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GEOMETRIC AND ALGEBRAIC SIGNIFICANCE OF SOME HURWITZ STABILISERS IN Br_n

Michael Lönne

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Abstract. We consider naturally generated subgroups E_n of Br_n. On the geometric side we show that E_n is the bifurcation braid monodromy group of the family of plane polynomial coverings of degree 3. On the algebraic side there is the Hurwitz action of the braid group Br_n on the n-fold Cartesian product Br_3^n of $\mathrm{Br}_3 = \langle a, b \,|\, aba = bab \rangle$. The stabiliser of finite alternating sequences of its generators a, b is expected to be given by E_n. We are able to prove this conjecture in a few cases and apply our results to give characterisations in the most basic instances of the *paths realised by degenerations* in families of polynomials as defined by Donaldson [7].

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1. Introduction

The braid group Br_n on n strands has first been considered by A. Hurwitz [9] and E. Artin [1]. Since then it has lead a prolific life in such diverse areas of mathematics as topology, combinatorics and algebraic geometry.

In his original approach A. Hurwitz investigates the action of Br_n on the n-fold Cartesian product of a symmetric group. This so called Hurwitz action can formally be defined on G^n for any group G and is determined by the action of the standard generators

$$\sigma_i(g_1, ..., g_n) = (g_1, ..., g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, ..., g_n)$$

With the *band generators* $\sigma_{ij} = \sigma_i \sigma_{i+1} \cdots \sigma_{j-1} (\sigma_i \sigma_{i+1} \cdots \sigma_{j-2})^{-1}$, see [2], we define a subgroup of Br_n,

$$E_n := \left\langle e_{ij} := \sigma_{ij}^{m_{ij}}, 1 \leq i < j \leq n, m_{ij} = \begin{cases} 1 & \text{if } i \equiv j\ (2) \\ 3 & \text{if } i \not\equiv j\ (2) \end{cases} \right\rangle,$$

which we want to relate to the Hurwitz action in case $G = \mathrm{Br}_3$ on one hand and to a geometric monodromy problem on the other hand.

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The Hurwitz action for G a Coxeter group or an Artin group has been studied a lot: Given a symmetric matrix M of positive integers m_{ij} there are the associated Coxeter group and Artin group

$$C = C_M := \langle s_1, ..., s_n \,|\, s_i^2 = (s_i s_j)^{m_{ij}} = 1 \rangle,$$
$$A = A_M := \langle a_1, ..., a_n \,|\, \underbrace{a_i a_j a_i ...}_{m_{ij}\text{ factors}} = \underbrace{a_j a_i a_j ...}_{m_{ij}\text{ factors}} \rangle.$$

The distinguished element $\mathcal{C} = (s_1, ..., s_n) \in C^n$ is fixed under the Hurwitz action of

$$E_M := \left\langle e_{ij} := \sigma_{ij}^{m_{ij}}, 1 \le i < j \le n \right\rangle \quad \subset \quad \mathrm{Br}_n .$$

In several important cases it is shown to equal the Hurwitz stabiliser group

$$S_{\mathcal{C}} := \{\beta \in \mathrm{Br}_n \,|\, \beta(s_1, ..., s_n) = (s_1, ..., s_n)\},$$

see [3,10,4,6], and presentations have been given in [12]. It is an immediate corollary [10], that in these cases E_M is also the Hurwitz stabiliser $S_{\mathcal{A}}$ of the distinguished element $\mathcal{A} = (a_1, ..., a_n) \in A^n$.

But there are examples of Coxeter systems which are sufficiently redundant, ie. $m_{ij} = 1$ for sufficiently many pairs ij, such that E_M is a proper subgroup of $S_{\mathcal{C}}$, see [3] and Section 2.

In this article we concentrate on the case of the matrices

$$M_n := \big(m_{ij}\big)_{1 \le i,j \le n}, \text{ with } m_{ij} = \begin{cases} 3 & \text{if} \quad i \not\equiv j \mod 2 \\ 1 & \text{if} \quad i \equiv j \mod 2 \end{cases}$$

In all cases we can handle either $E = S_A \subseteq S_C$ or $E \subseteq S_A \subset S_C$, so one may venture to hope that $E = S_A$ holds in general.

There is always a geometric flavour to these results: Birman and Wajnryb [3] applied their results to get a presentation for the mapping class group of surfaces with at most one boundary component. The stabilisers of the classical Coxeter systems for the finite Coxeter groups investigated by Catanese/Wajnryb (type A) and Dörner (type ADE) have been shown to be the fundamental groups of the bifurcation complements of the corresponding simple singularities [13].

On the geometric side our research was initiated by the article [7] in 'Mathematics: frontiers and perspectives'.

Given a plane polynomial $g_1 = g_1(x, y)$ the projection to the x-axis defines a function f_1 on the plane curve C_1 defined by g_1. Suppose now C_1 is smooth and f_1 simply branched, i.e. f_1 is a Morse function with singular values $b_0, ..., b_n \in \mathbb{C}$, which are called the branch points. Then Donaldson asks for the set of paths which can be realised by deformations of g_1, [7, p.56]. We call such paths *vanishing arcs*, since we have the following characterisation:

An embedded path γ with endpoints b_0, b_1 and disjoint from the branch set otherwise is a vanishing arc, if there is a smooth family $g_t, t \in [0,1]$ of polynomials and a homotopy $H = H(s,t)$ such that

i) the curves $C_t = g_t^{-1}(0)$ are smooth and f_t is simply branched for $t > 0$,
ii) the curve C_0 is smooth except for a single ordinary double point P and f_0 is simply branched on the smooth locus and injective on its critical points,
iii) H is a homotopy from γ to the constant map to $f_0(P)$ such that H_t meets the branch set of f_t in $H_t(\{0,1\})$.

So a vanishing arc can be contracted to a point relative to the branch locus.

The geometric content of such results is addressed in the second part of this article, where we show how Hurwitz stablisers occur naturally in the study of Donaldson's question (see also [11]):

To handle degenerations of plane polynomials we choose the set up of polynomial covers introduced by Hansen [8] and define the notion of bifurcation braid monodromy. The paper closes with a characterisation of sets of vanishing arcs in the spirit of [7].

2. Stabilisers of Coxeter systems

First we consider the Hurwitz action on Cartesian products of the symmetric group $\mathcal{S}_3$ on three elements. The Coxeter presentation for $\mathcal{S}_3$ is generated by transpositions s and t:

$$\mathcal{S}_3 \;\cong\; C := \langle s, t \,|\, s^2 = t^2 = (st)^3 = 1\rangle.$$

Any non-constant n-tuple $(s_1, ..., s_n)$ with $s_i \in \{s, t\}$ is not only an element in C^n but also a Coxeter system with Coxeter group C and matrix

$$M := \big(m_{ij}\big)_{1\le i,j\le n}, \text{ with } m_{ij} = \begin{cases} 3 & \text{if} \quad s_i \neq s_j \\ 1 & \text{if} \quad s_i = s_j \end{cases}$$

For some of these Coxeter systems the stabiliser group can be found with results of Birman and Wajnryb.

Lemma 2.1. *The stabiliser group of the Coxeter systems* $(s,t,t,...,t)$ *of length* $n \ge 3$ *and* $(s,s,t,...,t)$ *of length* $n \ge 4$ *can be given by*

$$\begin{aligned} S_C &= \langle \sigma_1^3, \sigma_2, ..., \sigma_{n-1}, \rangle, \quad n = 3,4, \\ S_C &= \langle \sigma_1^3, \sigma_2, ..., \sigma_{n-1}, (\sigma_4)\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_1\rangle, \quad n \ge 5, \end{aligned}$$

respectively

$$\begin{aligned} S_C &= \langle \sigma_1, \sigma_2^3, \sigma_3, ..., \sigma_{n-1}, (\sigma_3)\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2\sigma_1\rangle, \quad n = 4,5, \\ S_C &= \langle \sigma_1, \sigma_2^3, \sigma_3, ..., \sigma_{n-1}, (\sigma_5)\sigma_4\sigma_3\sigma_2^2\sigma_3\sigma_4^2\sigma_3\sigma_2, (\sigma_3)\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2\sigma_1\rangle, \quad n \ge 6. \end{aligned}$$

Remark: Here and elsewhere we use $(\sigma)\beta$ to denote the braid $\beta^{-1}\sigma\beta$ obtained by the action of a braid β on a braid σ by conjugation on the right with the convention that braid multiplication preceeds braid conjugation.

Proof. The first claim is actually proved in [3]. In the second case a generating set has been given in [14, Thm. 5] which is the same as ours except that it contains $(\sigma_1)\sigma_2\sigma_3^2\sigma_2$ instead of $(\sigma_3)\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2\sigma_1$. But they are equal up to conjugation by $\sigma_2^3\sigma_3\sigma_2^3$, hence our second claim is also true. □

Starting from this lemma we can get the stabiliser groups of the alternating Coxeter systems $\mathcal{C}_n := (s,t,s,t,...) \in C^n$ with matrix

$$M_n := \left(m_{ij}\right)_{1\le i,j\le n}, \text{ with } m_{ij} = \begin{cases} 3 & \text{if} \quad i \not\equiv j \mod 2 \\ 1 & \text{if} \quad i \equiv j \mod 2 \end{cases}$$

Lemma 2.2. *The stabiliser group $S_{\mathcal{C}_n}$ of an alternating Coxeter systems of length n is conjugated to the stabiliser group of a Coxeter system $(s,t,t,...,t)$ or $(s,s,t,...,t)$ of equal length. For small n we have:*

$$\begin{array}{lclcl} S_{(s,t,s)} & = & \sigma_1^{-1} & S_{(s,t,t)} & \sigma_1 \\ S_{(s,t,s,t)} & = & \sigma_2^{-1}\sigma_1 & S_{(s,t,t,t)} & \sigma_1^{-1}\sigma_2 \\ S_{(s,t,s,t,s)} & = & \sigma_3^{-1}\sigma_2\sigma_1^{-1} & S_{(s,t,t,t,t)} & \sigma_1\sigma_2^{-1}\sigma_3 \\ S_{(s,t,s,t,s,t)} & = & \sigma_4^{-1}\sigma_3\sigma_2^{-1} & S_{(s,s,t,t,t,t)} & \sigma_2\sigma_3^{-1}\sigma_4 \end{array}$$

Proof. The general claim is immediate from the observation that the alternating system is in the same Br_n orbit as at least one of the reference systems.

For the second claim we compute the action of the conjugating element on the alternating system of two transpositions s,t in the permutation group of 3 elements, with r denoting the third transposition.

$$\begin{array}{rcl} \sigma_1(\underline{s,t},s) & = & (r,s,s) \\ \\ \sigma_1^{-1}\sigma_2(s,\underline{t,s},t) & = & \sigma_1^{-1}(\underline{s,r},t,t) \\ & = & (r,t,t,t) \\ \\ \sigma_1\sigma_2^{-1}\sigma_3(s,t,\underline{s,t},s) & = & \sigma_1\sigma_2^{-1}(s,\underline{t,r},s,s) \\ & = & \sigma_1(\underline{s,r},s,s,s) \\ & = & (t,s,s,s,s) \\ \\ \sigma_2\sigma_3^{-1}\sigma_4(s,t,s,\underline{t,s},t) & = & \sigma_2\sigma_3^{-1}(s,t,\underline{s,r},t,t) \\ & = & \sigma_2(s,\underline{t,r},t,t,t) \\ & = & (s,s,t,t,t,t) \end{array}$$

Surely the stabiliser does not depend on the choice of two non-commuting transpositions and hence we are done. □

Having said that $E_n := E_{M_n} \subseteq S_{\mathcal{C}_n}$ we want to give a small set of generators now.

Lemma 2.3. *The group E_n is generated by $\sigma_1^3, \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}$, $i = 1, ..., n-2$.*

Proof. First note that the given generators are just $e_{12}, e_{i,i+2}$, $i = 1, ..., n-2$. For the remaining e_{ij} we have the following relations:

$$\begin{array}{rcll} e_{ij} & = & (e_{i,i+2})e_{i+2,i+4}^{-1}e_{i+4,i+6}^{-1}...e_{j-2,j}^{-1}, & i \equiv j \mod 2, \\ e_{ij} & = & (e_{12})e_{13}e_{24}...e_{i-1,i+1}e_{i+1,i+3}^{-1}e_{i+3,i+5}^{-1}...e_{j-2,j}^{-1}, & i \not\equiv j \mod 2. \end{array}$$ □

In fact we can choose two additional stabilisers to generate the stabiliser group $S_{\mathcal{C}_n}$. Our choice $\tau_1 = (\sigma_1)\sigma_2\sigma_3^{-1}\sigma_4, \tau_2 = (\sigma_2)\sigma_3\sigma_4^{-1}\sigma_5$ is motivated by their action on Artin systems, see Lemma 3.2.

Lemma 2.4. *The stabiliser groups $S_{\mathcal{C}_n}$ of the alternating Coxeter systems of lengths $n = 3, 4, 5, 6$ can be given as follows:*

$$\begin{array}{rclrcl} S_{(s,t,s)} & = & E_3, & S_{(s,t,s,t)} & = & E_4, \\ S_{(s,t,s,t,s)} & = & \langle E_5, \tau_1 \rangle, & S_{(s,t,s,t,s,t)} & = & \langle E_6, \tau_1, \tau_2 \rangle. \end{array}$$

Proof. To show that τ_1 belongs to the stabiliser of $\mathcal{C}_5$, respectively that τ_1, τ_2 belong to the stabiliser of $\mathcal{C}_6$, it obviously suffices to show $\tau_1 \in S_{\mathcal{C}_5}$:

$$\begin{array}{crl} & \sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_4 & (s,t,s,\underline{t,s}) \\ = & \sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1} & (s,t,\underline{s,r},t) \\ = & \sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2 & (s,\underline{t,r},t,t) \\ = & \sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_1 & (\underline{s,s},t,t,t) \\ = & \sigma_4^{-1}\sigma_3\sigma_2^{-1} & (s,\underline{s,t},t,t) \\ = & \sigma_4^{-1}\sigma_3 & (s,t,\underline{r,t},t) \\ = & \sigma_4^{-1} & (s,t,s,\underline{r,t}) \\ = & & (s,t,s,t,s) \end{array}$$

So the given groups are shown to be stabilising.

To prove the inverse implication we note that the stabiliser groups are generated by the following elements which are obtained from generators of Lemma 2.1 using conjugation as provided by Lemma 2.2:

$$\begin{array}{ll} n=3: & \sigma_1^3, (\sigma_2)\sigma_1, \\ n=4: & (\sigma_1^3)\sigma_2, (\sigma_2)\sigma_1^{-1}\sigma_2, (\sigma_3)\sigma_2, \\ n=5: & (\sigma_1^3)\sigma_2^{-1}\sigma_3, (\sigma_2)\sigma_1\sigma_2^{-1}\sigma_3, (\sigma_3)\sigma_2^{-1}\sigma_3, (\sigma_4)\sigma_3, \\ & \quad (\sigma_4)\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_1\sigma_1\sigma_2^{-1}\sigma_3, \\ n=6: & (\sigma_1)\sigma_2\sigma_3^{-1}\sigma_4, (\sigma_2^3)\sigma_3^{-1}\sigma_4, (\sigma_3)\sigma_2\sigma_3^{-1}\sigma_4, (\sigma_4)\sigma_3^{-1}\sigma_4, (\sigma_5)\sigma_4, \\ & \quad (\sigma_5)\sigma_4\sigma_3\sigma_2^2\sigma_3\sigma_4^2\sigma_3\sigma_2\sigma_2\sigma_3^{-1}\sigma_4, (\sigma_3)\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2\sigma_1\sigma_2\sigma_3^{-1}\sigma_4. \end{array}$$

The claim now follows from the fact that all these elements can be expressed by elements of the given groups:

$$
\begin{array}{llllll}
n=3: & & \sigma_1^3 & & (\sigma_2)\sigma_1 & \\
 & = & e_{12} & = & e_{13} & \\
n=4: & & (\sigma_1^3)\sigma_2 & & (\sigma_2)\sigma_1^{-1}\sigma_2 & & (\sigma_3)\sigma_2 \\
 & = & (e_{12})e_{23}^{-1}e_{13}^{-1} & = & (e_{13})e_{23} & = & e_{24} \\
n=5: & & (\sigma_1^3)\sigma_2^{-1}\sigma_3 & & (\sigma_2)\sigma_1\sigma_2^{-1}\sigma_3 & & (\sigma_3)\sigma_2^{-1}\sigma_3 \\
 & = & (e_{34})e_{13}^{-1} & = & (e_{24})e_{34}e_{13}^{-1} & = & (e_{24})e_{34} \\
 & & (\sigma_4)\sigma_3 & & (\sigma_4)\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_1\sigma_1\sigma_2^{-1}\sigma_3 & \\
 & = & e_{35} & = & (\tau_1)e_{35}e_{45}e_{34}e_{15}^{-1} & \\
n=6: & & (\sigma_2^3)\sigma_3^{-1}\sigma_4 & & (\sigma_3)\sigma_2\sigma_3^{-1}\sigma_4 & & (\sigma_2)\sigma_3^{-1}\sigma_4 \\
 & = & (e_{45})e_{24}^{-1} & = & (e_{35})e_{45}e_{24}^{-1} & = & (e_{35})e_{45} \\
 & & (\sigma_5)\sigma_4 & & (\sigma_5)\sigma_4\sigma_3\sigma_2^2\sigma_3\sigma_4^2\sigma_3\sigma_2\sigma_2\sigma_3^{-1}\sigma_4 & \\
 & = & e_{46} & = & (\tau_2)e_{46}e_{56}e_{45}e_{26}^{-1} & \\
 & & (\sigma_1)\sigma_2\sigma_3^{-1}\sigma_4 & & (\sigma_3)\sigma_2^{-1}\sigma_1^{-2}\sigma_2^2\sigma_1\sigma_2\sigma_3^{-1}\sigma_4 & \\
 & = & \tau_1 & = & e_{13} & \square
\end{array}
$$

While the claim of the preceding lemma has a generalisation to $n > 6$, the next result has not and is therefore special to $n = 5, 6$.

Lemma 2.5. *In case $n = 5, 6$ the group E_n is normal in the stabiliser group $S_{\mathcal{C}_n}$.*

Proof. It suffices to show that generators of E_n map to elements of E_n, $n = 5, 6$, when they are conjugated by τ_1, respectively by τ_1, τ_2: We take first all generators of E_5 and conjugate them by τ_1, and in the second step the additional generator e_{46} which we conjugate by τ_1 and all generators of E_6 conjugated by τ_2. The claim is proved, since we express all these elements by elements in E_5 and E_6 respectively.

$$
\begin{array}{lllllll}
n=5,6: & (e_{12})\tau_1^{-1} & = & (e_{45})e_{24}^{-1}, & (e_{13})\tau_1^{-1} & = & (e_{13})e_{12}^{-1}e_{24}e_{45}e_{24}^{-1} \\
 & (e_{24})\tau_1^{-1} & = & e_{24}, & (e_{35})\tau_1^{-1} & = & (e_{35})e_{15}e_{12}^{-1}e_{15}^{-1}e_{12}^{-1}e_{45} \\
n=6: & (e_{46})\tau_1^{-1} & = & e_{46}, & (e_{12})\tau_2^{-1} & = & (e_{56})e_{35}^{-1}e_{13}^{-1} \\
 & (e_{13})\tau_2^{-1} & = & e_{13}, & (e_{24})\tau_2^{-1} & = & (e_{24})e_{23}^{-1}e_{35}e_{56}e_{35}^{-1} \\
 & (e_{35})\tau_2^{-1} & = & e_{35}, & (e_{46})\tau_2^{-1} & = & (e_{46})e_{26}e_{23}^{-1}e_{26}^{-1}e_{23}^{-1}e_{56}
\end{array}
$$

$\square$

3. Stabilisers for redundant Artin systems

In analogy with the previous section we start with a standard presentation for the braid group on three strands generated by two braids a and b:

$$A := \langle a, b \,|\, aba = bab \rangle.$$

As we may expect the alternating Artin systems $\mathcal{A}_n := (a, b, a, b, ...)$ of length $n \geq 2$ have matrix M_n and Artin group A.

Lemma 3.1. *Let A act by elementwise conjugation on the n-fold Cartesian product A^n, then the stabiliser subgroup of the alternating Artin system $\mathcal{A}_n$ has trivial intersection with the subgroup $H = \langle a^2, b^2 \rangle$ of A if $n \geq 2$.*

Proof. Forgetting the second strand maps the pure braid group PBr_3 onto PBr_2 with kernel H, which hence is freely generated by a^2 and b^2. On the other hand if $n \geq 2$ any element in the stabiliser must actually belong to the center of $A \cong \mathrm{Br}_3$. But the intersection of the center of A with H must be contained in the center of H which is trivial since H is free. □

Lemma 3.2. *The braids τ_1, τ_2 act on elements of the H-orbit of (a, b, a, b, a, b) by overall conjugation with b^{-2} resp. a^{-2}.*

Proof. Since overall conjugation commutes with the braid action it suffices to check the claim for the action of τ_1, τ_2 on (a, b, a, b, a, b), and even one of these cases suffices by symmetry:

$$\begin{aligned} & \sigma_4^{-1}\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_4 \quad (a, b, a, b, a, b) \\ = \quad & (abbab^{-1}b^{-1}a^{-1}, b, b^{-1}aba^{-1}b, b, b^{-1}b^{-1}abb, b) \end{aligned}$$

and the last line equals $(b^{-2}ab^2, b, b^{-2}ab^2, b, b^{-2}ab^2, b)$ since

$$abbab^{-1}b^{-1}a^{-1} = aba^{-1}bab^{-1}a^{-1} = b^{-1}abbb^{-1}a^{-1}b = b^{-1}aba^{-1}b = b^{-1}b^{-1}abb. \qquad \square$$

Corollary 3.3. *The braid τ_1 acts trivially on alternating Coxeter systems but non-trivially on alternating Artin systems for $n \geq 5$.*

Proposition 3.4. *The groups $S_{\mathcal{C}_5}, S_{\mathcal{C}_6}$ are semi-direct products of their normal subgroups E_5 resp. E_6 and a free subgroup freely generated by τ_1 resp. τ_1, τ_2.*

Proof. The subgroup generated by τ_1 resp. τ_1, τ_2 acts freely on the H-orbit of (a, b, a, b, a) resp. (a, b, a, b, a, b), and therefore is a free subgroup. The claim is then immediate from the normality of the group E_n in $S_{\mathcal{C}_n}$ for $n = 5, 6$, see Lemma 2.5. □

Theorem 3.5. *Let $S_{\mathcal{A}_n}, S_{\mathcal{C}_n}$ be the stabilisers of the Artin resp. Coxeter system associated with the $n \times n$ matrix*

$$M_n := (m_{ij}),\ 1 \le i, j \le n, \ \text{with } m_{ij} = \begin{cases} 3 & \text{if} \quad i \not\equiv j \mod 2 \\ 1 & \text{if} \quad i \equiv j \mod 2 \end{cases}$$

and denote by E_n the subgroup of the braid group Br_n generated by elements $e_{ij} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^{m_{ij}} \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}, 1 \le i < j \le n$, then the following relations hold:

$$\begin{array}{ccccccl} E_n & = & S_{\mathcal{A}_n} & = & S_{\mathcal{C}_n} & \text{if} & n = 2,3,4, \\ E_n & = & S_{\mathcal{A}_n} & \subset & S_{\mathcal{C}_n} & \text{if} & n = 5,6, \\ E_n & \subseteq & S_{\mathcal{A}_n} & \subset & S_{\mathcal{C}_n} & \text{if} & n \ge 7. \end{array}$$

Proof. The inclusion E_M in $S_{\mathcal{A}_M}$ is a general fact again, whereas the inclusion $S_{\mathcal{A}_M} \subseteq S_{\mathcal{C}_M}$ is obvious. Equality $E_n = S_{\mathcal{C}_n}$ for $n = 2,3,4$ is shown in Lemma 2.4. Strict inclusion $S_{\mathcal{A}_n} \subset S_{\mathcal{C}_n}$ for $n \ge 5$ follows from Corollary 3.3. In case $n = 5,6$ finally each braid in $S_{\mathcal{C}_n}$ can be written as a product τe with $\tau \in \langle \tau_1, \tau_2 \rangle$ and $e \in E_n$ by Proposition 3.4. Hence it is immediate by Lemma 3.1 that E_n is the stabiliser group $S_{\mathcal{A}_n}$ in these cases. □

4. Conjugacy classes of simple braids

The aim of this section is to exhibit the set of simple braids in E_n as a single E_n conjugation class for $n \le 6$, where a braid is called simple if it is isotopic to a half twist associated to a path connecting two punctures.

We extend the conjugation action and its exponential notation to sets, so the conjugation orbit of e_{13} in E_n is denoted by $e_{13}^{E_n} := \{e_{13}\}^{E_n}$.

Lemma 4.1. *For $n \le 6$ there is a set of braids $T_{B/S}$ such that $\mathrm{Br}_n = T_{B/S} \cdot S_{\mathcal{C}_n}$ and for all $\gamma \in T_{B/S}$:*

$$e_{13}^{\gamma} \in E_n \quad \Rightarrow \quad e_{13}^{\gamma} \in e_{13}^{E_n}.$$

Proof. We defined $S_{\mathcal{C}_n}$ to be a stabiliser for the Hurwitz action of Br_n on the finite set $\mathcal{S}_3^n$, hence the stabiliser is of finite index. Our strategy is to construct T as a Schreier left transversal for the $S_{\mathcal{C}_n}$-cosets.

First note that the cosets are in bijection to the elements of the Br_n-orbit of the Coxeter system. This orbit actually is contained in $\{s,t,r\}^n$, the set of n-tuples of transpositions in $\mathcal{S}_3$. Hence we may assume that the Schreier transversal contains only positive braids, since σ_i^{-1} and σ_i^2 act the same way on all elements of $\{s,t,r\}^n$.

Such transversal $T_{B/S}$ for $n = 6$ can be found, e.g. by a short symbolic computation, containing 240 elements. Since $E_n = S_{\mathcal{A}}$ we extract the list $\{\gamma | e_{13}^{\gamma} \in E_n\}$ as the list of all elements γ, such that e_{13}^{γ} stabilises $\mathcal{A}_6$.

Thus we get in fact 18 elements e_{13}^{γ}, which can be shown to be conjugated in E_n to elements $e_{i,i+2}$ and hence to e_{13}:

$(e_{13})\sigma_3\sigma_1 = (e_{24})e_{13}^{-1}$
$(e_{13})\sigma_3\sigma_2\sigma_2 = (e_{13})e_{23}e_{24}^{-1}$
$(e_{13})\sigma_3\sigma_3\sigma_1\sigma_2 = (e_{24})e_{12}e_{34}$
$(e_{13})\sigma_3\sigma_3\sigma_2\sigma_1\sigma_1 = (e_{24})e_{12}^{-1}e_{24}^{-1}$
$(e_{13})\sigma_2\sigma_2\sigma_3\sigma_3\sigma_1 = (e_{24})e_{23}$
$(e_{13})\sigma_3\sigma_3\sigma_4\sigma_4\sigma_2 = (e_{12})e_{23}e_{35}^{-1}e_{45}$
$(e_{13})\sigma_3\sigma_4\sigma_4\sigma_5\sigma_2 = (e_{13})e_{23}e_{35}^{-1}e_{45}e_{46}e_{24}$
$(e_{13})\sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_1 = (e_{35})e_{45}e_{13}^{-1}e_{46}$
$(e_{13})\sigma_3\sigma_4\sigma_5\sigma_1\sigma_1\sigma_2 = (e_{13})e_{12}e_{35}^{-1}e_{45}e_{13}^{-1}e_{12}^{-1}e_{46}$
$(e_{13})\sigma_3\sigma_4\sigma_4\sigma_5\sigma_5\sigma_2\sigma_3\sigma_1 = (e_{46})e_{56}e_{24}e_{35}e_{56}$
$(e_{13})\sigma_1\sigma_2 = (e_{13})e_{12}^{-1}$
$(e_{13})\sigma_2\sigma_2\sigma_3 = (e_{13})e_{23}e_{24}$
$(e_{13})\sigma_3\sigma_4\sigma_1\sigma_1 = (e_{24})e_{34}e_{35}e_{12}$
$(e_{13})\sigma_3\sigma_2\sigma_2\sigma_3\sigma_1 = (e_{24})e_{34}e_{13}^{-2}$
$(e_{13})\sigma_2\sigma_2\sigma_3\sigma_4\sigma_1 = (e_{24})e_{34}e_{35}$
$(e_{13})\sigma_3\sigma_3\sigma_4\sigma_5\sigma_2 = (e_{13})e_{23}e_{35}^{-1}e_{45}e_{46}$
$(e_{13})\sigma_3\sigma_4\sigma_4\sigma_2\sigma_1\sigma_1 = (e_{35})e_{45}e_{13}^{-1}$
$(e_{13})\sigma_3\sigma_4\sigma_2\sigma_3\sigma_3\sigma_1 = (e_{24})e_{34}e_{35}^{-1}$

□

Lemma 4.2. *Let $\tau \in S_{\mathcal{C}_5}$, resp. $S_{\mathcal{C}_6}$ be a freely reduced word in τ_1, resp. τ_1, τ_2, then*

$$e_{13}^{\tau} \in e_{13}^{E_n}.$$

Proof. For the at most four words of length one we have:

$$\begin{aligned}
(e_{13})\tau_1 &= (e_{13})e_{35}^{-1}e_{45}e_{35}e_{45}e_{13}e_{12}^{-1}\\
(e_{13})\tau_1^{-1} &= (e_{13})e_{12}^{-1}e_{23}e_{45}e_{24}^{-1}\\
(e_{13})\tau_2 &= e_{13}\\
(e_{13})\tau_2^{-1} &= e_{13}
\end{aligned}$$

To conclude by induction on the word length we next consider a word τ which is the concatenation of a word τ' and a letter τ''. By induction hypothesis $e_{13}^{\tau'} = e_{13}^{e}$ for some $e \in E_n$ and by the normality of E_n in $\langle E_n, \tau_1\rangle$, resp. $\langle E_n, \tau_1, \tau_2\rangle$ we can write $e\tau'' = \tau'' e'$ for a suitable $e' \in E_n$. Hence

$$e_{13}^{\tau} = e_{13}^{\tau'\tau''} = e_{13}^{e\tau''} = e_{13}^{\tau'' e'} \in e_{13}^{E_n}. \qquad \square$$

Proposition 4.3. *The intersection of the conjugacy class of half twists in Br_n, $n \le 6$, with E_n coincides with the conjugacy class of e_{13} in E_n.*

Proof. One inclusion is obvious. So we pick any $\beta \in \mathrm{Br}_n$ such that $e_{13}^{\beta} \in E_n$, which may be factorised as $\beta = \gamma\tau e$ with $\gamma \in T_{B/S}$, $\tau \in \langle \tau_1, \tau_2\rangle$, $e \in E_n$. Since τe normalises E_n, we get $e_{13}^{\gamma} \in E_n$ and $e_{13}^{\gamma} = e_{13}^{e'}$ for some $e' \in E_n$ by Lemma 4.1.

By Lemma 2.5 and Lemma 4.2 there are $e'', e''' \in E_n$ such that:

$$e_{13}^{\beta} = e_{13}^{\gamma\tau e} = e_{13}^{e'\tau e} = e_{13}^{\tau e'' e} = e_{13}^{e''' e'' e} \in e_{13}^{E_n}. \qquad \square$$

5. Bundles and monodromy

We owe the following exposition of polynomial covers to [5] and [8].

On a connected topological manifold X a simple Weierstrass polynomial of degree d is a map $f : X \times \mathbb{C} \to \mathbb{C}$ given by

$$f(x,z) := z^d + \sum_{i=1}^{d} c_i(x) z^{d-i},$$

with continuous coefficient maps $c_i : X \to \mathbb{C}$, and with no multiple roots for any $x \in X$. Given such a function f the first coordinate projection map onto X may be restricted to the subspace

$$Y_f := \{(x,z) \in X \times \mathbb{C} \,|\, f(x,z) = 0\}$$

defining a d-fold cover π_f onto X, the polynomial cover associated to f, or to the complement $X \times \mathbb{C} \setminus Y_f$ defining a fibre bundle over X with fibre diffeomorphic to a d-punctured disc, the punctured disc bundle associated to f.

A finite unramified cover is called polynomial if it is equivalent to a polynomial cover for some simple Weierstrass function as above. Any cover $\pi : Y \to X$ gives rise to a monodromy homomorphism from $\pi_1(X,x)$ to the symmetric group $\mathcal{S}(\pi^{-1}(x))$, which serves for a natural characterisation of polynomial covers:

Proposition 5.1 ([8]). *An unramified cover of degree d is polynomial if and only if its monodromy homomorphism to the symmetric group $\mathcal{S}_d$ lifts along the natural homomorphism* $\mathrm{Br}_d \to \mathcal{S}_d$.

There is a natural way to get from a Coxeter system of length n of a symmetric group $\mathcal{S}_d$ to a finite cover: Given an n-punctured disc and a geometric basis for its fundamental group, which is a choice of free generators $\gamma_1, ..., \gamma_n$ such that

i) the generator are represented by disjoint paths homotopic to positive loops around single punctures,
ii) the product $\gamma_1 \cdots \gamma_n$ is freely homotopic to the positive boundary of the disc.

A homomorphism to $\mathcal{S}_d$ is obtained by assigning to these generators the elements of the Coxeter system. The preimage of any subgroup isomorphic to $\mathcal{S}_{d-1}$ determines a subgroup of the fundamental group in a unique conjugacy class and thus a well-defined finite cover of the punctured disc.

The corresponding result associates with an Artin system of length n for the braid group Br_d a d-punctured disc bundle, once a geometric basis for the fundamental group of an n-punctured disc has been chosen. Here the basis elements are mapped to the generators of the Artin system, so a homomorphism to the braid group is obtained.

Since the space of monic polynomials is an Eilenberg-MacLane space for the braid group, there is a smooth classifying map for this homomorphism. Pulling back the tautological simple Weierstrass polynomial we get a simple Weierstrass polynomial on the n-punctured disc, and the associated punctured disc bundle is the one we aim for.

6. Families of polynomials on the plane

We enter now the realm of complex geometry where there is an abundance of covers and bundles as defined in the previous section.

Example 6.1. Given a branched cover $p : Y \to X$ of a complex manifold X with branch locus B, then the restriction $Y - p^{-1}(B) \to X - B$ is a finite topological cover. Its monodromy

$$\pi_1(X - B, x) \longrightarrow \mathcal{S}(p^{-1}(x))$$

is also called the monodromy of the branched cover.

Example 6.2. Given a plane curve $C \in \mathbb{C}^2$ and a projection $p : \mathbb{C}^2 \to \mathbb{C}$ such that $p|_C$ is a finite branched cover, then restricted to the preimage of the complement of the branch locus $p|_C$ is a polynomial cover and $p|_{\mathbb{C}^2 - C}$ is a punctured disc bundle. The corresponding structure homomorphism to the braid group is called the braid monodromy.

The second example leads a straight way to the following generalised notion of braid monodromy:

Definition 6.3. Suppose $D \subset T \times \mathbb{C}$ is a divisor in the trivial line bundle over a complex manifold T such that the map $p|_D$ induced from the first projection $p = pr_1$ onto T is a finite cover with branch locus $B \subset T$, then the restriction of p to $T - (D \cup p^{-1}(B))$, the intersection of the complement of D and the preimage of the branch complement, is a punctured disc bundle and its structure homomorphism is called the braid monodromy of D.

In favourable circumstances this notion can be used to assign a braid monodromy to a family of polynomials.

Definition 6.4. A map $f : T \times \mathbb{C}^2 \to \mathbb{C}$ is called a family of plane polynomials admissible with respect to a projection $p : \mathbb{C}^2 \to \mathbb{C}$ if

i) the restriction f_t to each plane $\{t\} \times \mathbb{C}^2$ is a polynomial,
ii) the zero divisor $Z_f = f^{-1}(0)$ and the singular values divisor V_f are branched covers for the appropriate maps.

$$\begin{array}{ccc} Z_f & \to & T \times \mathbb{C}^2 \\ & & \downarrow \\ & & T \times \mathbb{C} \end{array} \qquad\qquad \begin{array}{ccc} V_f & \to & T \times \mathbb{C} \\ & & \downarrow \\ & & T \end{array}$$

In this case the generalised braid monodromy of $V_f \subset T \times \mathbb{C}$ is called the bifurcation braid monodromy of the family.

Lemma 6.5. *The bifurcation braid monodromy of the family of plane polynomials $p_\lambda(x,y) = y^3 - 3\lambda y + 2x$ is generated by the cube of the twist on the two singular values.*

Proof. The divisor of singular values is given by the equation $\lambda^3 = x^2$, hence the bifurcation braid monodromy is the well-known braid monodromy of a generically projected simple cusp. □

Lemma 6.6. *The bifurcation braid monodromy of the family $p_\lambda(x,y) = y^2 - x^2 + \lambda$ is the full braid group* Br_2.

Proof. The divisor of singular values is given by the equation $x^2 = \lambda$, hence the bifurcation braid monodromy is the well-known braid monodromy of a vertical tangency point on a smooth double cover. □

Definition 6.7. The bifurcation braid monodromy group of a plane polynomial p_0 with zero set a simple cover branched at n points by a linear projection $p : \mathbb{C}^2 \to \mathbb{C}$ is the subgroup of Br_n generated by the images of the bifurcation braid monodromy of all families of plane polynomials containing p_0 which are admissible w.r.t. p.

Proposition 6.8. *The bifurcation braid monodromy of any generic polynomial deformation equivalent to $y^2 - x^k$ is the full braid group* Br_k.

Proof. It suffices to consider the family $y^2 - x^k + kx + \lambda$. Its singular value divisor is given by the equation $x^k - kx = \lambda$ of which the braid monodromy is as claimed. □

Thus having dealt with the easiest cases we now want to investigate polynomials with branch degree three, which are in fact intimately related to the alternating Artin systems considered in the first part of this paper.

Lemma 6.9. *The polynomial cover and its complement fibration for the polynomial $y^3 - 3y + 2x^k$ are associated to the alternating Artin system of length $2k$ for a natural choice of basis of the fundamental group of the branch complement.*

Proof. By straightforward computation the fibre at $x = 0$ is $\mathbb{C}^2$ with punctures at $y = -\sqrt{3}, 0, \sqrt{3}$ which is regular, non-regular fibres occur at $x^{2k} = 1$ exactly and along rays $x = r\zeta, r \in [0,1], \zeta$ primitive with $\zeta^{2k} = 1$ the points $-\sqrt{3}, 0$ respectively $0, \sqrt{3}$ get closer and merge finally according to $\zeta^k = -1$ resp. $\zeta^k = 1$.
So the elements of the star-shaped basis are assigned alternatingly the twists of the intervals $[-\sqrt{3}, 0], [0, \sqrt{3}]$ which constitute the generator set for an alternating Artin system of length $2k$ generating Br_3. □

Number the branch points of the polynomial cover given by the polynomial $y^3 - 3y + 2x^k$ according to increasing arg starting with $x_1 = 1$.

Notice that the branch locus for the polynomial cover given by $y^3 - 3p(x)y + 2q(x)$ is described by the equation $p^3(x) = q^2(x)$.

Lemma 6.10. *The family $y^3 - 3y + 2(x^k - \lambda), \lambda \in [-1, 1]$ degenerates at $\lambda = \pm 1$ only, branch points are confined to straight rays, and even resp. odd indexed branch points merge at zero for $\lambda \to 1$ resp. $\lambda \to -1$.*

Proof. The branch points solve the equation

$$(x^k - \lambda)^2 = 1 \quad \Longleftrightarrow \quad x^k = \lambda + 1 \quad \vee \quad x^k = \lambda - 1.$$

The claim follows. □

Lemma 6.11. *The families $y^3 - 3y + 2(x^k \pm 1 - \mu k x), \mu \in \mathbb{C}$ small, have an associated branch locus divisor locally isomorphic to that of the family $y^2 - x^k$.*

Proof. The branch locus is given by the equation

$$\begin{aligned} & (x^k \pm 1 - \mu k x)^2 = 1 \\ \Leftrightarrow \quad & (x^k - \mu k x)(x^k - \mu k x \mp 2) = 0 \end{aligned}$$

The corresponding divisor consists for small μ of a smooth unbranched part and the divisor associated to $y^2 - x^k$. □

Lemma 6.12. *The family $y^3 - 3(1 - \lambda)y + 2(x^k - i\lambda), \lambda \in [0, 1]$ degenerates at $\lambda = 1$ only, all branch points are on a circle of modulus depending on λ, and pairs $x_\nu, x_{\nu+1}$, where ν is even, merge at k distinct points for $\lambda \to 1$.*

Proof. The branch points solve the equation

$$(x^k - i\lambda)^2 = (1 - \lambda)^3 \quad \Leftrightarrow \quad x^k = i\lambda \pm \sqrt{(1 - \lambda)^3}$$

and one may check that $arg(x_\nu)$ is strictly increasing resp. decreasing with $\lambda \to 1$ for odd resp. even index. □

Lemma 6.13. *The family $y^3 + \mu y + 2(x^k - i - \mu), \mu \in \mathbb{C}$ small, has an associated branch locus divisor isomorphic to k copies of the branch locus divisor of the family $y^3 - 3\lambda y + 2x$ locally at $\mu = 0$.*

Proof. The branch locus is given by the equation $(x^k - i - \mu)^2 = -\mu^3$. Hence up to invertible factors this equation reads at each root α of the left hand side: $(x - \alpha)^2 = \mu^3$. Since the number of roots is k the claim follows. □

Lemma 6.14. *There is a family of local deformations of $y^3 - 2(x^k - i)$ such that generically all branch points are simple except for a single double point.*

Proof. At $\lambda = 1$ the polynomial is of the form $y^3 + 2(x^k - i)$, hence the singular values for the projection are at $x^k = i$. On may choose an arbitrary one of these roots say α and define a perturbation $y^3 - 3\varepsilon(x - \alpha)y + 2(x^k - i)$. The singular values are now the zero locus of

$$\varepsilon(x-\alpha)^3 - (x^k - i)^2 =: (x-\alpha)^2(\varepsilon(x-\alpha) - p_\alpha^2(x)).$$

Assume there is another double root $x(\varepsilon)$ for the ε-family then

$$\begin{aligned} & \varepsilon(x(\varepsilon) - \alpha) - p_\alpha^2(x(\varepsilon)) &=& 0 & \quad (1) \\ \wedge \quad & \varepsilon - 2p'_\alpha(x(\varepsilon))p_\alpha(x(\varepsilon)) &=& 0 & \quad (2) \end{aligned}$$

but on the other hand with equation (1) also its derivative must vanish:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varepsilon}\left(\varepsilon(x(\varepsilon) - \alpha) - p_\alpha^2(x(\varepsilon))\right) \\ &= x(\varepsilon) - \alpha + \varepsilon x'(\varepsilon) - 2p(x(\varepsilon))p'(x(\varepsilon))x'(\varepsilon) \\ &\overset{(2)}{=} x(\varepsilon) - \alpha + \varepsilon x'(\varepsilon) - \varepsilon x'(\varepsilon) \\ &= x(\varepsilon) - \alpha. \end{aligned}$$

contrary to the assumption that $x(\varepsilon)$ is a root different from α. □

Lemma 6.15. *There is a family of polynomials which contains in its interior a family parameterised by $\lambda \in [0,1]$ such that $\lambda = 0$ yields the polynomial $y^3 - 3y + 2x^k$ and for $\lambda \to 1$ the family meets its only degeneration for which all points branch points remain distinct except for the merging pair x_1, x_2. Moreover the branch locus divisor has a cusp over $\lambda = 1$.*

Proof. We have to combine the families of the preceding lemmas into a family of two complex parameters with α chosen to be the root at which x_1, x_2 merge. Then the interval can be mapped to the parameter space in such a way as to yield the desired properties.

The final claim follows from Lemma 6.13. □

Our objective is now reached since the corresponding monodromies generate the group E_n.

Proposition 6.16. *The bifurcation braid monodromy group of the plane polynomials $y^3 - 3y + 2x^k$ contains a subgroup in the conjugacy class of the standard isotropy group E_n.*

Proof. The case $k = 1$ is Lemma 6.5. In case $k \geq 2$ we consider the family $y^3 - 3\ell(x)y + 2(x^k - \mu)$ with $\ell(x)$ linear. We compute the braid monodromy with respect to the natural choice of geometric basis of Lemma 6.9 with only a slightest move of the reference point from the origin to a point in the sector defined by the rays of x_{2k} and x_1.

It suffices to show that the generators $e_{12}, e_{\nu,\nu+2}$ of E_M are contained in the braid monodromy. The triple twist e_{12} is obtained by going around a degeneration as given by the family of Lemma 6.15. The twists $e_{\nu,\nu+2}, \nu$ even resp. odd are generators for the full braid group on the even resp. odd indexed branch points. They are realised by an appropriate deformation in the family of Lemma 6.11. □

7. Vanishing arcs and the Donaldson problem

The objective of this section is to shed some light on the Donaldson problem of characterising the vanishing arcs among all isotopy classes of paths in the base of a polynomial covering. We first sharpen the necessary criterion of [7]. Next we give a sufficient criterion in terms of the bifurcation braid monodromy. Finally – after a short digression to the situation for finite coverings – we make ends meet in favourable cases with the help of our algebraic results.

First we recall Donaldson's original definition of admissible paths, to which we add our more restrictive notion of braid admissible paths. Given a path γ in the base we consider also small punctured discs D_0 and D_1 around its endpoints.

Definition 7.1. A path γ in the base of a polynomial covering is called *admissible*, if the monodromy of the associated finite cover along the boundary of D_0 is the same transposition as the monodromy along the boundary of D_1 after parallel transport along γ.

Definition 7.2. A path γ is called *braid admissible*, if the braid monodromy along the boundary of D_0 is the same braid as the monodromy along the boundary of D_1 after parallel transport along γ.

Remark 7.3. Since the finite cover is supposed to be simple, the monodromy of any simple loop around a single branch point is a transposition which determines and is determined by the pair of point, which concurs for the degeneration into the singular fibre. Hence a path γ is admissible, if and only if the concurrent pairs coincide for both ends.

To characterize braid-admissibility one has to take into account the ambient fibre. A path γ is braid admissible, if and only if the concurrent pairs coincide for both ends and there is a path connecting the pair in each fibre punctured at the zeroes of g, which varies smoothly along γ.

Lemma 7.4. *A vanishing arc is admissible and braid admissible.*

Proof. A vanishing arc arises in a smoothing of a single ordinary double point. So locally the divisor of critical values consists of only smooth components without vertical tangents except for a single smooth component locally isomorphic to a double cover branched at a single point.

Hence the vanishing arc is braid admissible and thus admissible. □

Theorem 7.5. *The set of vanishing arcs for a plane polynomial $y^3-3y+2x^k$ is the orbit of a single vanishing arc under the action of the bifurcation braid monodromy group.*

Proof. Any vanishing arc is obtained by a degeneration of the reference polynomial along an embedded path in the bifurcation complement with second endpoint a generic point on the degeneration divisor. Since versal families of the given polynomials are versal for the plane curve singularity y^3+x^k, the degeneration divisor is irreducible so we may assume within each class of such a path to have chosen one which ends at a specified point. Hence a pair of vanishing arcs defines an element of the fundamental group by concatenation of the corresponding paths. The braid associated to this loop maps one of the arcs into the other. □

This result should be seen in contrast to the situation where instead of a family of polynomial covers we consider (abstract) finite covers. Then the set of corresponding vanishing arcs is much larger as it is even invariant under the action of $S_{\mathcal{C}_{2k}}$ and coincides – as remarked in [7] – with the set of admissible arcs.

We can finally characterise vanishing arcs as braid admissible isotopy classes in case the x-degree of our polynomial is sufficiently small:

Theorem 7.6. *The set of vanishing arcs for the polynomials $y^3-3y+2x^k$, $k=2,3$ is the orbit of the chord between the branch points x_1,x_3 under the bifurcation braid monodromy and coincides with the set of braid admissible paths.*

Proof. For the family of Lemma 6.11 the chord between x_1,x_3 is a vanishing arc and Theorem 7.5 then implies that its orbit under the bifurcation braid monodromy group E_{2k} is the set of vanishing arcs. By Lemma 7.4 it is a subset of the braid admissible arcs.

On the other hand if we are given a braid admissible path then performing a half twist on it does not change the monodromy of the polynomial covering, hence it corresponds to a half twist in $S_{\mathcal{A}_{2k}}$. This group coincides with E_{2k} in the given cases, so the half twist can actually be given as e_{13}^e with some $e\in E_{2k}$ by Proposition 4.3. We conclude that the given path is the e-translate of our chord and hence is a vanishing arc. □

Remark 7.7. In general a braid admissible isotopy class gives rise to a half twist contained in $S_{\mathcal{A}_{2k}}$ and it is an open question whether $S_{\mathcal{A}_{2k}}$ equals E_{2k} and whether its half twists are contained in a single E_{2k} conjugation class.

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Michael Lönne
Mathematisches Institut, Universität Bayreuth
Universitätsstraße 30, 95447 Bayreuth, Germany
e-mail: Michael.Loenne@uni-bayreuth.de