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A NOTE ON GORENSTEIN GLOBAL DIMENSION OF PULLBACK RINGS

Driss Bennis

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Abstract. The study of global dimension of pullback rings has been subject of several interesting works and has been served to solve many open problems. In this paper, we attempt to extend some results on the global dimension of pullback rings to the Gorenstein setting. As a particular case we discuss the transfer of the notion of Gorenstein rings in some particular pullback constructions.

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1. Introduction

Throughout the paper all rings are associative with identity, and all modules are, if not specified otherwise, left modules. Let R be a ring and let M be an R-module. The notation M_R (resp., ${}_RM$) means that M is a right (resp., left) R-module. The projective (resp., injective, and flat) dimension of an R-module M is denoted by $\mathrm{pd}(M)$ (resp., $\mathrm{id}(M)$ and $\mathrm{fd}(M)$).

An R-module M is said to be *Gorenstein projective* if there exists an exact sequence of projective modules

$$\mathbf{P} = \ \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence $\mathbf{P}$ exact whenever Q is a projective R-module.

We say that a module M has *Gorenstein projective dimension at most a positive integer* n, and we write $\mathrm{Gpd}(M) \leq n$, if there exists an exact sequence of modules

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is Gorenstein projective.

The Gorenstein projective dimension is analogous to the classical projective dimension and shares some of its principal properties (see [14,15,18,30] for more details). They are linked by the well-known fact that, for every module M, $\mathrm{Gpd}(M) \leq \mathrm{pd}(M)$ with equality $\mathrm{Gpd}(M) = \mathrm{pd}(M)$ when $\mathrm{pd}(M)$ is finite. We say that the Gorenstein projective dimension is a refinement of the projective dimension.

The origin of Gorenstein projective dimension dates back to the sixties of the last century when Auslander [1] introduced it for finitely generated modules over Noetherian rings and developed it with Bridger in [2]. The Gorenstein projective dimension was first called G-dimension by Auslander. Later, Enochs and Jenda [19,20] gave the current extension of the G-dimension to arbitrary modules over rings (that are not necessarily Noetherian), and named it as Gorenstein projective dimension. The same authors [19,20] defined the Gorenstein injective module and the Gorenstein injective dimension as dual notions of their respective Gorenstein projective ones. Namely, a module M is said to be *Gorenstein injective*, if there exists an exact sequence of injective modules

$$\mathbf{I} = \ \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$$

such that $M \cong \mathrm{Im}(I_0 \to I^0)$ and such that $\mathrm{Hom}(E, -)$ leaves the sequence $\mathbf{I}$ exact whenever E is an injective module. We say that a module M has *Gorenstein injective dimension at most a positive integer* n, and we write $\mathrm{Gid}(M) \leq n$, if there exists an exact sequence of modules

$$0 \to M \to G_0 \to \cdots \to G_n \to 0,$$

where each G_i is Gorenstein injective. Also, the same authors with Torrecillas [22] introduced the Gorenstein flat modules and the Gorenstein flat dimension, such that a module M is called *Gorenstein flat* if there exists an exact sequence of flat modules

$$\mathrm{F} = \ \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots ,$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that $I \otimes -$ leaves the sequence F exact whenever I is an injective right module. We say that M has *Gorenstein flat dimension at most a positive integer* n, and we write $\mathrm{Gfd}(M) \leq n$, if there exists an exact sequence of modules

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is Gorenstein flat.

The Gorenstein homological dimensions over Noetherian rings have been subject to an extensive study (see for instance [14,15,18]). It turned out ultimately that they are similar to (and refinements of) the classical homological dimensions. In 2004, Holm [30] generalized several well-known results on Gorenstein dimensions over Noetherian rings to arbitrary rings, and then the Gorenstein homological dimensions theory witnessed a new impetus. Now, the study of Gorenstein dimensions is known as *Gorenstein homological algebra.* A principle guiding in the study of Gorenstein homological dimensions has been formulated in the following metatheorem [29, page V]: *Every result in classical homological algebra has a counter part in Gorenstein homological algebra.* In line with this, several classical results on global homological dimensions were extended to global Gorenstein homological dimensions (see [10,9,8]). Namely, it is proved in [10, Theorem 1.1] that for a ring R:

$$\sup\{\mathrm{Gpd}_R(M) \mid M \textit{ is an } R\textit{−module}\} = \sup\{\mathrm{Gid}_R(M) \mid M \textit{ is an } R\textit{−module}\}.$$

The common value of the terms of this equality is called, when we consider left (resp., right) R-modules, left (resp., right) Gorenstein global dimension of R, and denoted by $l.\mathrm{Ggldim}(R)$ (resp., $r.\mathrm{Ggldim}(R)$). Also, the (left and right) Gorenstein weak dimension of a ring R, $\mathrm{Gwdim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \textit{ is an } R\textit{−module}\}$, is investigated. It is also proved that the Gorenstein global and weak dimensions are refinements of the classical global and weak dimensions of rings, respectively. Thus, $\mathrm{Ggldim}(R) \leq \mathrm{gldim}(R)$ and $\mathrm{Gwdim}(R) \leq \mathrm{wdim}(R)$, with equalities $\mathrm{Ggldim}(R) = \mathrm{gldim}(R)$ and $\mathrm{Gwdim}(R) = \mathrm{wdim}(R)$ when $\mathrm{wdim}(R)$ is finite ([10, Corollary 1.2]).

In this paper, we investigate Gorenstein global dimension in pullback constructions. Recall that a commutative square of ring homomorphisms

$$(\square) \qquad \begin{array}{ccc} R & \xrightarrow{i_1} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow[j_2]{} & R' \end{array}$$

is said to be a pullback square, if given $r_1 \in R_1$ and $r_2 \in R_2$ with $j_1(r_1) = j_2(r_2)$ there exists a unique element $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$. In the above pullback diagram $(\square)$, we assume that j_2 is surjective, so that results of Milnor [33] apply. The ring R is called a pullback of R_1 and R_2 over R'. The most useful particular cases of pullback rings are constructed as follows: Let I be an ideal of a ring T. A subring D of the quotient ring T/I is of the form R/I where R is a subring of T, which contained I as an ideal. Then, R is a pullback ring of T and D over T/I issued from the following pullback diagram of canonical homomorphisms:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \twoheadrightarrow & D = R/I \\ \downarrow & & \downarrow \\ T & \twoheadrightarrow & T/I \end{array}$$

Following [12], R is called the ring of the (T, I, D) construction. This construction includes the well-known "$D+M$" construction and in general the $D+I$-construction (for more details about these constructions, see [12,23,25]).

These constructions have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. In the same direction, the study of global dimension of pullback rings leads to interesting examples (see [17,26,27,32]). In [32, Theorem 2], an interesting upper bound on the global dimension of pullback rings is established as follows: Consider a pullback diagram of type ($\square$). Then,

$$l.\text{gldim}(R) \leq \max_i\{l.\text{gldim}(R_i) + \text{fd}(R_i)_R\}.$$

In this paper, we attempt to extend this result to the setting of Gorenstein dimensions, such that we get:

$$l.\text{Ggldim}(R) \leq \max_i\{l.\text{Ggldim}(R_i) + \text{Gfd}(R_i)_R\}.$$

Naturally, to establish this inequality, one would like to mimic Kirkman and Kuzmanovich's proof of [32, Theorem 2]. But, this seems impossible. In fact, the key result for proving [32, Theorem 2] is the following result.

Proposition 1.1. ([32, Proposition 3]) *Consider a pullback diagram of type* ($\square$)*. Let M be an R-module such that* $\text{Tor}^R_{n_i+m}(R_i, M) = 0$ *for* $m \geq 1$, $i = 1,\ 2$*. Then*

$$\text{pd}_R(M) \leq \max_i\{n_i + \text{pd}_{R_i}(R_i \otimes_R \text{Im}\, f_{n_i})\}$$

where $\cdots \longrightarrow P_{k+1} \overset{f_{k+1}}{\longrightarrow} P_k \overset{f_k}{\longrightarrow} \cdots \longrightarrow P_1 \overset{f_1}{\longrightarrow} P_0 \overset{f_0}{\longrightarrow} M \longrightarrow 0$ *is a projective resolution of M.*

The proof of this result is a direct consequence of the following well-known Milnor's result [33] (see also [24]):

Theorem 1.2 ([33])**.** *Consider a pullback diagram of type* ($\square$)*. An R-module M is projective if and only if the R_1-module $R_1 \otimes_R M$ and the R_2-module $R_2 \otimes_R M$ are projective.*

However, the Gorenstein counterpart of Milnor's result fails from Example 1.4 below. Before giving this example, we need to recall the following notions and results. A ring R is called n-Gorenstein if it is Noetherian with $\mathrm{id}({}_RR) \leq n$ and $\mathrm{id}(R_R) \leq n$; and R is said to be Iwanaga-Gorenstein if it is n-Gorenstein for some positive integer n (see [31] and [18, Section 9.1]). Notice that 0-Gorenstein rings are just the well-known quasi-Frobenius rings. Thus, a ring R is quasi-Frobenius if and only if R is Noetherian and, for every ideal I, $\mathrm{Ann}(\mathrm{Ann}(I)) = I$ where $\mathrm{Ann}(I)$ denotes the annihilator of I (see [35] for more details about this kind of rings).

Lemma 1.3. ([18, Theorem 12.3.1]) *Let R be a Noetherian ring and n a positive integer. The following statements are equivalent:*

1. *R is n-Gorenstein.*
2. *$l.\mathrm{Ggldim}(R) \leq n$.*
3. *$r.\mathrm{Ggldim}(R) \leq n$.*
4. *$\mathrm{Gpd}_R(R/I) \leq n$ for every left and every right ideal I of R.*

The implication (4) $\Longrightarrow$ (1) is a simple consequence of [36, Theorem 9.11]. In fact, for a not necessarily Noetherian ring R with finite left (resp., right) Gorenstein global dimension, one can show, using [36, Theorem 9.11] and [10, Lemma 2.1], that $l.\mathrm{Ggldim}(R) \leq n$ (resp., $r.\mathrm{Ggldim}(R) \leq n$) if and only if $\mathrm{Gpd}_R(R/I) \leq n$ for every left (resp., right) ideal I of R.

Recall that the trivial extension of a ring R by an R-module M is the ring denoted by $R \ltimes M$ whose underling group is $A \times M$ with multiplication given by $(r,m)(r',m') = (rr', rm' + r'm)$ (see [25]).

Example 1.4. Let $\mathbb{R}$ denotes the field of real numbers and let $\mathbb{C}$ denotes the field of complex numbers. Consider the following pullback diagram

$$\begin{array}{ccc} \mathbb{R} \ltimes \mathbb{C} & \twoheadrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{C} \ltimes \mathbb{C} & \twoheadrightarrow & \mathbb{C} \end{array}$$

The ideal $0 \ltimes \mathbb{R}$ of $\mathbb{R} \ltimes \mathbb{C}$ is not Gorenstein projective. However, the $\mathbb{R}$-module $0 \ltimes \mathbb{R} \otimes_{\mathbb{R}\ltimes\mathbb{C}} \mathbb{R}$ and the $\mathbb{C} \ltimes \mathbb{C}$-module $0 \ltimes \mathbb{R} \otimes_{\mathbb{R}\ltimes\mathbb{C}} \mathbb{C} \ltimes \mathbb{C}$ are both Gorenstein projective.

Proof. First, note that $\mathbb{C} \ltimes \mathbb{C}$ is quasi-Frobenius over which all modules are Gorenstein projective, and so the second assertion holds true.
It is easy to show that all ideals of $\mathbb{R} \ltimes \mathbb{C}$ are either isomorph to $(0 \ltimes R)^2$ or to the

ideal $0 \ltimes \mathbb{R}$. Then, if the ideal $0 \ltimes \mathbb{R}$ is Gorenstein projective, so are all ideals of $\mathbb{R} \ltimes \mathbb{C}$. Then $\mathbb{R} \ltimes \mathbb{C}$ is quasi-Frobenius (by Lemma 1.3 and since $\mathbb{R} \ltimes \mathbb{C}$ is Artinian). But this contradict the fact that $\mathrm{Ann}(\mathrm{Ann}(0 \ltimes \mathbb{R})) = 0 \ltimes \mathbb{C}$. Therefore, the ideal $0 \ltimes \mathbb{R}$ of $\mathbb{R} \ltimes \mathbb{C}$ is not Gorenstein projective. □

In view of this, we deduce the difficulty to follow the classical way to extend Kirkman and Kuzmanovich's result [32, Theorem 2]. However, in Section 2, we show that Proposition 1.1 allows us to give a situation where Kirkman and Kuzmanovich's result has a Gorenstein counterpart (see Theorem 2.1), and also allows as to give an upper bound to the global dimension of coherent pullback rings (see Theorem 2.3). In Example 2.4, we give a pullback construction where Theorem 2.3 can be applied.

It is worthwhile reminding that the study of Gorenstein global dimension of Noetherian pullback rings turns to the transfer of the notion of Gorenstein rings in pullback constructions (see Lemma 1.3). In Section 3, we investigate the transfer of the notion of Gorenstein rings in some particular pullback constructions. In Theorem 3.4, we establish a Gorenstein counterpart of the classical fact that domains of global dimension 1 are Noetherian, namely Dedekind. Thus, a domain has Gorenstein global dimension 1 if and only if it is 1-Gorenstein. These kind of domains are extensively studied and characterized by several notions (see [3, Section 6]). The most useful one is the notion of divisorial ideal: an ideal I of a domain R is called divisorial (or reflexive) if $(I^{-1})^{-1} := I_v = I$. The class of domains in which each nonzero ideal is divisorial has been studied, independently and with different methods, by H. Bass [3] and W. Heinzer [28]. Following S. Bazzoni and L. Salce [5,6], these domains are now called divisorial domains. From [3, Theorem 6.3], [28, Corollary 4.3], we have:

Proposition 1.5. *A domain R is 1-Gorenstein if and only if it is a Noetherian divisorial domain.*

Combining the above result with [4, Theorem 2.1(m)], we deduce the following result on 1-Gorenstein domains in "$D + M$" constructions.

Proposition 1.6. *Let V be a discrete valuation domain of the form $K + M$, $R = D + M$, where D is a proper subring of K. Then, R is 1-Gorenstein (i.e., $l.\mathrm{Ggldim}(R) = r.\mathrm{Ggldim}(R) = 1$) if and only if D is a field and the degree of K over D is two.*

Later, using Mimouni's paper [34], we generalize this result to more general cases (see Theorem 3.5 and Corollary 3.8).

2. Gorenstein global dimension of pullback rings

We begin with the first situation where we can get a Gorenstein counterpart of Kirkman and Kuzmanovich's result [32, Theorem 2].

Theorem 2.1. *Consider a pullback diagram of type* ($\square$). *If* $l.\mathrm{Ggldim}(R) < \infty$, *then*

$$l.\mathrm{Ggldim}(R) \leq \max_i\{l.\mathrm{Ggldim}(R_i) + \mathrm{Gfd}(R_i)_R\}.$$

Proof. Assume that $\max_i\{l.\mathrm{Ggldim}(R_i)+\mathrm{Gfd}(R_i)_R\}$ is finite, and set $\mathrm{Gfd}(R_i)_R = n_i$ for $i = 1, 2$. Let I be an injective R-module and consider a projective resolution of I:

$$\cdots \to P_2 \to P_1 \to P_0 \to I \to 0$$

By [10, Lemma 2.1], $\mathrm{pd}_R(I) < \infty$. Since $l.\mathrm{Ggldim}(R) < \infty$, $\mathrm{pd}_R(I_{n_i}) < \infty$ for $i = 1, 2$, where $I_{n_i} = \mathrm{Im}(P_{n_i} \to P_{n_i-1})$. Consider a positive integer $m_i \geq 1$ with $m_i \geq \mathrm{pd}_R(I_{n_i})$. Then, we have the following projective resolution of I_{n_i}:

$$0 \to I_{n_i+m_i} \to P_{n_i+m_i-1} \to \cdots \to P_{n_i} \to I_{n_i} \to 0$$

So, for $m \geq 1$, $\mathrm{Tor}^R_m(R_i, I_{n_i}) = \mathrm{Tor}^R_{n_i+m}(R_i, I) = 0$ (since $\mathrm{Gfd}(R_i)_R = n_i$ and I is an injective R-module). Thus, the following sequence

$$0 \to R_i \otimes_R I_{m_i+n_i} \to \cdots \to R_i \otimes_R P_{n_i} \to R_i \otimes_R I_{n_i} \to 0$$

is exact and it is a projective resolution of the R_i-module $R_i \otimes_R I_{n_i}$. This means that $R_i \otimes_R I_{n_i}$ has finite projective dimension which is at most $l.\mathrm{Ggldim}(R_i)$. Thus, from Proposition 1.1, $\mathrm{pd}_R(I) \leq \max_i\{l.\mathrm{Ggldim}(R_i)+\mathrm{Gfd}(R_i)_R\}$. Therefore, by [10, Lemma 2.1], $l.\mathrm{Ggldim}(R) \leq \max_i\{l.\mathrm{Ggldim}(R_i) + \mathrm{Gfd}(R_i)_R\}$. $\square$

For commutative coherent rings, we give an upper bound to the global dimension of coherent pullback rings without assuming that $l.\mathrm{Ggldim}(R) < \infty$. For that, we need the following result.

In [8], the authors gave a result that allows to show where Gorenstein global dimension of coherent ring is finite (see [8, Proposition 2.5 and 2.11] and [8, Example 2.8]). Recall that a ring R is called n-perfect for some positive integer n, if $\mathrm{pd}(F) \leq n$ for every flat R-module F (see [16,21]), and a commutative ring R is said to be n-FC for some positive integer n, if it is coherent and $\mathrm{FP}{-}\mathrm{id}_R(R) \leq n$, where $\mathrm{FP}{-}\mathrm{id}(M)$ denotes, for an R-module M, the FP-injective dimension, which is defined to be the least positive integer n for which $\mathrm{Ext}_R^{n+1}(P, M) = 0$ for all finitely presented R-modules P. From [13, Theorem 7] (see also [8]), for a positive integer n and a commutative coherent ring R, $\mathrm{Gwdim}(R) \leq n$ if and only if R is n-FC. Then, the inequality of [8, Theorem 2.1] can be written as follows:

Lemma 2.2. ([8, Theorem 2.1]) *For two positive integers n and m, if a commutative ring R is n-FC and m-perfect, then* $\mathrm{Ggldim}(R) \leq n + m$.

Using this result with Proposition 1.1, we establish an upper bound on the global dimension of coherent pullback rings.

Theorem 2.3. *Consider a pullback diagram of type ($\square$) of commutative rings. If R is coherent, then*

$$\mathrm{Ggldim}(R) \leq \max_i\{\mathrm{gldim}(R_i) + \mathrm{Gpd}_R(R_i)\}.$$

Proof. Assume that $\max_i\{\mathrm{gldim}(R_i) + \mathrm{Gpd}_R(R_i)\}$ is finite. Let $\mathrm{Gpd}_R(R_i) = n_i$ and let $\mathrm{gldim}(R_i) = m_i$ for $i = 1, 2$. Then, $\mathrm{Ext}_R^{n_i+k}(R_i, R) = 0$ for every $k \geq 1$. Consider an injective resolution of the R-module R:

$$0 \to R \to I_0 \to I_1 \to I_2 \to \cdots$$

We have, for every $k \geq 1$ and for $i = 1, 2$, $\mathrm{Ext}_R^m(R_i, K_{n_i}) = \mathrm{Ext}_R^{n_i+k}(R_i, R) = 0$, where $K_{n_i} = \mathrm{Im}(I_{n_i} \to I_{n_i+1})$. So we obtain the following exact sequence for $i = 1, 2$:

$$0 \to \mathrm{Hom}_R(R_i, K_{n_i}) \to \mathrm{Hom}_R(R_i, I_{n_i+1}) \to \cdots \to \mathrm{Hom}_R(R_i, I_{n_i+m_i}) \to 0.$$

Since $\mathrm{gldim}(R_i) = m_i$ is finite, $\mathrm{Hom}_R(R_i, I_{n_i+m})$ is an injective R_i-module, where $m = \max\{m_1, m_2\}$. Then, by [24], I_{n_i+m} is an injective R-module, which means that $\mathrm{id}_R(R) \leq n_i + m$ and so R is $n_i + m$-FC.
On the other hand, consider a flat R-module F and a projective resolution of M as in Proposition 1.1. Since $\mathrm{gldim}(R_i) = m_i$, $\mathrm{pd}(_{R_i}(R_i \otimes_R \mathrm{Im}\, f_{n_i})) \leq m_i$. Then, by Proposition 1.1, $\mathrm{pd}_R(F) \leq \max_i\{\mathrm{pd}(_{R_i}(R_i \otimes_R \mathrm{Im}\, f_{n_i}))\} \leq m$, where $m = \max_i\{m_1, m_2\}$. This means that R is m-perfect. By Lemma 2.2, $\mathrm{Ggldim}(R)$ is finite and so $\mathrm{Ggldim}(R) \leq \max_i\{\mathrm{Ggldim}(R_i) + \mathrm{Gfd}(R_i)\}$ from Theorem 2.1. Therefore, the desired inequality follows since over rings of finite Gorenstein global dimension every Gorenstein projective module is Gorenstein flat and so $\mathrm{Gfd}(R_i) \leq \mathrm{Gpd}(R_i)$ (see [10, proof of Corollary 1.2 (1)]). $\square$

As an example where Theorem 2.3 can be applied, we give the following example due to Vasconcelos [37, Page 29]:

Example 2.4. *Let V be a valuation domain with a principal maximal ideal aV and let R be a subring of $V \times V$ of pairs (x, y) with $x - y \in aV$. Then,* $\mathrm{Ggldim}(R) \leq \mathrm{Ggldim}(V)$.

Proof. First note that R is the pullback ring arising from the following pullback diagram:

$$\begin{array}{ccc} R & \xrightarrow{i_2} & V \\ \downarrow{\scriptstyle i_2} & & \downarrow \\ V & \twoheadrightarrow & V/aV \end{array}$$

where $i_1(x, y) = (0, y)$ and $i_2(x, y) = (x, 0)$. Then, V as R-module via i_1 is isomorph to $0 \times aV$, and V as R-module via i_2 is isomorph to $aV \times 0$.
Consider the following short exact sequences:

$$0 \longrightarrow 0 \times aV \hookrightarrow R \xrightarrow{(a,0)} aV \times 0 \longrightarrow 0$$

$$0 \longrightarrow aV \times 0 \hookrightarrow R \xrightarrow{(0,a)} aV \times 0 \longrightarrow 0$$

Using the same argument as in the proof of [7, Example 2.4] and the fact that R is a local ring, we get that $aV \times 0 \cong V$ and $0 \times aV \cong V$ are Gorenstein projective R-modules.
On the other hand, from [37, proof of Theorem], V is a coherent ring. Indeed, since V is a valuation ring, every finitely generated ideal I of R is of the form $I = (x, 0)R \oplus (0, y)R$ for appropriate x and y, and since we have the following short exact sequences:

$$0 \longrightarrow 0 \times aV \hookrightarrow R \xrightarrow{(x,0)} (x, 0)R \longrightarrow 0$$

$$0 \longrightarrow aV \times 0 \hookrightarrow R \xrightarrow{(0,y)} (0, y)R \longrightarrow 0$$

I is finitely presented and so R is coherent. Therefore, Theorem 2.3 can be applied, and thus $\mathrm{Ggldim}(R) \leq \mathrm{Ggldim}(V)$. □

3. Gorenstein pullback rings

Throughout this section all rings are, except in Theorem 3.4, commutative.

This section investigates Gorenstein global dimension of Noetherian pullback rings. From Lemma 1.3, this is equivalent to the notion of Gorenstein rings in pullback constructions. Then, using properties of n-Gorenstein rings, we establish more results on Gorenstein global dimension of Noetherian pullback rings.

Theorem 3.1. *Consider a pullback diagram of type* (□). *If R is Noetherian and* $\max_i\{\mathrm{gldim}(R_i) + \mathrm{Gfd}_R(R_i)\} = n$ *is finite, then R is n-Gorenstein.*

Proof. Recall that a Noetherian ring is n-Gorenstein if and only if every injective module has projective dimension at most n. Then, consider an injective R-module I and let

$$\cdots \to P_2 \to P_1 \to P_0 \to I \to 0$$

be a projective resolution of I. Consider $I_{n_i} = \mathrm{Im}(P_{n_i} \to P_{n_i-1})$, where $n_i = \mathrm{Gfd}_R(R_i)$ for $i = 1, 2$. Since I is an injective R-module,

$$\mathrm{Tor}^R_m(R_i, I_{n_i}) = \mathrm{Tor}^R_{n_i+m}(R_i, I) = 0$$

for every $m \geq 1$. Thus, from Proposition 1.1,

$$\mathrm{pd}_R(I) \leq \max_i\{n_i + \mathrm{pd}_{R_i}(R_i \otimes_R I_{n_i})\} \leq \max_i\{\mathrm{Ggldim}(R_i) + \mathrm{Gfd}_R(R_i)\} = n.$$

Therefore, by [18, Theorem 9.1.11], R is n-Gorenstein. □

Using the notion of Krull dimension of rings, we can determine the Gorenstein global dimension of certain Noetherian pullback rings. In fact, it is well-known that the Krull dimension, $\dim(R)$, of a Iwanaga-Gorenstein ring R is finite. Namely, if R is a Iwanaga-Gorenstein ring, then $\mathrm{id}_R(R) = \dim(R)$ [3, Corollary 3.4]. On the other hand, the Krull dimension of a ring R of the (T, I, D) construction was investigated by Cahen in [12, Section 2], such that, from [12, Section 2, Corollary 2], $\dim(R) = \dim(T) + \dim(D)$ if T is local with a unique maximal ideal I.
Also recall that R is Noetherian if and only if T is Noetherian, $D = R/I$ is a subfield of T/I such that the degree of T/I over D is finite [12, Section 1, Corollary 1].
Using these facts with [14, Theorem 1.4.9 and Proposition 5.2.9], we get:

Proposition 3.2. *Consider a Noetherian ring R of the (T, I, D) construction where T is local with a unique maximal ideal I. If $\mathrm{Gfd}_R(R/I)$ is finite, then R is an n-Gorenstein local ring, where $n = \mathrm{Gfd}_R(R/I) = \dim(R) = \dim(T)$.*

Example 3.3. *Let V be a discrete valuation domain with a principal maximal ideal aV and let R be a subring of $V \times V$ of pairs (x, y) with $x - y \in aV$. Then,* $\mathrm{Ggldim}(R) = 1$.

Proof. Flows from Example 2.4, Proposition 3.2, and since R is of the $(V\times V, aV \times aV, V/aV)$ construction (see [37, the note before Theorem 3.4]). □

We close the paper with some results on 1-Gorenstein domains.

The following result shows that domains of Gorenstein global dimension 1 are Noetherian and so they are just 1-Gorenstein domains. This is an extension of the well-known classical case; that is, the domains of global dimension 1 are Noetherian. Namely, they are just Dedekind domains.

Theorem 3.4. *Let R be an integral domain. Then, the following statements are equivalent:*

(1) $l.\mathrm{Ggldim}(R) \leq 1$.
(2) $r.\mathrm{Ggldim}(R) \leq 1$.
(3) *R is 1-Gorenstein.*

Proof. The implications (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) follow from [18, Theorem 12.3.1]. We prove the implication (1) $\Rightarrow$ (3). The implication (2) $\Rightarrow$ (3) has a similar proof. Consider a left ideal I of R and let $x \neq 0$ be any element in I. We have to prove that the quotient ring R/xR is Noetherian. In fact, this implies that the ideal I/xR of R/xR is finitely generated and so is the ideal I of R. Therefore, R is Noetherian. Then, it remains to prove that the quotient ring R/xR is Noetherian. Consider an R/xR-module M. As an R-module, M can not be Gorenstein projective (deny M embeds in a free R-module, and this contradicts the fact that $xM = 0$). Thus, there exists, from [30, Proposition 2.18], a short exact sequence of R-modules

$$0 \to G_1 \to G_0 \to M \to 0,$$

where G_1 is free and G_0 is Gorenstein projective. Tensorising this sequence by R/xR we get the following exact sequence of R/xR-module:

$$\mathrm{Tor}_1^R(R/xR, G_0) \to \mathrm{Tor}_1^R(R/xR, M) \to G_1/xG_1 \to G_0/xG_0 \to M \to 0.$$

From [11, Examples (1), p. 102], $\mathrm{Tor}_1^R(R/xR, M) = M$ since $xM = 0$ and $\mathrm{Tor}_1^R(R/xR, G_0) = 0$ since as a Gorenstein projective R-module, G_0 embeds in a free R-module, and so x is a G_0-regular element. Then, M embeds in the free R/xR-module G_1/xG_1. Therefore, by Faith and Walker's theorem [35, Theorem7.56], R/xR is quasi-Frobenius and it is then Noetherian. □

As mentioned in the end of the introduction, the 1-Gorenstein domains are also characterized by the notion of divisorial ideals (see Proposition 1.5). This gives another tool to show when a pullback domain has Gorenstein global dimension 1 (i.e., it is 1-Gorenstein). We close the paper with the following results on 1-Gorenstein domains R of the (T, M, D) constructions.

The following result generalizes Proposition 1.6.

Theorem 3.5. *Consider a domain R of the (T, M, D) construction such that T is a domain and M is a maximal ideal of T. Then, the following statements are equivalent:*

(1) *R is 1-Gorenstein.*
(2) *T is 1-Gorenstein, $M^{-1} = T$, D is a subfield of $T/M = K$, $[K : D] = 2$, and every ideal J of R_M, with JT_M is not principal in T_M, is an ideal of T_M.*

The proof would use the following results:

First note that the local case of Theorem 3.5 above can be deduced from Mimouni's results [34, Lemma 3.2 and Theorem 3.4,(ii)⇔(iii)], such that we get:

Lemma 3.6. *Consider a domain R of the (T, M, D) construction such that T is a local domain with a unique maximal ideal M. Then, the following statements are equivalent:*

(1) *R is 1-Gorenstein.*
(2) *T is 1-Gorenstein, $M^{-1} = T$, D is a subfield of $T/M = K$, $[K : D] = 2$, and every ideal J of R, with JT is not principal in T, is an ideal of T.*

The following result is well-known. It is in fact a simple consequence of [3, Corollary 2.3 (ii)] (see also [18, Corollary 3.2.6]).

Lemma 3.7. *Let S be a Noetherian ring. Then, for a positive integer n, the following statements are equivalent:*

(1) *S is n-Gorenstein.*
(2) *S_m is n-Gorenstein for every maximal ideal m of R.*

Proof of Theorem 3.5. (1) ⇒ (2). Assume that R is a 1-Gorenstein domain. Then, by [12, Section 1, Corollary 1], D is a field and T is Noetherian. Thus, $M^{-1} = T$, and $[K : k] = 2$ (by [34, Lemma 3.2]). To show that T is 1-Gorenstein, we use Lemma 3.7, such that we prove that every localization of T is 1-Gorenstein. Let N be a maximal ideal of T. If $N = M$, apply Lemma 3.6 to the diagram $(\square_M)$:

$$\begin{array}{ccc} R_M & \longrightarrow & k \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K \end{array}$$

we get that T_M is 1-Gorenstein. For the case where $N \neq M$, we have $T_N = R_{N\cap R}$, which is 1-Gorenstein by Lemma 3.7.

The last statement is a consequence of Lemmas 3.6 applied to the diagram $(\square_M)$.
(2) $\Rightarrow$ (1). A similar argument as in (1)$\Rightarrow$(2) works. $\square$

As a direct consequence of Theorem 3.5, we have the following result. Compare it with Proposition 1.6 and [34, Corollary 3.9].

Corollary 3.8. *Consider a domain R of the (T, M, D) construction such that T is a Dedekind domain and M is a maximal ideal of T. Then, the following statements are equivalent:*

(1) *R is 1-Gorenstein.*
(2) *D is a subfield of $T/M = K$ with $[K : D] = 2$.*

To construct a concrete examples, we can use the polynomial rings as follows:

Corollary 3.9. *Consider a domain R of the $(K[X], M, D)$ construction, where $K[X]$ is a polynomial ring of one indeterminate X over a field K. Then, the following statements are equivalent:*

(1) *R is 1-Gorenstein.*
(2) *D is a subfield of K with $[K : D] = 2$.*

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Driss Bennis
Department of Mathematics,
Faculty of Science and Technology of Fez, Box 2202,
University S. M. Ben Abdellah Fez, Morocco,
e-mail: driss_bennis@hotmail.com