures on $\overline{C_\alpha \otimes M}$ are given by

$$(c \cdot 1_{<-1,\alpha>} \otimes m \cdot 1_{<0,0>}) \cdot a = c \cdot a_{<-1,\alpha>} \otimes m \cdot a_{<0,0>}, \tag{3. 2}$$

$${}^l\rho^{G(M)}_\beta(c\cdot 1_{<-1,\alpha>} \otimes m\cdot 1_{<0,0>}) = c_{(1,\beta)} \otimes c_{(2,\beta^{-1}\alpha)} \cdot 1_{<-1,\beta^{-1}\alpha>} \otimes m\cdot 1_{<0,0>}, \tag{3. 3}$$

for all $c \in C_\alpha$, $a \in A$, $m \in M$ and $\alpha, \beta \in \pi$. The unit of the adjoint pair (F^C, G) is

$$\rho : 1_{{}^{\pi-C}\mathcal{U}(H)_A} \to G \circ F^C$$

defined by $\rho_M : M \to G(M)$, $\rho_M(m) = \bigoplus_{\alpha\in\pi} m_{<-1,\alpha>} \otimes m_{<0,0>}$, for all $m \in M$. Since A is a right $A-$module, one has that $\bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes A}$ is a weak Doi-Hopf π-module via

$$(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \cdot b = c \cdot b_{<-1,\alpha>} \otimes ab_{<0,0>}, \tag{3. 4}$$

$${}^l\rho^{G(A)}_\beta(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) = c_{(1,\beta)} \otimes c_{(2,\beta^{-1}\alpha)} \cdot 1_{<-1,\beta^{-1}\alpha>} \otimes a1_{<0,0>} \tag{3. 5}$$

for all $a \in A$, $c \in C_\alpha$ and $\beta \in \pi$.

Lemma 3.1. *The vector space* $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$ *is a right* π*-* C*-comodule via*

$$\begin{aligned}&{}^r\rho_\beta^{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>})\\ =\;& c_{(1,\alpha\beta^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}1_{<0,0>}\otimes c_{(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}),\end{aligned}$$

for any $\beta,\alpha\in\pi$*,* $c\in C_\alpha$ *and* $a\in A$*.*

Proof. First, we can prove that $(id_{G(A)}\otimes\varepsilon_C)\circ^r\rho_e^{G(A)}=id_{G(A)}$. In fact, for all $c\in C_\alpha$ and $a\in A$, we have

$$\begin{aligned}&(id_{G(A)}\otimes\varepsilon_C)\circ^r\rho_e^{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>})\\
=\;&\varepsilon(c_{(2,e)}S_e(a_{<-1,e>}))c_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\otimes a_{<0,0>}1_{<0,0>}\\
\overset{(2.27)}{=}\;&\varepsilon(c_{(2,e)}\varepsilon_e^t(S_e(a_{<-1,e>})))c_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\otimes a_{<0,0>}1_{<0,0>}\\
\overset{(W10)}{=}\;&\varepsilon(c_{(2,e)}\varepsilon_e^t\varepsilon_e^s(a_{<-1,e>}))c_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\otimes a_{<0,0>}1_{<0,0>}\\
\overset{(2.21)}{=}\;&\varepsilon(c_{(2,e)}\varepsilon_e^t(1'_{<-1,e>}))c_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\otimes a1'_{<0,0>}1_{<0,0>}\\
\overset{(2.27,2.18)}{=}\;&\varepsilon(c_{(2,e)}\varepsilon_e^s(1'_{<-1,e>}))c_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\otimes a1'_{<0,0>}1_{<0,0>}\\
\overset{(W3,2.26)}{=}\;&\varepsilon((c\cdot\varepsilon_\alpha^s(1'_{<-1,e>}))_{(2,e)})(c\cdot\varepsilon_\alpha^s(1'_{<-1,e>}))_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\otimes a1'_{<0,0>}1_{<0,0>}\\
\overset{(2.5)}{=}\;&c\cdot\varepsilon_\alpha^s(1'_{<-1,e>})1_{<-1,\alpha>}\otimes a1'_{<0,0>}1_{<0,0>}\\
\overset{(2.18)}{=}\;&c\cdot 1'_{<-1,\alpha>}1_{<-1,\alpha>}\otimes a1'_{<0,0>}1_{<0,0>}\\
\overset{(2.19)}{=}\;&c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>}.\end{aligned}$$

For all $c\in C_\alpha$ and $a\in A$, we have

$$\begin{aligned}&({}^r\rho_\gamma^{G(A)}\otimes id_{C_\beta})\circ^r\rho_\beta^{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>})\\
=\;&c_{(1,\alpha\beta^{-1})(1,\alpha\beta^{-1}\gamma^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}\gamma^{-1}>}\otimes a_{<0,0><0,0>}1_{<0,0>}\\
&\otimes c_{(1,\alpha\beta^{-1})(2,\gamma)}S_{\gamma^{-1}}(a_{<0,0><-1,\gamma^{-1}>})\otimes c_{(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>})\\
\overset{(2.4)}{=}\;&c_{(1,\alpha\beta^{-1}\gamma^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}\gamma^{-1}>}\otimes a_{<0,0><0,0>}1_{<0,0>}\\
&\otimes c_{(2,\gamma\beta)(1,\gamma)}S_{\gamma^{-1}}(a_{<0,0><-1,\gamma^{-1}>})\otimes c_{(2,\gamma\beta)(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>})\\
\overset{(2.16)}{=}\;&c_{(1,\alpha\beta^{-1}\gamma^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}\gamma^{-1}>}\otimes a_{<0,0>}1_{<0,0>}\\
&\otimes c_{(2,\gamma\beta)(1,\gamma)}S_{\gamma^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>(2,\gamma^{-1})})\otimes c_{(2,\gamma\beta)(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>(1,\beta^{-1})})\\
\overset{(W9)}{=}\;&c_{(1,\alpha\beta^{-1}\gamma^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}\gamma^{-1}>}\otimes a_{<0,0>}1_{<0,0>}\otimes\\
&c_{(2,\gamma\beta)(1,\gamma)}S_{\beta^{-1}\gamma^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>})_{(1,\gamma)}\otimes c_{(2,\gamma\beta)(2,\beta)}S_{\beta^{-1}\gamma^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>})_{(2,\beta)}\end{aligned}$$

and also

$$
\begin{aligned}
& (id_{G(A)}\otimes\Delta_{\gamma,\beta})\circ^{r}\rho_{\gamma\beta}^{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>})\\
= \ & (id_{G(A)}\otimes\Delta_{\gamma,\beta})(c_{(1,\alpha\beta^{-1}\gamma^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}\gamma^{-1}>}\\
& \otimes a_{<0,0>}1_{<0,0>}\otimes c_{(2,\gamma\beta)}S_{(\gamma\beta)^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>}))\\
= \ & c_{(1,\alpha\beta^{-1}\gamma^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}\gamma^{-1}>}\otimes a_{<0,0>}1_{<0,0>}\\
& \otimes c_{(2,\gamma\beta)(1,\gamma)}S_{(\gamma\beta)^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>})_{(1,\gamma)}\otimes c_{(2,\gamma\beta)(2,\beta)}S_{(\gamma\beta)^{-1}}(a_{<-1,\beta^{-1}\gamma^{-1}>})_{(2,\beta)}.
\end{aligned}
$$

Thus we prove that $({}^{r}\rho_{\gamma}^{G(A)}\otimes id_{C_\beta})\circ^{r}\rho_{\beta}^{G(A)}=(id_{G(A)}\otimes\Delta_{\gamma,\beta})\circ^{r}\rho_{\gamma\beta}^{G(A)}$.
The proof is completed. □

The vector space

$$\bigoplus_{\beta\in\pi}\bigoplus_{\gamma\in\pi}\overline{C_\beta\otimes C_\gamma\otimes A}=G\circ F^{C}(\bigoplus_{\gamma\in\pi}\overline{C_\gamma\otimes A})=GF^{C}G(A)=G^{2}(A)$$

is also an object in ${}^{\pi-C}\mathcal{U}(H)_A$, i.e.,

$$
\begin{aligned}
& G\circ F^{C}(\bigoplus_{\gamma\in\pi}\overline{C_\gamma\otimes A})\\
= \ & \{\bigoplus_{\beta\in\pi}(c_\beta\cdot 1'_{<-1,\beta>}\otimes(\bigoplus_{\gamma\in\pi}d_\gamma\cdot 1_{<-1,\gamma>}\otimes a1_{<0,0>})\cdot 1'_{<0,0>})|\forall a\in A\}\\
= \ & \{\bigoplus_{\beta\in\pi}\bigoplus_{\gamma\in\pi}(c_\beta\cdot 1_{<-1,\beta>}\otimes d_\gamma\cdot 1_{<0,0><-1,\gamma>}\otimes a1_{<0,0><0,0>})|\forall a\in A\}
\end{aligned}
$$

and also have

$$
\begin{cases}
(c\cdot 1_{<-1,\alpha>}\otimes d\cdot 1_{<0,0><-1,\gamma>}\otimes a1_{<0,0><0,0>})\cdot b\\
\qquad\qquad = c\cdot b_{<-1,\alpha>}\otimes d\cdot b_{<0,0><-1,\gamma>}\otimes ab_{<0,0><0,0>},\\
{}^{l}\rho_{\beta}^{G^{2}(A)}(c\cdot 1_{<-1,\alpha>}\otimes d\cdot 1_{<0,0><-1,\gamma>}\otimes a1_{<0,0><0,0>})\\
\qquad = c_{(1,\beta)}\otimes c_{(2,\beta^{-1}\alpha)}\cdot 1_{<-1,\beta^{-1}\alpha>}\otimes d\cdot 1_{<0,0><-1,\gamma>}\otimes a1_{<0,0><0,0>},\\
{}^{r}\rho_{\beta}^{G^{2}(A)}(c\cdot 1_{<-1,\alpha>}\otimes d\cdot 1_{<0,0><-1,\gamma>}\otimes a1_{<0,0><0,0>})\\
\qquad\qquad = c\cdot 1_{<-1,\alpha>}\otimes d_{(1,\gamma\beta^{-1})}\cdot 1_{<0,0><-1,\gamma\beta^{-1}>}\\
\qquad\qquad\qquad \otimes a_{<0,0>}1_{<0,0><0,0>}\otimes d_{(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}),
\end{cases}
$$

for all $c\in C_\alpha$, $d\in C_\gamma$, $a,b\in A$ and $\beta,\alpha,\gamma\in\pi$.

Now, let $F_A:{}^{\pi-C}\mathcal{U}(H)_A\to{}^{\pi-C}\mathcal{U}$ be the other forgetful functor, which forgets the A-action and

$$\hat{G}:{}^{\pi-C}\mathcal{U}\to{}^{\pi-C}\mathcal{U}(H)_A,\ \hat{G}(N)=\{n\hat{\otimes}a=\varepsilon_C(n_{<-1,e>}\cdot 1_{<-1,e>})n_{<0,0>}\otimes 1_{<0,0>}a\}$$

its left adjoint, where for $N \in^{\pi-C} \mathcal{U}$, $\hat{G}(N) \in^{\pi-C} \mathcal{U}(H)_A$ via the structures: for all $\beta \in \pi$, $n \in N$ and $a, b \in A$,

$$(n\hat{\otimes}a) \cdot b = n\hat{\otimes}ab, \tag{3. 6}$$

$$\rho_\beta^{\hat{G}(N)}(n\hat{\otimes}a) = n_{<-1,\beta>} \cdot a_{<-1,\beta>} \otimes n_{<0,0>} \otimes a_{<0,0>}. \tag{3. 7}$$

Since $\mathcal{C} = \bigoplus_{\alpha\in\pi} C_\alpha$ is a left π-C-comodule via ${}^l\rho_\beta^{\mathcal{C}}(c) = c_{(1,\beta)} \otimes c_{(2,\beta^{-1}\alpha)}$, for all $c \in C_\alpha$ and $\beta \in \pi$. We have $\hat{G}(\mathcal{C}) = \bigoplus_{\alpha\in\pi} \hat{G}(C_\alpha)$, where

$$\begin{aligned}
\hat{G}(C_\alpha) \quad &= \quad \{\varepsilon_C(c_{(1,e)} \cdot 1_{<-1,e>})c_{(2,\alpha)} \otimes 1_{<0,0>}a\} \\
&\overset{(2.25)}{=} \{c \cdot \varepsilon_\alpha^t(1_{<-1,e>}) \otimes 1_{<0,0>}\} \\
&\overset{(2.18,W10)}{=} \{c \cdot S_{\alpha^{-1}}(\varepsilon_{\alpha^{-1}}^s(1_{<-1,e>})) \otimes 1_{<0,0>}a\} \\
&\overset{(2.18)}{=} \{c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a\}.
\end{aligned}$$

We can view $\hat{G}(\mathcal{C})$ as a weak Doi-Hopf π- module via

$$(c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a) \cdot b = c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}ab, \tag{3. 8}$$

$$\begin{aligned}
&{}^l\rho_\beta^{\hat{G}(\mathcal{C})}(c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a) \\
= \quad & c_{(1,\beta)} \cdot a_{<-1,\beta>} \otimes c_{(2,\beta^{-1}\alpha)} \cdot S_{\beta^{-1}\alpha}(1_{<-1,\beta^{-1}\alpha>}) \otimes 1_{<0,0>}a_{<0,0>},
\end{aligned} \tag{3. 9}$$

for all $c \in C_\alpha$, $a, b \in A$ and $\alpha, \beta \in \pi$.

From the discussion above, we have two types of weak Doi-Hopf π-modules, i.e., weak Doi-Hopf π-module $\bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes A}$ via (3.4) and (3.5) and weak Doi-Hopf π-module $\hat{G}(\mathcal{C})$ via (3.8) and (3.9). The following proposition will reveal the relation between them.

Proposition 3.2. *two types of weak Doi-Hopf π-modules $\bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes A}$ and $\hat{G}(\mathcal{C})$ are isomorphic in the category ${}^{\pi-C}\mathcal{U}(H)_A$.*

Proof. We construct the maps as follows: for all $c \in C_\alpha$, $a \in A$ and $\alpha \in \pi$,

$$u : \bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes A} \to \bigoplus_{\alpha\in\pi} \hat{G}(C_\alpha),$$

$$u(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) = c \cdot S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}) \otimes a_{<0,0>}, \tag{3. 10}$$

$$v : \bigoplus_{\alpha\in\pi} \hat{G}(C_\alpha) \to \bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes A},$$

$$v(c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a) = c \cdot a_{<-1,\alpha>} \otimes a_{<0,0>}. \tag{3. 11}$$

For all $c \in C_\alpha$, $a \in A$ and $\alpha \in \pi$, since

$$
\begin{aligned}
& v \circ u(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \\
= \;& v(c \cdot S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}) \otimes a_{<0,0>}) \\
= \;& c \cdot S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>})a_{<0,0><-1,\alpha>} \otimes a_{<0,0><0,0>} \\
\overset{(2.16)}{=} \;& c \cdot S_{\alpha^{-1}}(a_{<-1,e>(1,\alpha^{-1})})a_{<-1,e>(2,\alpha)} \otimes a_{<0,0>} \\
\overset{(2.14)}{=} \;& c\varepsilon^s_\alpha(a_{<-1,e>}) \otimes a_{<0,0>} \overset{(2.21)}{=} c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}.
\end{aligned}
$$

So we have $v \circ u = id_{G(A)}$. On the other hand, for all $c \in C_\alpha$, $a \in A$ and $\alpha \in \pi$, since

$$
\begin{aligned}
& u \circ v(c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a) \\
= \;& c \cdot a_{<-1,\alpha>}S_{\alpha^{-1}}(a_{<0,0><-1,\alpha^{-1}>}) \otimes a_{<0,0><0,0>} \\
\overset{(2.16)}{=} \;& c \cdot a_{<-1,e>(1,\alpha)}S_{\alpha^{-1}}(a_{<-1,e>(2,\alpha^{-1})}) \otimes a_{<0,0>} \\
\overset{(2.15)}{=} \;& c \cdot \varepsilon^t_\alpha(a_{<-1,e>}) \otimes a_{<0,0>} \overset{(2.22)}{=} c \cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>}) \otimes 1_{<0,0>}a.
\end{aligned}
$$

Thus we have $u \circ v = id_{\hat{G}(\mathcal{C})}$. The proof that u is both an A-module map and a π-C-comoodule map is straightforward. □

4. Integral of a weak Doi-Hopf π-Datum

Definition 4.1. Let (H, A, C) be a weak Doi-Hopf π -datum. A family of k-linear maps $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ are called an integral of (H, A, C) if

$$c_{(1,\alpha)}\otimes\theta_\beta(c_{(2,\beta)})(d) = d_{(2,\alpha)}(\theta_{\alpha\beta}(c)(d_{(1,(\alpha\beta)^{-1})}))_{<-1,\alpha>}\otimes(\theta_{\alpha\beta}(c)(d_{(1,(\alpha\beta)^{-1})}))_{<0,0>}, \tag{4. 1}$$

for all $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}, d \in C_{\beta^{-1}}$. An integral $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ is called total if

$$\sum_{\alpha\in\pi} \theta_\alpha(c_{(1,\alpha)})(c_{(2,\alpha^{-1})}) = \varepsilon_C(c \cdot 1_{<-1,e>})1_{<0,0>}, \tag{4. 2}$$

for all $\alpha \in \pi, c \in C_e$.

We shall now prove that the existence of an integral $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}$ permits the deformation of a k-linear map between two weak Doi-Hopf π-modules until it becomes a π-C-colinear map.

Proposition 4.2. *Let (H, A, C) be a weak Doi-Hopf π-datum, $N \in^{\pi-C} \mathcal{U}$, $M \in^{\pi-C} \mathcal{U}(H)_A$ and $u : N \to M$ a k-linear map. Suppose that there exists $\theta = \{\theta_\alpha : C_\alpha \to Hom(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ an integral. Then*

(1) The map

$$\tilde{u} : N \to M,\ \tilde{u}(n) = \sum_{\alpha\in\pi} u(n_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>}),$$

for all $n \in N$, is left π-C-colinear,

(2) If θ is a total integral and $f : M \to N$ is a morphism in ${}^{\pi-C}\mathcal{U}(H)_A$ which is a k-split injection (resp. a k-split surjection), then f has a π-C-colinear retraction (resp. a section).

Proof. (1) For $n \in N, \beta \in \pi$, we have

$$\begin{aligned}
& \rho^M_\beta \circ \tilde{u}(n) \\
= \; & \sum_{\alpha\in\pi} u(n_{<0,0>})_{<0,0><-1,\beta>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>})_{<-1,\beta>'} \\
& \otimes u(n_{<0,0>})_{<0,0><0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>})_{<0,0>'} \\
\overset{(2.6)}{=} \; & \sum_{\alpha\in\pi} u(n_{<0,0>})_{<-1,\alpha^{-1}\beta>(2,\beta)}\theta_\alpha(n_{<-1,\alpha>}) \\
& (u(n_{<0,0>})_{<-1,\alpha^{-1}\beta>(1,\alpha^{-1})})_{<-1,\beta>'} \\
& \otimes u(n_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}\beta><-1,\alpha^{-1}>})_{<0,0>'} \\
\overset{\nu=\alpha^{-1}\beta}{=} \; & \sum_{\nu\in\pi} u(n_{<0,0>})_{<-1,\nu>(2,\beta)}\theta_{\beta\nu^{-1}}(n_{<-1,\beta\nu^{-1}>}) \\
& (u(n_{<0,0>})_{<-1,\nu>(1,\nu\beta^{-1})})_{<-1,\beta>'} \\
& \otimes u(n_{<0,0>})_{<0,0>}\theta_{\beta\nu^{-1}}(n_{<-1,\beta\nu^{-1}>})(u(n_{<0,0>})_{<-1,\nu><-1,\beta\nu^{-1}>})_{<0,0>'} \\
\overset{(4.1)}{=} \; & \sum_{\nu\in\pi} n_{<-1,\beta\nu^{-1}>(1,\beta)} \otimes u(n_{<0,0>})_{<0,0>} \\
& \theta_{\nu^{-1}}(n_{<-1,\beta\nu^{-1}>(2,\nu^{-1})})(u(n_{<0,0>})_{<-1,\nu>}) \\
\overset{\alpha=\nu^{-1}}{=} \; & \sum_{\alpha\in\pi} n_{<-1,\beta\alpha>(1,\beta)} \otimes u(n_{<0,0>})_{<0,0>} \\
& \theta_\alpha(n_{<-1,\beta\alpha>(2,\alpha)})(u(n_{<0,0>})_{<-1,\alpha^{-1}>}) \\
\overset{(2.6)}{=} \; & \sum_{\alpha\in\pi} n_{<-1,\beta>} \otimes u(n_{<0,0><0,0>})_{<0,0>} \\
& \qquad\qquad \theta_\alpha(n_{<0,0><-1,\alpha>})(u(n_{<0,0><0,0>})_{<-1,\alpha^{-1}>}) \\
= \; & (id \otimes \tilde{u}) \circ \rho^N_\beta(n).
\end{aligned}$$

Hence $\tilde{u}$ is a left π-C-colinear.

(2) Let $u : N \to M$ be a k-linear retraction (resp. section) of f. Then $\tilde{u} : N \to M$, is a left π-C-colinear retraction (resp.section) of f. Assume first that u is a retraction of f. Then, for $m \in M$, one has

$$\begin{aligned}
(\tilde{u}\circ f)(m) &= \sum_{\alpha\in\pi} u(f(m)_{<0,0>})_{<0,0>}\theta_\alpha(f(m)_{<-1,\alpha>})(u(f(m)_{<0,0>})_{<-1,\alpha^{-1}>}) \\
&= \sum_{\alpha\in\pi} u(f(m_{<0,0>}))_{<0,0>}\theta_\alpha(m_{<-1,\alpha>})(u(f(m_{<0,0>}))_{<-1,\alpha^{-1}>}) \\
&= \sum_{\alpha\in\pi} m_{<0,0><0,0>}\theta_\alpha(m_{<-1,\alpha>})(m_{<0,0><-1,\alpha^{-1}>}) \\
&\overset{(2.6)}{=} \sum_{\alpha\in\pi} m_{<0,0>}\theta_\alpha(m_{<-1,e>(1,\alpha)})(m_{<-1,e>(2,\alpha^{-1})}) \\
&\overset{(4.2)}{=} m_{<0,0>}\cdot 1_{<0,0>}\varepsilon(m_{<-1,e>}\cdot 1_{<-1,e>}) = m.
\end{aligned}$$

Hence $\tilde{u} : N \to M$ is a left π-C-colinear retraction of f.

On the other hand, if u is a section of f , then, for $n \in N$, we have

$$\begin{aligned}
(f\circ\tilde{u})(n) &= \sum_{\alpha\in\pi} f(u(n_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>})) \\
&= \sum_{\alpha\in\pi} f(u(n_{<0,0>})_{<0,0>})\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>}) \\
&= \sum_{\alpha\in\pi} f(u(n_{<0,0>}))_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(f(u(n_{<0,0>}))_{<-1,\alpha^{-1}>}) \\
&= \sum_{\alpha\in\pi} n_{<0,0><0,0>}\theta_\alpha(n_{<-1,\alpha>})(n_{<0,0><-1,\alpha^{-1}>}) \\
&\overset{(2.6)}{=} \sum_{\alpha\in\pi} n_{<0,0>}\theta_\alpha(n_{<-1,e>(1,\alpha)})(n_{<-1,e>(2,\alpha^{-1})}) \\
&\overset{(4.2)}{=} n_{<0,0>}\cdot 1_{<0,0>}\varepsilon(n_{<-1,e>}\cdot 1_{<-1,e>}) = n,
\end{aligned}$$

i.e., $\tilde{u} : N \to M$ is a left π-C-colinear section of f. Thus the proof is completed. □

For weak Doi-Hopf π-modules $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$ and $\bigoplus_{\beta\in\pi}\bigoplus_{\gamma\in\pi}\overline{C_\beta\otimes C_\gamma\otimes A}$, we define a map as follows:

$$\begin{aligned}
&\rho_{G(A)} : \bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\to\bigoplus_{\beta\in\pi}\bigoplus_{\gamma\in\pi}\overline{C_\beta\otimes C_\gamma\otimes A}, \\
&\rho_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>}) \\
= &\bigoplus_{\gamma\in\pi} c_{(1,\gamma)}\cdot 1_{<-1,\gamma>}\otimes c_{(2,\gamma^{-1}\alpha)}\cdot 1_{<0,0><-1,\gamma^{-1}\alpha>}\otimes a1_{<0,0><0,0>}, \quad (4.\ 3)
\end{aligned}$$

for all $c \in C_\alpha$, $a \in A$ and $\alpha \in \pi$.

Lemma 4.3. *$\rho_{G(A)}$ is a morphism in the category ${}^{\pi-C}\mathcal{U}^{\pi-C}$.*

Proof. It is sufficient to prove that for all $\lambda \in \pi$, the following identities

$$ {}^{l}\rho_{\lambda}^{G^2(A)} \circ \rho_{G(A)} = (id_{C_\lambda} \otimes \rho_{G(A)}) \circ^{l} \rho_{\lambda}^{G(A)} $$

and

$$ {}^{r}\rho_{\lambda}^{G^2(A)} \circ \rho_{G(A)} = (\rho_{G(A)} \otimes id_{C_\lambda}) \circ^{r} \rho_{\lambda}^{G(A)} $$

hold. Now we shall check the first. In fact, for all $\lambda, \alpha \in \pi$ and $c \in C_\alpha$, we have

$$ \begin{aligned} & {}^{l}\rho_{\lambda}^{G^2(A)} \circ \rho_{G(A)}(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \\ = & \bigoplus_{\gamma\in\pi} c_{(1,\gamma)(1,\lambda)} \otimes c_{(1,\gamma)(2,\lambda^{-1}\gamma)} \cdot 1_{<-1,\lambda^{-1}\gamma>} \\ & \qquad \otimes c_{(2,\gamma^{-1}\alpha)} \cdot 1_{<0,0><-1,\gamma^{-1}\alpha>} \otimes a1_{<0,0><0,0>} \\ \overset{(2.1)}{=} & \bigoplus_{\gamma\in\pi} c_{(1,\lambda)} \otimes c_{(2,\lambda^{-1}\alpha)(1,\lambda^{-1}\gamma)} \cdot 1_{<-1,\lambda^{-1}\gamma>} \\ & \qquad \otimes c_{(2,\lambda^{-1}\alpha)(2,\gamma^{-1}\alpha)} \cdot 1_{<0,0><-1,\gamma^{-1}\alpha>} \otimes a1_{<0,0><0,0>} \\ \overset{\gamma=\lambda\omega}{=} & \bigoplus_{\omega\in\pi} c_{(1,\lambda)} \otimes c_{(2,\lambda^{-1}\alpha)(1,\omega)} \cdot 1_{<-1,\omega>} \\ & \qquad \otimes c_{(2,\lambda^{-1}\alpha)(2,\omega^{-1}\lambda^{-1}\alpha)} \cdot 1_{<0,0><-1,\omega^{-1}\lambda^{-1}\alpha>} \otimes a1_{<0,0><0,0>} \\ = & (id_{C_\lambda} \otimes \rho_{G(A)}) \circ^{l} \rho_{\lambda}^{G(A)}(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}). \end{aligned} $$

Similarly, we can check the other. Thus the proof of the Lemma is completed. □

Theorem 4.4. *Let (H, A, C) be a weak Doi-Hopf π-datum. The following conditions are equivalent:*

(1) There exists $\theta = \{\theta_\alpha : C_\alpha \to Hom(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ a total integral,

(2) the natural transformation

$$ \rho : F_A \circ 1_{\pi-C\mathcal{U}(H)_A} \to F_A \circ G \circ F^C $$

splits,

(3) The maps

$$ \begin{aligned} & \rho_{G(A)} : \bigoplus_{\alpha\in\pi} \overline{C_\alpha \otimes A} \to \bigoplus_{\beta\in\pi}\bigoplus_{\alpha\in\pi} \overline{C_\beta \otimes C_\alpha \otimes A}, \\ & \rho_{G(A)}(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \\ = & \bigoplus_{\beta\in\pi} c_{(1,\beta)} \cdot 1_{<-1,\beta>} \otimes c_{(2,\beta^{-1}\alpha)} \cdot 1_{<0,0><-1,\beta^{-1}\alpha>} \otimes a1_{<0,0><0,0>}, \end{aligned} \tag{4. 4} $$

for all $c \in C_\alpha$, $a \in A$ and $\alpha \in \pi$, splits in ${}^{\pi-C}\mathcal{U}^{\pi-C}$.

Proof. (1) ⇒ (2). Let $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ be a total integral. We have to construct a natural transformation ξ that splits ρ. Let $M \in^{\pi-C} \mathcal{U}(H)_A$

and $u_M : G(M) \to M$ be the k- linear retraction of $\rho_M : M \to G(M)$ given by

$$u_M(\bigoplus_{\alpha\in\pi} c_\alpha \cdot 1_{<-1,\alpha>} \otimes m_\alpha \cdot 1_{<0,0>}) = \varepsilon_C(c_e \cdot 1_{<-1,e>}) m_e \cdot 1_{<0,0>}.$$

In fact, for all $m \in M$, we have

$$\begin{aligned} u_M \circ \rho_M(m) &= u_M(\bigoplus_{\alpha\in\pi} m_{<-1,\alpha>} \otimes m_{<0,0>}) \\ &= \varepsilon_C(m_{<-1,e>}) m_{<0,0>} = m. \end{aligned}$$

We define

$$\xi_M = \tilde{u}_M : G(M) \to M,$$

$$\begin{aligned} &\xi_M(\bigoplus_{\alpha\in\pi} c_\alpha \cdot 1_{<-1,\alpha>} \otimes m_\alpha \cdot 1_{<0,0>}) \\ = &\sum_{\alpha\in\pi} m_{\alpha<0,0>} \cdot 1_{<0,0><0,0>} \theta_\alpha(c_\alpha \cdot 1_{<-1,\alpha>})(m_{\alpha<-1,\alpha^{-1}>} \cdot 1_{<0,0><-1,\alpha^{-1}>}). \end{aligned}$$

It follows from Proposition 4.2 that the map ξ_M is a left π-C-colinear retraction of ρ_M.

It remains to prove that $\xi = \{\xi_M | M \in^{\pi-C} \mathcal{U}(H)_A\}$ is a natural transformation. Let $f : M \to N$ be a morphism in ${}^{\pi-C}\mathcal{U}(H)_A$. We have to prove that the diagram

$$\begin{array}{ccc} G(M) & \xrightarrow{\quad \xi_M \quad} & M \\ \Big\downarrow G(f) & & \Big\downarrow f \\ G(N) & \xrightarrow{\quad \xi_N \quad} & N \end{array}$$

is commutative. Using that f is right A-linear, we have

$$\begin{aligned} &(f \circ \xi_M)(\bigoplus_{\alpha\in\pi} c_\alpha \cdot 1_{<-1,\alpha>} \otimes m_\alpha \cdot 1_{<0,0>}) \\ = \; & f(\sum_{\alpha\in\pi} m_{\alpha<0,0>} \cdot 1_{<0,0><0,0>} \\ &\qquad \theta_\alpha(c_\alpha \cdot 1_{<-1,\alpha>})(m_{\alpha<-1,\alpha^{-1}>} \cdot 1_{<0,0><-1,\alpha^{-1}>})) \\ = \; & f(\sum_{\alpha\in\pi} m_{\alpha<0,0>}) \cdot 1_{<0,0><0,0>} \\ &\qquad \theta_\alpha(c_\alpha \cdot 1_{<-1,\alpha>})(m_{\alpha<-1,\alpha^{-1}>} \cdot 1_{<0,0><-1,\alpha^{-1}>}) \end{aligned}$$

and using that f is left π- C-colinear

$$\begin{aligned} &(\xi_N \circ (G(f)))(\bigoplus_{\alpha\in\pi} c_\alpha \cdot 1_{<-1,\alpha>} \otimes m_\alpha \cdot 1_{<0,0>}) \\ = \; & \xi_N(\bigoplus_{\alpha\in\pi} c_\alpha \cdot 1_{<-1,\alpha>} \otimes f(m_\alpha \cdot 1_{<0,0>})) \end{aligned}$$

$$= \sum_{\alpha\in\pi} f(m_\alpha)_{<0,0>}\cdot 1_{<0,0><0,0>}\theta_\alpha(c_\alpha\cdot 1_{<-1,\alpha>})$$
$$(f(m_\alpha)_{<-1,\alpha^{-1}>}\cdot 1_{<0,0><-1,\alpha^{-1}>})$$
$$= \sum_{\alpha\in\pi} f(m_{\alpha<0,0>})\cdot 1_{<0,0><0,0>}$$
$$\theta_\alpha(c_\alpha\cdot 1_{<-1,\alpha>})(m_{\alpha<-1,\alpha^{-1}>}\cdot 1_{<0,0><-1,\alpha^{-1}>}).$$

I.e., ξ is a natural transformation that splits ρ.

(2) $\Rightarrow$ (3).Assume that for any $M\in^{\pi-C}\mathcal{U}(H)_A$. $\rho_M : M\to G(M)$ splits in the category ${}^{\pi-C}\mathcal{U}$ of left π-C-comodules and the character of the splitting is functorial. In particular,

$$\rho_{G(A)} : \bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\to\bigoplus_{\beta\in\pi}\bigoplus_{\alpha\in\pi}\overline{C_\beta\otimes C_\alpha\otimes A},$$
$$\rho_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>})$$
$$= \bigoplus_{\beta\in\pi} c_{(1,\beta)}\cdot 1_{<-1,\beta>}\otimes c_{(2,\beta^{-1}\alpha)}\cdot 1_{<0,0><-1,\beta^{-1}\alpha>}\otimes a1_{<0,0><0,0>} \quad (4.\ 5)$$

splits in ${}^{\pi-C}\mathcal{U}$ and let

$$\xi_{G(A)} : \bigoplus_{\beta\in\pi}\bigoplus_{\alpha\in\pi}\overline{C_\beta\otimes C_\alpha\otimes A}\to\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$$

be the left π-C-colinear retraction of $\rho_{G(A)}$. Using the naturality of ξ, we will prove that ξ is a right π-C-colinear, where $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$ and $\bigoplus_{\beta\in\pi}\bigoplus_{\alpha\in\pi}\overline{C_\beta\otimes C_\alpha\otimes A}$ are right π- C-comodules in section 3.

First, let V be a vector space and $M\in^{\pi-C}\mathcal{U}(H)_A$. Then $M\otimes V\in^{\pi-C}\mathcal{U}(H)_A$ via the structures arising from the ones of M, i.e.,

$$(m\otimes v)\cdot a = ma\otimes v,\ \rho_\alpha^{M\otimes V} = \rho_\alpha^M\otimes id_V,$$

for all $\alpha\in\pi, m\in M, a\in A$, and $v\in V$. Let $v\in V$ and $g_v : M\to M\otimes V,\ g_v(m) = m\otimes v$. Then g_v is a morphism in ${}^{\pi-C}\mathcal{U}(H)_A$. From the naturality of ξ, we obtain that $g_v\circ\xi_M = \xi_{M\otimes V}\circ G(g_v)$. Hence

$$\begin{aligned}\xi_{M\otimes V}(c\cdot 1_{<-1,\alpha>}\otimes m\cdot 1_{<0,0>}\otimes v) &= g_v(\xi_M(c\cdot 1_{<-1,\alpha>}\otimes m\cdot 1_{<0,0>}))\\ &= \xi_M(c\cdot 1_{<-1,\alpha>}\otimes m\cdot 1_{<0,0>})\otimes v,\end{aligned}$$

for all $\alpha\in\pi, c\in C_\alpha$ and $m\in M, v\in V$. Thus we prove that $\xi_{M\otimes V} = \xi_M\otimes id_V$.

In particular, let us take $M = G(A)$ and $V = \bigoplus_{\beta\in\pi} C_\beta$ viewed only as a vector space. Then $G(A)\otimes\bigoplus_{\beta\in\pi} C_\beta = \bigoplus_{\alpha\in\pi}\bigoplus_{\beta\in\pi}\overline{C_\alpha\otimes A}\otimes C_\beta\in^{\pi-C}\mathcal{U}(H)_A$ via the structures:

$$(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>}\otimes d)\cdot b = c\cdot b_{<-1,\alpha>}\otimes ab_{<0,0>}\otimes d, \quad (4.\ 6)$$

$$\begin{aligned} & {}^{l}\rho_{\gamma}^{G(A)\otimes\bigoplus_{\beta\in\pi}C_{\beta}}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>}\otimes d) \\ = \ & c_{(1,\gamma)}\otimes c_{(2,\gamma^{-1}\alpha)}\cdot 1_{<-1,\gamma^{-1}\alpha>}\otimes a1_{<0,0>}\otimes d, \end{aligned} \quad (4.\ 7)$$

for all $\alpha,\beta,\gamma\in\pi, c\in C_{\alpha}, a\in A$ and $d\in C_{\beta}$. With these structures, the map

$$\begin{aligned} & f={}^{r}\rho^{G(A)}: G(A)\to G(A)\otimes\bigoplus_{\beta\in\pi}C_{\beta} \\ & {}^{r}\rho^{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>}) \\ = \ & \bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}1_{<0,0>}\otimes c_{(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}), \end{aligned}$$

for all $a\in A$, $\alpha\in\pi$ and $c\in C_{\alpha}$, is a morphism in ${}^{\pi-C}\mathcal{U}(H)_A$. In fact, for all $a,b\in A$, $\alpha\in\pi$ and $c\in C_{\alpha}$, we have

$${}^{r}\rho^{G(A)}((c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>})\cdot b) = c\cdot b_{<-1,\alpha>}\otimes ab_{<0,0>}$$

$$\begin{aligned} = \ & \bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot b_{<-1,\alpha>(1,\alpha\beta^{-1})}1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}b_{<0,0><0,0>}1_{<0,0>} \\ & \otimes c_{(2,\beta)}\cdot b_{<-1,\alpha>(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}b_{<0,0><-1,\beta^{-1}>}) \\ \overset{(2.16)}{=} \ & \bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot b_{<-1,\alpha\beta^{-1}>(1,\alpha)(1,\alpha\beta^{-1})}1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}b_{<0,0>}1_{<0,0>} \\ & \otimes c_{(2,\beta)}\cdot b_{<-1,\alpha\beta^{-1}>(1,\alpha)(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}b_{<-1,\alpha\beta^{-1}>(2,\beta^{-1})}) \\ \overset{(2.1)}{=} \ & \bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot b_{<-1,\alpha\beta^{-1}>(1,\alpha\beta^{-1})}1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}b_{<0,0>}1_{<0,0>} \\ & \otimes c_{(2,\beta)}\cdot b_{<-1,\alpha\beta^{-1}>(2,e)(1,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}b_{<-1,\alpha\beta^{-1}>(2,e)(2,\beta^{-1})}) \\ \overset{(2.15)}{=} \ & \bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot b_{<-1,\alpha\beta^{-1}>(1,\alpha\beta^{-1})}1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}b_{<0,0>}1_{<0,0>} \\ & \otimes c_{(2,\beta)}\cdot \varepsilon_{\beta}^{t}(b_{<-1,\alpha\beta^{-1}>(2,e)})S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}) \\ \overset{(W4)}{=} \ & \bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot b_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}b_{<0,0>}\otimes c_{(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}) \\ \overset{(4.6)}{=} \ & (\bigoplus_{\beta\in\pi}c_{(1,\alpha\beta^{-1})}\cdot 1_{<-1,\alpha\beta^{-1}>}\otimes a_{<0,0>}1_{<0,0>}\otimes c_{(2,\beta)}S_{\beta^{-1}}(a_{<-1,\beta^{-1}>}))\cdot b. \end{aligned}$$

Thus f is A-linear. We are left to prove that f is π-C-colinear. It is sufficient to check that for all $\gamma\in\pi$, ${}^{l}\rho_{\gamma}^{G(A)\otimes\bigoplus_{\beta\in\pi}C_{\beta}}\circ f=(id_{C_{\gamma}}\otimes f)\circ{}^{l}\rho_{\gamma}^{G(A)}$ holds. As a matter of fact, for all $\gamma \in \pi$, $a \in A$, $\alpha \in \pi$ and $c \in C_\alpha$, since

$$
\begin{aligned}
& {}^{l}\rho^{G(A)\otimes\bigoplus_{\beta\in\pi} C_\beta} \circ f(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \\
= \;& c_{(1,\gamma)} \otimes f(c_{(2,\gamma^{-1}\alpha)} \cdot 1_{<-1,\gamma^{-1}\alpha>} \otimes a1_{<0,0>}) \\
= \;& \bigoplus_{\beta\in\pi} c_{(1,\gamma)} \otimes c_{(2,\gamma^{-1}\alpha)(1,\gamma^{-1}\alpha\beta^{-1})} \cdot 1_{<-1,\gamma^{-1}\alpha\beta^{-1}>} \\
& \otimes a_{<0,0>}1_{<0,0>}) \otimes c_{(2,\gamma^{-1}\alpha)(2,\beta)} S_{\beta^{-1}}(a_{<-1,\beta^{-1}>})
\end{aligned}
$$

and

$$
\begin{aligned}
& {}^{l}\rho_\gamma^{G(A)\otimes\bigoplus_{\beta\in\pi} C_\beta} \circ f(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \\
= \;& {}^{l}\rho_\gamma^{G(A)\otimes\bigoplus_{\beta\in\pi} C_\beta} (\bigoplus_{\beta\in\pi} c_{(1,\alpha\beta^{-1})} \cdot 1_{<-1,\alpha\beta^{-1}>} \\
& \qquad \otimes a_{<0,0>}1_{<0,0>} \otimes c_{(2,\beta)} S_{\beta^{-1}}(a_{<-1,\beta^{-1}>})) \\
= \;& \bigoplus_{\beta\in\pi} c_{(1,\alpha\beta^{-1})(1,\gamma)} \otimes c_{(1,\alpha\beta^{-1})(2,\gamma^{-1}\alpha\beta^{-1})} \cdot 1_{<-1,\gamma^{-1}\alpha\beta^{-1}>} \\
& \qquad \otimes a_{<0,0>}1_{<0,0>} \otimes c_{(2,\beta)} S_{\beta^{-1}}(a_{<-1,\beta^{-1}>})) \\
= \;& \bigoplus_{\beta\in\pi} c_{(1,\gamma)} \otimes c_{(2,\gamma^{-1}\alpha)(1,\gamma^{-1}\alpha\beta^{-1})} \cdot 1_{<-1,\gamma^{-1}\alpha\beta^{-1}>} \\
& \qquad \otimes a_{<0,0>}1_{<0,0>} \otimes c_{(2,\gamma^{-1}\alpha)(2,\beta)} S_{\beta^{-1}}(a_{<-1,\beta^{-1}>})).
\end{aligned}
$$

Thus f is π-C-colinear. From the naturality of ξ, we gain the following commutative diagram

$$
\begin{array}{ccc}
G^2(A) & \xrightarrow{\quad \xi_{G(A)} \quad} & G(A) \\
\downarrow{\scriptstyle G(f) = id \otimes f} & & \downarrow{\scriptstyle f} \\
G(G(A) \otimes \bigoplus_{\alpha\in\pi} C_\alpha) & \xrightarrow{\quad \xi_{G(A)} \otimes id_{\bigoplus_{\alpha\in\pi} C_\alpha} \quad} & G(A) \otimes \bigoplus_{\alpha\in\pi} C_\alpha
\end{array} \qquad (*)
$$

I.e., ξ is also a right π-C-colinear.

(3) $\Rightarrow$ (1). From (2), we have that

$$
\rho_{G(A)} : G(A) \to \bigoplus_{\beta\in\pi} \bigoplus_{\alpha\in\pi} \overline{C_\beta \otimes C_\alpha \otimes A},
$$

$$
\begin{aligned}
& \rho_{G(A)}(c \cdot 1_{<-1,\alpha>} \otimes a1_{<0,0>}) \\
= \;& \bigoplus_{\beta\in\pi} c_{(1,\beta)} \cdot 1_{<-1,\beta>} \otimes c_{(2,\beta^{-1}\alpha)} \cdot 1_{<0,0><-1,\beta^{-1}\alpha>} \otimes a1_{<0,0><0,0>}
\end{aligned} \qquad (4.\ 8)
$$

splits in ${}^{\pi-C}\mathcal{U}^{\pi-C}$. Let

$$
\xi_{G(A)} : \bigoplus_{\alpha\in\pi} \bigoplus_{\beta\in\pi} C_\alpha \otimes C_\beta \otimes A \to \bigoplus_{\alpha\in\pi} C_\alpha \otimes A,
$$

be a split of $\rho_{G(A)}$ in ${}^{\pi-C}\mathcal{U}^{\pi-C}$. In particular,

$$\xi_{G(A)}(\bigoplus_{\beta\in\pi} c_{(1,\beta)}\cdot 1_{<-1,\beta>}\otimes c_{(2,\beta^{-1}\alpha)}\cdot 1_{<0,0><-1,\beta^{-1}\alpha>}\otimes a1_{<0,0><0,0>})$$
$$= c\cdot 1_{<-1,\alpha>}\otimes a1_{<0,0>}. \tag{4. 9}$$

We define

$$\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A),$$
$$\theta_\alpha(c)(d) = (\varepsilon_C\otimes id)\circ P_{\overline{C_e\otimes A}}\circ\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes d\cdot 1_{<0,0><-1,\alpha^{-1}>}\otimes 1_{<0,0><0,0>}), \tag{4. 10}$$

for all $\alpha\in\pi, c\in C_\alpha, d\in C_{\alpha^{-1}}$, where $P_{\overline{C_e\otimes A}}$ is the natural projective map from $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$ onto $\overline{C_e\otimes A}$, we will prove that θ is a total integral, i.e., E.q (4.1)and (4.2) hold. For all $c\in C_e$, we have

$$\begin{aligned}
&\sum_{\alpha\in\pi}\theta_\alpha(c_{(1,\alpha)})(c_{(2,\alpha^{-1})})\\
=\ & (\varepsilon_C\otimes id)\circ P_{\overline{C_e\otimes A}}\circ\xi_{G(A)}(\bigoplus_{\alpha\in\pi} c_{(1,\alpha)}\cdot 1_{<-1,\alpha>}\\
&\qquad \otimes c_{(2,\alpha^{-1})}\cdot 1_{<0,0><-1,\alpha^{-1}>}\otimes 1_{<0,0><0,0>})\\
\overset{(4.9)}{=}\ & (\varepsilon_C\otimes id)\circ P_{\overline{C_e\otimes A}}(c\cdot 1_{<-1,\alpha>}\otimes 1_{<0,0>}) = \varepsilon_C(c\cdot 1_{<-1,\alpha>})1_{<0,0>}.
\end{aligned}$$

So E.q (4.2) holds.

Now, we are left to check the other E.q(4.1). For all $\alpha,\beta\in\pi$ and $c\in C_{\alpha\beta}, d\in C_{\beta^{-1}}$, the left hand side of E.q (4.1) is

$$\begin{aligned}
& c_{(1,\alpha)}\otimes\theta_\beta(c_{(2,\beta)})(d)\\
=\ & c_{(1,\alpha)}\otimes(\varepsilon\otimes id)P_{\overline{C_e\otimes A}}\xi_{G(A)}(c_{(2,\beta)}\cdot 1_{<-1,\beta>}\\
& \otimes d\cdot 1_{<0,0><-1,\beta^{-1}>}\otimes 1_{<0,0><0,0>})\\
=\ & (id_{C_\alpha}\otimes(\varepsilon\otimes id)\circ P_{\overline{C_e\otimes A}})\rho_\alpha^{G(A)}(\xi_{G(A)}(c\cdot 1_{<-1,\alpha\beta>}\\
& \otimes d\cdot 1_{<0,0><-1,\beta^{-1}>}\otimes 1_{<0,0><0,0>}))\\
=\ & (P_{\overline{C_\alpha\otimes A}})\circ\xi_{G(A)}(c\cdot 1_{<-1,\alpha\beta>}\otimes d\cdot 1_{<0,0><-1,\beta^{-1}>}\otimes 1_{<0,0><0,0>}),
\end{aligned}$$

where $P_{\overline{C_\alpha\otimes A}}$ is the natural projective map from $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$ onto $\overline{C_\alpha\otimes A}$. In order to compute the right hand side of E.q (4.1) , we adopt the temporary notation

$$\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes d_{(1,(\alpha\beta)^{-1})}\cdot 1_{<0,0><-1,(\alpha\beta)^{-1}>}\otimes 1_{<0,0><0,0>}) = \bigoplus_{\gamma\in\pi} p_\gamma\otimes q_\gamma.$$

Now,

$$\begin{aligned}
& d_{(2,\alpha)}\theta_{\alpha\beta}(c)(d_{(1,(\alpha\beta)^{-1})})_{<-1,\alpha>}\otimes\theta_{\alpha\beta}(c)(d_{(1,(\alpha\beta)^{-1})})_{<0,0>}\\
&\quad = d_{(2,\alpha)}q_{e<-1,\alpha>}\otimes\varepsilon_C(p_e)q_{e<0,0>}.
\end{aligned}$$

Hence (4.1) is equivalent to

$$\begin{aligned}
&(P_{\overline{C_\alpha\otimes A}})\circ\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes d\cdot 1_{<0,0><-1,\beta^{-1}>}\otimes 1_{<0,0><0,0>})\\
= \;&(\cdot\otimes id_A)(d_{(2,\alpha)}\otimes id_{H_\alpha}\otimes id_A)(\varepsilon_C\otimes\rho^A_\alpha)(P_{\overline{C_e\otimes A}})\cdot\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\\
&\otimes d_{(1,(\alpha\beta)^{-1})}\cdot 1_{<0,0><-1,(\alpha\beta)^{-1}>}\otimes 1_{<0,0><0,0>}), \qquad (4.\ 11)
\end{aligned}$$

for all $\alpha,\beta\in\pi$ and $c\in C_{\alpha\beta}, d\in C_{\beta^{-1}}$. Denoting

$$\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes d\cdot 1_{<0,0><-1,\beta^{-1}>}\otimes 1_{<0,0><0,0>})=\bigoplus_{\nu\in\pi}D_\nu\otimes A_\nu$$

and evaluating the diagram at($*$) ,we obtain

$$\begin{aligned}
&\bigoplus_{\gamma\in\pi}\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\otimes d_{(1,(\underline{\gamma})^{-1})}\cdot 1_{<0,0><-1,(\underline{\gamma})^{-1}>}\otimes 1_{<0,0><0,0>})\otimes d_{(2,\gamma)}\\
= \;&\bigoplus_{\gamma\in\pi}\bigoplus_{\nu\in\pi}D_{\nu(1,\nu\gamma^{-1})}\cdot 1_{<-1,\nu\gamma^{-1}>}\otimes A_{\nu<0,0>}1_{<0,0>}\otimes D_{\nu(2,\gamma)}S_{\gamma^{-1}}(A_{\nu<-1,\gamma^{-1}>}),
\end{aligned}$$

for all $\alpha,\beta\in\pi$, $c\in C_a$ and $d\in C_{\beta^{-1}}$. Hence, for all $\alpha\in\pi$, we have

$$\begin{aligned}
&(d_{(2,\alpha)}\otimes id_A)(\varepsilon_C\otimes id_A)(P_{C_e\otimes A})\circ\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\\
&\otimes d_{(1,(\underline{\alpha})^{-1})}\cdot 1_{<0,0><-1,(\alpha\beta)^{-1}>}\otimes 1_{<0,0><0,0>})\\
= \;&D_{\alpha(2,\alpha)}S_{\alpha^{-1}}(A_{\alpha<-1,\alpha^{-1}>})\varepsilon_C(D_{\alpha(1,e)}\cdot 1_{<-1,e>})\otimes A_{\alpha<0,0>}1_{<0,0>}\\
\overset{(2.25)}{=} \;&D_\alpha\cdot\varepsilon^t_\alpha(1_{<-1,e>})S_{\alpha^{-1}}(A_{\alpha<-1,\alpha^{-1}>})\otimes A_{\alpha<0,0>}1_{<0,0>}\\
\overset{(W12)}{=} \;&D_\alpha\cdot S_{\alpha^{-1}}(1_{<-1,\alpha^{-1}>})S_{\alpha^{-1}}(A_{\alpha<-1,\alpha^{-1}>})\otimes A_{\alpha<0,0>}1_{<0,0>}\\
\overset{(W7,3.1)}{=} \;&D_\alpha\cdot S_{\alpha^{-1}}(A_{\alpha<-1,\alpha^{-1}>})\otimes A_{\alpha<0,0>}.
\end{aligned}$$

Now we apply ρ^A_α to the second factor of both sides, we have

$$\begin{aligned}
&(id_{C_\alpha}\otimes\rho^A_\alpha)\circ(d_{(2,\alpha)}\otimes id_A)(\varepsilon_C\otimes id_A)(P_{C_e\otimes A})\circ\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\\
&\otimes d_{(1,(\alpha\beta)^{-1})}\cdot 1_{<0,0><-1,(\alpha\beta)^{-1}>}\otimes 1_{<0,0><0,0>})\\
&=D_\alpha\cdot S_{\alpha^{-1}}(A_{\alpha<-1,\alpha^{-1}>})\otimes A_{\alpha<0,0><-1,\alpha>}\otimes A_{\alpha<0,0><0,0>}.
\end{aligned}$$

Let the second factor act on the first one, we gain

$$\begin{aligned}
&(\cdot\otimes id_A)\circ(d_{(2,\alpha)}\otimes id_{H_\alpha}\otimes id_A)\circ(id_{C_\alpha}\otimes\rho^A_\alpha)\circ(d_{(2,\alpha)}\otimes id_A)\\
&(\varepsilon_C\otimes id_A)(P_{C_e\otimes A})\circ\xi_{G(A)}(c\cdot 1_{<-1,\alpha>}\\
&\otimes d_{(1,(\alpha\beta)^{-1})}\cdot 1_{<0,0><-1,(\alpha\beta)^{-1}>}\otimes 1_{<0,0><0,0>})\\
= \;&D_\alpha\cdot S_{\alpha^{-1}}(A_{\alpha<-1,e>(1,\alpha^{-1})})A_{\alpha<-1,e>(2,\alpha)}\otimes A_{\alpha<0,0>}\\
\overset{(2.14)}{=} \;&D_\alpha\cdot\varepsilon^s_\alpha(A_{\alpha<-1,e>})\otimes A_{\alpha<0,0>}\overset{(2.21)}{=}D_\alpha\otimes A_\alpha.
\end{aligned}$$

Since

$$(\cdot \otimes id_A) \circ (d_{(2,\alpha)} \otimes id_{H_\alpha} \otimes id_A) \circ (id_{C_\alpha} \otimes \rho_\alpha^A) \circ (d_{(2,\alpha)} \otimes id_A)$$
$$= (\cdot \otimes id_A)(d_{(2,\alpha)} \otimes id_{H_\alpha} \otimes id_A)\rho_\alpha^A,$$

we can gain E.q (4.11). To sum up, θ is a total integral. □

From the proof of Theorem 4.4, if there exists

$$\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$$

a total integral, then the natural transformation

$$\xi : F_A \circ G \circ F^C \to F_A \circ 1_{\pi-C}\mathcal{U}(H)_A$$

splits ρ, so we have

$$\xi_M \circ F_A(\rho_M) = I_{F_A(M)}$$

for any $M \in^{\pi-C} \mathcal{U}(H)_A$, by Rafael$'$s theorem, F^C is F_A-separable. Conversely, if F^C is F_A-separable, then there exists a natural transformation

$$\xi : F_A \circ G \circ F^C \to F_A \circ 1_{\pi-C}\mathcal{U}(H)_A$$

such that

$$\xi_M \circ F_A(\rho_M) = I_{F_A(M)}$$

for any $M \in^{\pi-C} \mathcal{U}(H)_A$, we have that ξ splits ρ, by Theorem 4.4, there exists $\theta = \{\theta_\alpha : C_\alpha \to \mathrm{Hom}(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ a total integral. Immediately, we have the following conclusion.

Corollary 4.5. *Let (H, A, C) be a weak Doi-Hopf π-datum. The following statements are equivalent:*

(1)F^C is F_A-separable,

(2) There exists $\theta = \{\theta_\alpha : C_\alpha \to Hom(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ a total integral,

Let $\pi = \{e\}$ be a trivial group, the weak Hopf π-coalgebras is just the weak Hopf algebras. Combining Theorem 4.4 and Corollary 4.5, we have the following result.

Corollary 4.6. *Let (H, A, C) be a weak Doi-Hopf datum. The following statements are equivalent:*

(1) F^C is F_A-separable,

(2) There exists $\theta : C \to Hom(C, A)$ a total integral,

(3) the natural transformation

$$\rho : F_A \circ 1_{C}\mathcal{U}(H)_A \to F_A \circ G \circ F^C$$

splits,

(4) the map

$$\rho^{l}_{\overline{C\otimes A}}:\overline{C\otimes A}\to\overline{C\otimes C\otimes A},$$

$$\rho_{\overline{C\otimes A}}(c\cdot 1_{<-1>}\otimes a1_{<0>})=c_{(1)}\otimes c_{(2)}\cdot 1_{<-1>}\otimes a1_{<0>}$$

splits in ${}^{C}\mathcal{U}^{C}$ *, the category of* C*-bicomodules. Consequently, if one of the equivalent conditions holds, any weak Doi-Hopf module is injective as a left* C*-comodule.*

We shall prove now the main applications of the existence of a total integral.

Theorem 4.7. *Let* (H,A,C) *be a weak Doi-Hopf* π*-datum and suppose that there exists* $\theta=\{\theta_\alpha: C_\alpha\to Hom(C_{\alpha^{-1}},A)\}$ *a total integral. Then for any* $M\in^{\pi-C}\mathcal{U}(H)_A$*, the map*

$$f:\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\otimes M\to M,$$

$$\begin{aligned}&f(\bigoplus_{\alpha\in\pi}c_\alpha\cdot 1_{<-1,\alpha>}\otimes a_\alpha\cdot 1_{<0,0>}\otimes m)\\ =&\sum_{\alpha\in\pi}m_{<0,0>}\theta_\alpha(c_\alpha S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))(m_{<-1,\alpha^{-1}>})a_{<0,0>},\end{aligned}$$

for all $m\in M$ *is a k-split epimorphism in* ${}^{\pi-C}\mathcal{U}(H)_A$*. In particular,* $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}$ *is a generator in the category* ${}^{\pi-C}\mathcal{U}(H)_A$*.*

Proof. $\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\otimes M$ is viewed as an object in ${}^{\pi-C}\mathcal{U}(H)_A$ with the structures as follows,

$$(c\cdot 1_{<-1,\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m)\cdot b=c\cdot b_{<-1,\alpha>}\otimes ab_{<0,0>}\otimes m,$$

$$\rho_\beta^{\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\otimes M}(c\cdot 1_{<-1,\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m)=c_{(1,\beta)}\otimes c_{(2,\beta^{-1}\alpha)}\cdot 1_{<-1,\beta^{-1}\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m,$$

for all $\alpha,\beta\in\pi$, $c\in C_\alpha$, $a,b\in A$ and $m\in M$. First, we shall prove that f is a split surjection. Let

$$g:M\to\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\otimes M,$$

$$g(m)=\bigoplus_{\alpha\in\pi}m_{<-1,\alpha>}\cdot 1_{<-1,\alpha>}\otimes 1_{<0,0>}\otimes m_{<0,0>},$$

for all $m \in M$. Then g is left π- C-colinear (but is not right A-linear) and for $m \in M$, we have

$$\begin{aligned}
(f\circ g)(m) &= f(\bigoplus_{\alpha\in\pi} m_{<-1,\alpha>}\cdot 1_{<-1,\alpha>}\otimes 1_{<0,0>}\otimes m_{<0,0>})\\
&= \sum_{\alpha\in\pi} m_{<0,0><0,0>}\theta_\alpha(m_{<-1,\alpha>})(m_{<0,0><-1,\alpha^{-1}>})\\
&\overset{(2.6)}{=} \sum_{\alpha\in\pi} m_{<0,0>}\theta_\alpha(m_{<-1,e>(1,\alpha)})(m_{<-1,e>(2,\alpha^{-1}>})\\
&\overset{(4.2)}{=} m_{<0,0>}\cdot 1_{<0,0>}\varepsilon_C(m_{<-1,e>}\cdot 1_{<-1,e>}) = m.
\end{aligned}$$

Thus g is a left π- C-colinear section of f. For $a, b \in A, c \in C_\alpha, m \in M$, we have

$$\begin{aligned}
& f((c\cdot 1_{<-1,\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m)\cdot b)\\
=\ & f(c\cdot b_{<-1,\alpha>}\otimes ab_{<0,0>}\otimes m)\\
=\ & m_{<0,0>}\theta_\alpha(c\cdot b_{<-1,\alpha>}S_{\alpha^{-1}}(b_{<0,0><-1,\alpha^{-1}>})S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))\\
& (m_{<-1,\alpha^{-1}>})a_{<0,0>}b_{<0,0><0,0>}\\
\overset{(2.16,2.15)}{=}\ & m_{<0,0>}\theta_\alpha(c\cdot \varepsilon^t_\alpha(b_{<-1,e>})S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))(m_{<-1,\alpha^{-1}>})a_{<0,0>}b_{<0,0>}\\
\overset{(2.22)}{=}\ & m_{<0,0>}\theta_\alpha(cS_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))(m_{<-1,\alpha^{-1}>})a_{<0,0>}b\\
=\ & f(c\cdot 1_{<-1,\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m)\cdot b,
\end{aligned}$$

i.e., f is right A-linear. It remains to prove that f is also left π- C-colinear. First, for all $\beta \in \pi$, $c_\alpha \in C_\alpha$, $a \in A$ and $m \in M$, we compute

$$\begin{aligned}
& (id_C\otimes f)\circ\rho_\beta^{\bigoplus_{\alpha\in\pi}\overline{C_\alpha\otimes A}\otimes M}(c\cdot 1_{<-1,\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m)\\
& = (c_{(1,\beta)}\otimes m_{<0,0>}\theta_{\beta^{-1}\alpha}(c_{(2,\beta^{-1}\alpha)}S_{\alpha^{-1}\beta}(a_{<-1,\alpha^{-1}\beta>}))(m_{<-1,\alpha^{-1}\beta>})a_{<0,0>})
\end{aligned}$$

and

$$\begin{aligned}
& \rho^M_\beta(f(c\cdot 1_{<-1,\alpha>}\otimes a\cdot 1_{<0,0>}\otimes m))\\
=\ & \rho^M_\beta(m_{<0,0>}\theta_\alpha(c_\alpha S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))(m_{<-1,\alpha^{-1}>})a_{<0,0>})\\
=\ & (m_{<0,0><-1,\beta>}(\theta_\alpha(c_\alpha S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))(m_{<-1,\alpha^{-1}>}))_{<-1,\beta>}\\
& \qquad a_{<0,0><-1,\beta>}\otimes m_{<0,0><0,0>}(\theta_\alpha(c_\alpha S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}>}))\\
& \qquad\qquad (m_{<-1,\alpha^{-1}>}))_{<0,0>}a_{<0,0><0,0>})\\
\overset{(2.6)}{=}\ & (m_{<-1,\alpha^{-1}\beta>(2,\beta)}\theta_\alpha(c_\alpha S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}\beta>(1,\alpha^{-1})}))\\
& (m_{<-1,\alpha^{-1}\beta>(1,\alpha^{-1})})_{<-1,\beta>}a_{<-1,\alpha^{-1}\beta>(2,\beta)}\otimes m_{<0,0>}
\end{aligned}$$

$$
\begin{aligned}
&(\theta_\alpha(c_\alpha S_{\alpha^{-1}}(a_{<-1,\alpha^{-1}\beta>(1,\alpha^{-1})}))(m_{<-1,\alpha^{-1}>}))_{<0,0>}a_{<0,0>}\\
\stackrel{(\nu=\alpha^{-1}\beta)}{=}\ & (m_{<-1,\nu>(2,\beta)}\theta_{\beta\nu^{-1}}(c_{\beta\nu^{-1}}S_{\nu\beta^{-1}}(a_{<-1,\nu>(1,v\beta^{-1})}))\\
&(m_{<-1,\nu>(1,v\beta^{-1})})_{<-1,\beta>}a_{<-1,\nu>(2,\beta)}\otimes m_{<0,0>}\\
&\theta_{\beta\nu^{-1}}(c_{\beta\nu^{-1}}S_{\nu\beta^{-1}}(a_{<-1,\nu>(1,v\beta^{-1})}))(m_{<-1,\nu>(1,v\beta^{-1})})_{<0,0>}a_{<0,0>})\\
\stackrel{(4.1)}{=}\ & ((c_{\beta\nu^{-1}}S_{\nu\beta^{-1}}(a_{<-1,\nu>(1,v\beta^{-1})}))_{(1,\beta)}a_{<-1,\nu>(2,\beta)}\otimes\\
&m_{<0,0>}\theta_{\nu^{-1}}((c_{\beta\nu^{-1}}S_{\nu\beta^{-1}}(a_{<-1,\nu>(1,v\beta^{-1})}))_{(2,\nu^{-1})})(m_{<-1,\nu>})a_{<0,0>})\\
\stackrel{(W9)}{=}\ & (c_{\beta\nu^{-1}(1,\beta)}S_{\beta^{-1}}(a_{<-1,\nu>(1,\nu\beta^{-1})(2,\beta^{-1})})a_{<-1,\nu>(2,\beta)}\otimes\\
&m_{<0,0>}\theta_{\nu^{-1}}(c_{\beta\nu^{-1}(2,\nu^{-1})}S_\nu(a_{<-1,\nu>(1,\nu\beta^{-1})(1,v)}))(m_{<-1,\nu>})a_{<0,0>})\\
\stackrel{(2.1)}{=}\ & (c_{\beta\nu^{-1}(1,\beta)}S_{\beta^{-1}}(a_{<-1,\nu>(2,e)(1,\beta^{-1})})a_{<-1,\nu>(2,e)(2,\beta)}\otimes\\
&m_{<0,0>}\theta_{\nu^{-1}}(c_{\beta\nu^{-1}(2,\nu^{-1})}S_\nu(a_{<-1,\nu>(1,v)}))(m_{<-1,\nu>})a_{<0,0>})\\
\stackrel{(2.14)}{=}\ & (c_{\beta\nu^{-1}(1,\beta)}\varepsilon^s_\beta(a_{<-1,\nu>(2,e)})\otimes m_{<0,0>}\\
&\theta_{\nu^{-1}}(c_{\beta\nu^{-1}(2,\nu^{-1})}S_\nu(a_{<-1,\nu>(1,v)}))(m_{<-1,\nu>})a_{<0,0>})\\
\stackrel{(W11)}{=}\ & (c_{\beta\nu^{-1}(1,\beta)}S_{\beta^{-1}}(1_{(2,\beta^{-1})})\otimes m_{<0,0>}\\
&\theta_{\nu^{-1}}(c_{\beta\nu^{-1}(2,\nu^{-1})}S_\nu(a_{<-1,\nu>}1_{(1,\nu)}))(m_{<-1,\nu>})a_{<0,0>})\\
\stackrel{(W9)}{=}\ & ((c_{\beta\nu^{-1}(1,\beta)})\otimes m_{<0,0>}\theta_{\nu^{-1}}(c_{\beta\nu^{-1}(2,\nu^{-1})}S_\nu(a_{<-1,\nu>}))(m_{<-1,\nu>})a_{<0,0>}),
\end{aligned}
$$

i.e., f is left π-C-colinear. Hence, we proved that f is an epimorphism in ${}^{\pi-C}\mathcal{U}(H)_A$ and has a π-C-colinear section. □

Remark 4.8. The Theorem 4.7 can be followed in a new way as follows: From the forgetful functors $F_A :^{\pi-C} \mathcal{U}(H)_A \to^{\pi-C} \mathcal{U}(H)$ and $F^C :^{\pi-C} \mathcal{U}(H)_A \to \mathcal{U}(H)_A$, we have the functor $\mathfrak{H} = F^C \circ F_A :^{\pi-C} \mathcal{U}(H)_A \to Vect_k$. the natural transformation $\mathcal{P}$ is constructed as follows: for $M, N \in^{\pi-C} \mathcal{U}(H)$, we define

$$\mathcal{P}_{M,N} : \mathrm{Hom}_{\pi-c\mathcal{U}(H)}(M,N) \to \mathrm{Hom}_{Vect_k}(M,N), \mathcal{P}_{M,N}(f) = F^C(f).$$

Notice $\mathcal{P}\circ F_A = \mathfrak{H}$, i.e., F_A is $\mathfrak{H}$-separable functor. From the proof of Theorem 4.4, we know that $F_A(f)$ has a right inverse in ${}^{\pi-C}\mathcal{U}(H)$, by the Maschke′s theorem for $\mathfrak{H}$-separable functor(see[7,Prop.2.4]), $\mathfrak{H}(f)$ has a right inverse in $Vect_k$.

5. The Maschke-type Theorem For weak Doi-Hopf π-modules

In the section, we give the Maschke-type Theorem for weak Doi-Hopf π-modules. First, we need the following Lemma.

Lemma 5.1. *Let (H, A, C) be a weak Doi-Hopf π-datum, $N, M \in^{\pi-C} \mathcal{U}(H)_A$ and $u : N \to M$ a A-linear map. Suppose that there exists $\theta = \{\theta_\alpha : C_\alpha \to Hom(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ an integral such that A-centralising condition, i.e., for all $\alpha \in \pi$, $a \in A$,$c \in C_\alpha$ and $d \in C_{\alpha^{-1}}$,*

$$a_{<0,0><0,0>}\theta_\alpha(c \cdot a_{<-1,\alpha>})(d \cdot a_{<0,0><-1,\alpha^{-1}>}) = \theta_\alpha(c)(d)a. \qquad (5.\ 1)$$

Then the map

$$\tilde{u} : N \to M,\ \tilde{u}(n) = \sum_{\alpha\in\pi} u(n_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>}),$$

for all $n \in N$, is left π-C-colinear and A-linear.

Proof. From the Proposition 4.2, $\tilde{u}$ is a left π-C-colinear. We are left to check that $\tilde{u}$ is right A-linear. In fact, for all $n \in N$ and $a \in A$, Since

$$\begin{aligned}
\tilde{u}(n \cdot a) &= \sum_{\alpha\in\pi} u(n_{<0,0>} \cdot a_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>} \cdot a_{<-1,\alpha>}) \\
&\qquad (u(n_{<0,0>} \cdot a_{<0,0>})_{<-1,\alpha^{-1}>}) \\
&= \sum_{\alpha\in\pi} u(n_{<0,0>})_{<0,0>} \cdot a_{<0,0><0,0>}\theta_\alpha(n_{<-1,\alpha>} \cdot a_{<-1,\alpha>}) \\
&\quad (u(n_{<0,0>})_{<-1,\alpha^{-1}>} \cdot a_{<0,0><-1,\alpha^{-1}>}) \\
&= \sum_{\alpha\in\pi} u(n_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>})a.
\end{aligned}$$

So the proof of the lemma is completed. □

Theorem 5.2. *Let (H, A, C) be a weak Doi-Hopf π-datum, and $N, M \in^{\pi-C} \mathcal{U}(H)_A$, and $v : M \to N$ a morphism in ${}^{\pi-C}\mathcal{U}(H)_A$. Suppose that there exists $\theta = \{\theta_\alpha : C_\alpha \to Hom(C_{\alpha^{-1}}, A)\}_{\alpha\in\pi}$ an integral such that A-centralising condition. If the map v is a split injection in $\mathcal{U}_A$, then u have a retraction in ${}^{\pi-C}\mathcal{U}(H)_A$.*

Proof. Since v is a split injection in $\mathcal{U}_A$, there exist a morphism $u : N \to M$ in $\mathcal{U}_A$ such that $uv = id_M$. Define $\tilde{u}$ as follows

$$\tilde{u}(n) = \sum_{\alpha\in\pi} u(n_{<0,0>})_{<0,0>}\theta_\alpha(n_{<-1,\alpha>})(u(n_{<0,0>})_{<-1,\alpha^{-1}>}).$$

From Lemma 5.1, $\tilde{u}$ is a morphism in ${}^{\pi-C}\mathcal{U}(H)_A$. Next, we shall check $\tilde{u}v = id_M$. For $m \in M$, one has

$$\begin{aligned}
& \tilde{u}v(m) \\
= & \sum_{\alpha\in\pi} u(v(m)_{<0,0>})_{<0,0>}\theta_\alpha(v(m)_{<-1,\alpha>})(u(v(m)_{<0,0>})_{<-1,\alpha^{-1}>}) \\
= & \sum_{\alpha\in\pi} u(v(m_{<0,0>}))_{<0,0>}\theta_\alpha(m_{<-1,\alpha>})(u(v(m_{<0,0>}))_{<-1,\alpha^{-1}>}) \\
= & \sum_{\alpha\in\pi} m_{<0,0><0,0>}\theta_\alpha(m_{<-1,\alpha>})(m_{<0,0><-1,\alpha^{-1}>}) \\
= & \sum_{\alpha\in\pi} m_{<0,0>}\theta_\alpha(m_{<-1,e>(1,\alpha)})(m_{<-1,e>(2,\alpha^{-1})}) \\
= & m.
\end{aligned}$$

So the theorem is proved. □

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References

[1] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras, I. Integral theory and C*-structure*, J. Algebra, 221 (1999), 385-438.

[2] G. Böhm, *Doi-Hopf modules over weak Hopf algebras*, Comm. Algebra, 28 (2000), 4687-4698.

[3] S. Caenepeel, G. Militaru and Zhu Shenglin, *Crossed modules and Doi-Hopf modules*, Israel J. Math., 100 (1997), 221-248.

[4] S. Caenepeel and M. De Lombaerde, *A categorical approach to Turaev′s Hopf group-coalgebras*, Comm. Algebra, 34(7) (2006), 2631-2657.

[5] S. Caenepeel, G. Militaru and Zhu Shenglin, *A Maschke-type theorem for Doi-Hopf modules. Applications*, J. Algebra, 187 (1997), 388-412.

[6] S. Caenepeel, Bogdan Ion, G. Militaru and Zhu Shenglin, *Separable functors for the Category of Doi-Hopf modules, Applications,* Adv. Math., 145 (1999),239-290.

[7] S. Caenepeel, G. Military, *Maschke functors, semisimple functors and separable functors of the second kind. Applications*, J. Pure Appl. Algebra, 178 (2003), 131-157.

[8] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, *Separable functors applied to graded rings*, J. Algebra, 123 (1989), 397-413

[9] Y. Doi, *Unifying Hopf modules,* J. Algebra, 153(2) (1992), 373-385.

[10] C. Menini and G. Militaru, *Integral, Quantum galois extensions, and the Affineness criterion for quantum Yetter-Drinfeld modules,* J. Algebra, 247 (2002), 467-508.

[11] D. E. Radford and J. Towber, *Yetter-Drinfeld categories associated to an arbitrary bialgebra*, J. Pure Appl. Algebra, 87(3) (1993), 259-279.

[12] Shen Bingliang and Wang Shuanhong, *Blattner-Cohen-Montgomery′s Duality Theorem for (Weak) Group Smash Products*, Comm. Algebra, 36(6) (2008), 2387-2409.

[13] M. E. Sweedler, Hopf Algebras, New York: Benjamin, 1969.

[14] V. G. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*, Preprint GT/0005291,(2000).

[15] A. Van Daele, *Multiplier Hopf algebras*, Trans. Amer. Math. Soc., 342(2) (1994), 917-932.

[16] A. Van Daele, *An algebraic framework for group duality*, Adv. Math., 140 (1998),323-366.

[17] A. Van Daele and Wang Shuanhong, *New braided crossed categories and Drinfeld quantum double for weak Hopf π-coalgebras,* Comm. Algebra, 36 (2008), 2341-2386.

[18] A. Van Daele and Wang Shuanhong, *Multiplier unifying Hopf algebras*. Submitted for publication.

[19] A. Virelizier, *Hopf group-coalgebras*, J. Pure Appl. Algebra, 171 (2002), 75-122.

[20] A. Virelizier, *Graded quantum groups and quasitriangular Hopf group-coalgebras,* Comm. Algebra, 33 (9)(2005), 3029-3050.

[21] Wang Shuanhong, *Group twisted smash products and Doi-Hopf modules for T-coalgebras*, Comm. Algebra, 32(9) (2004), 3437-3458 .

[22] Wang Shuanhong, *Group entwining structures and group coalgebra Galois extensions*, Comm. Algebra, 32(9) (2004), 3417-3436.

[23] Wang Shuanhong, *A Maschke-Type Theorem for Hopf π-Comodules,* Tsukuba J. Math., 28(2) (2004), 377-388.

[24] Wang Shuanhong, *Morita Contexts, π-Galois Extensions for Hopf π-coalgebras*, Comm. Algebra, 34(2) (2006), 521-546.

[25] D. N. Yetter, *Quantum groups and representations of monoidal categories*, Math. Proc. Cambridge Philos. Soc., 108(2) (1990), 261-290.

[26] M. Zunino, *Double construction for crossed Hopf coalgebra*, J. Algebra, 278 (2004), 43-75.

[27] M. Zunino, *Yetter-Drinfeld modules for Turaev crossed structures*, J. Pure Appl. Algebra, 193(1-3) (2004), 313-343.

[28] M. D. Rafael, *Separable functors revisited*, Comm. Algebra ,18 (1990), 1445-1459.

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