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ON THE EXTENDED TOTAL GRAPH OF MODULES OVER COMMUTATIVE RINGS

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ABSTRACT. Let M be a module over a commutative ring R and U a nonempty proper subset of M. In this paper, the extended total graph, denoted by $ET_U(M)$, is presented, where U is a multiplicative-prime subset of M. It is the graph with all elements of M as vertices, and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $rm + sn \in U$ for some $r, s \in R \setminus (U : M)$. We also study the two (induced) subgraphs $ET_U(U)$ and $ET_U(M \setminus U)$, with vertices U and $M \setminus U$, respectively. Among other things, the diameter and the girth of $ET_U(M)$ are also studied.

Mathematics Subject Classification (2010): 13C13, 05C75, 13A15 Keywords: Total graph, prime submodule, multiplicative-prime subset

1. Introduction

Throughout this paper, R is a commutative ring with nonzero identity and M is a unitary R-module. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [1], [2], [3], [5], [6], [7], [8], and [9]). Anderson and Badawi in [4] defined a nonempty proper subset H of R to be a *multiplicative-prime* subset of R if the following two conditions hold: (i) $rs \in H$ for every $r \in H$ and $s \in R$; (ii) if $rs \in H$ for some $r, s \in R$, then either $r \in H$ or $s \in H$. They introduced the notion of the generalized total graph of a commutative ring $GT_H(R)$ with the vertices all elements of R, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in H$, where H is a *multiplicative-prime* subset of R.

Let R be a commutative ring and U be a nonempty subset of an R-module M. The subset $\{r \in R : rM \subseteq U\}$ will be denoted by $(U :_R M)$ or (U : M). It is clear that if U is a submodule of M, then (U : M) is an ideal of R. We define a nonempty subset U of M to be a *multiplicative-prime* subset of M if the following two conditions hold: (i) $rm \in U$ for every $r \in R$ and $m \in U$; (ii) if $sx \in U$ for some $s \in R$ and $x \in M$, then $x \in U$ or $s \in (U : M)$. Note that if U is a multiplicative-prime submodule of M, then U is necessarily a prime submodule of M. One can show that if U is a multiplicative-prime subset of M, then (U:M) is a multiplicative-prime subset of R.

The total graph of a module M with respect to a multiplicative-prime subset U(denoted by $GT_U(M)$) was introduced in [10]. The set of vertices of $GT_U(M)$ is all elements of M, and two distinct vertices m and n adjacent whenever $m + n \in U$. In this paper, we introduce an extension of the graph $GT_U(M)$, denoted by $ET_U(M)$, such that its vertex set consists of all elements of M and for distinct $m, n \in M$, the vertices m and n are adjacent if and only if $rm + sn \in U$ for some $r, s \in R \setminus (U : M)$, where U is a multiplicative-prime subset of M.

Let $ET_U(U)$ be the (induced) subgraph of $ET_U(M)$ with vertex set U, and let $ET_U(M \setminus U)$ be the (induced) subgraph $ET_U(M)$ with vertices consisting of $M \setminus U$. Obviously, the total graph $GT_U(M)$ is a subgraph of $ET_U(M)$. It follows that each edge (path) of $GT_U(M)$ is an edge (path) of $ET_U(M)$. The study of $ET_U(M)$ breaks naturally into two cases depending on whether or not U is a submodule of M. In the second section, we handle the case when U is a submodule of M. For every case, we characterize the girth and diameter of $ET_U(M)$, $ET_U(U)$ and $ET_U(M \setminus U)$.

We begin with some notation, and definitions. For a graph Γ , by $E(\Gamma)$ and $V(\Gamma)$, we mean the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two of its distinct vertices. At the other extreme, we say that a graph is totally disconnected if no two vertices of this graph are adjacent. The distance between two distinct vertices a and b, denoted by d(a, b), is the length of a shortest path connecting them (if such a path does not exist, then $d(a,b) = \infty$). We also define d(a,a) = 0. The diameter of a graph Γ , denoted by diam(Γ), is equal to sup{ $d(a, b) : a, b \in V(\Gamma)$ }. A graph is complete if it is connected with diameter less than or equal to one. The girth of a graph Γ , denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ , provided Γ contains a cycle; otherwise; $\operatorname{gr}(\Gamma) = \infty$. We denote the complete graph on n vertices by K^n and the complete bipartite graph on m and n vertices by $K^{m,n}$ (we allow m and n to be infinite cardinals). For a graph Γ , the degree of a vertex v in Γ , denoted by deg(v), is the number of edges of Γ incident with v. We say that two (induced) subgraphs Γ_1 and Γ_2 of Γ are disjoint if Γ_1 and Γ_2 have no common vertices and no vertex of Γ_1 is adjacent(in Γ) to some vertex of Γ_2 .

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2. The case when U is a submodule of M

In this section, we study the case when U is a submodule of M. It is clear that if U is a submodule of M, then U is a prime submodule of M. If U = M, then it is clear that $ET_U(M)$ is a complete graph and $ET_U(M)$ is a disconnected graph when U = 0 and $|M| \ge 2$. So we may assume that $U \ne 0$ and $U \ne M$.

First, we begin with the following example that shows we may have $ET_U(M) \neq GT_U(M)$.

Example 2.1. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{10}$. Set $U = \{\overline{0}, \overline{5}\}$. It is clear that U is a submodule of M and $(U:M) = 5\mathbb{Z}$. Since $\overline{1} + \overline{3} = \overline{4} \notin U$, so $\overline{1} - \overline{3}$ is not an edge in $GT_U(M)$. But $2(\overline{1}) + 1(\overline{3}) = \overline{5} \in U$ and $2, 1 \in R \setminus (U:M)$. Thus $\overline{1} - \overline{3}$ is an edge in $ET_U(M)$. Hence $ET_U(M) \neq GT_U(M)$.

The main goal of this section is a general structure theorem (Theorem 2.4) for $ET_U(M \setminus U)$ when U is a submodule of M. But first, we record the trivial observation that if U is a submodule of M, then $ET_U(U)$ is a complete subgraph of $ET_U(M)$ and is disjoint from $ET_U(M \setminus U)$. Thus we will concentrate on the subgraph $ET_U(M \setminus U)$ throughout this section.

Theorem 2.2. Let M be a module over a commutative ring R and U be a prime submodule of M. Then $ET_U(U)$ is a complete subgraph of $ET_U(M)$ and is disjoint from $ET_U(M \setminus U)$. In particular, $ET_U(U)$ is connected and $ET_U(M)$ is disconnected.

Proof. Let $m, n \in U$. Then it is clear that $m + n \in U$ since U is a submodule of M. If $x \in U$ is adjacent to $y \in M \setminus U$, then $rx + sy \in U$ for some $r, s \in R \setminus (U : M)$. This implies that $sy \in U$; so $y \in U$ or $s \in (U : M)$ since U is a prime submodule, which is a contradiction. The "in particular" statement is clear.

Theorem 2.3. Let M be a module over a commutative ring R and U be a prime submodule of M. Let G be an induced subgraph of $ET_U(M \setminus U)$, and m and m' be distinct nonadjacent vertices of G that are connected by a path in G. Then there exists a path in G of length 2 between m and m'. In particular, if $ET_U(M \setminus U)$ is connected, then $diam(ET_U(M \setminus U)) \leq 2$.

Proof. (1) Let m_1, m_2, m_3 and m_4 be distinct vertices of G. It suffices to show that if there is a path $m_1 - m_2 - m_3 - m_4$ from m_1 to m_4 , then m_1 and m_4 are adjacent. Now, $r_1m_1 + r_2m_2, r'_2m_2 + r'_3m_3, r_3m_3 + r_4m_4 \in U$ for some $r_1, r_2, r'_2, r'_3, r_3, r_4 \in R \setminus (U:M)$. Hence $(r_1r_3r'_2)m_1 + (r_2r'_3r_4)m_4 = r_3r'_2(r_1m_1 + r_2m_2) - r_2r_3(r'_2m_2 + r'_3m_3)m_4 = r_3r'_2(r_1m_1 + r_2m_3) + r_3r_3(r'_2m_3 + r_3m_3)m_4$ $r'_3m_3) + r_2r'_3(r_3m_3 + r_4m_4) \in U$, and $r_1r_3r'_2, r_2r'_3r_4 \notin (U:M)$ since (U:M) is a prime ideal of R. Thus m_1 and m_4 are adjacent. So if $ET_U(M \setminus U)$ is connected, then $diam(ET_U(M \setminus U)) \leq 2$.

Now, we give the main theorem of this section. Since $ET_U(U)$ is a complete subgraph of $ET_U(M)$ by Theorem 2.2, the next theorem gives a complete description of $ET_U(M \setminus U)$. Let $|U| = \alpha$. We allow α to be an infinite cardinal. Compare the next theorem with [10, Theorem 3.5].

Theorem 2.4. Let M be a module over a commutative ring R, U be a prime submodule of M, and $|U| = \alpha$.

- (1) If $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$, then $ET_U(M \setminus U)$ is the union of complete subgraphs.
- (2) If $r + s \notin (U:M)$ for all $r, s \in R \setminus (U:M)$, then $ET_U(M \setminus U)$ is the union of totally disconnected subgraphs and some connected subgraphs.

Proof. (1) Suppose that $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$. For $m, m' \in M \setminus U$, we write $m \sim m'$ if and only if $tm + t'm' \in U$ and $t + t' \in (U : M)$ for some $t, t' \in R \setminus (U : M)$. It is straightforward to check that \sim is an equivalence relation on $M \setminus U$ since U is a prime submodule. For $m \in M \setminus U$, we denote the equivalence class which contains m by [m]. Now let $m \in M \setminus U$. If $[m] = \{m\}$, then $r(m + u_1) + s(m + u_2) = (r + s)m + ru_1 + su_2 \in U$ for every $u_1, u_2 \in U$ since $r + s \in (U : M)$. Then m + U is a complete subgraph of $ET_U(M \setminus U)$ with at most α vertices. Now let $|[m]| = \nu$ and $m' \in [m]$. Then $tm + t'm' \in U$ and $t + t' \in (U : M)$ for some $t, t' \in R \setminus (U : M)$. So $t(m + u_1) + t'(m' + u_2) = tm + t'm' + tu_1 + t'u_2 \in U$ for every $u_1, u_2 \in U$. Thus m + U is part of the complete graph k^{μ} , where $\mu \leq \alpha\nu$. (2) Assume that $r + s \notin (U : M)$ for all $r, s \in R \setminus (U : M)$. Let

$$A_m = \{m' \in M \setminus U : rm + sm' \in U \text{ for some } r, s \in R \setminus (U : M)\}$$

be the set of all vertices adjacent to m. If $A_m = \emptyset$, then $pm + qm' \notin U$ for every $m' \in M \setminus U$ and every $p, q \in R \setminus (U : M)$. In this case, we show that m + U is a totally disconnected subgraph of $ET_U(M \setminus U)$. If $r(m + m_1) + s(m + m_2) \in U$ for some $r, s \in R \setminus (U : M)$ and $m_1, m_2 \in U$, then $(r + s)m \in U$. Since U is a prime submodule of M and $m \notin U$, then $r + s \in (U : M)$, which is a contradiction. Therefore m + U is a totally disconnected subgraph of $ET_U(M \setminus U)$. Now, we may assume that $A_m \neq \emptyset$. Then $rm + sm' \in U$ for some $r, s \in R \setminus (U : M)$ and $m' \in M \setminus U$. Thus $r(m + u_1) + s(m' + u_2) = rm + sm' + ru_1 + su_2 \in U$ for every $u_1, u_2 \in U$; hence each element of m + U is adjacent to each element of m' + U.

If $|A_m| = \nu$, then we have a connected subgraph of $ET_U(M \setminus U)$ with at most $\alpha \nu$ vertices. So $ET_U(M \setminus U)$ is the union of totally disconnected subgraphs and some connected subgraphs.

Now it is easy to compute the diameter and the girth of $ET_U(M \setminus U)$ using Theorem 2.4.

Theorem 2.5. Let M be a module over a commutative ring R such that U is a prime submodule of M.

- (1) $diam(ET_U(M \setminus U)) = 0$ if and only if $U = \{0\}$ and |M| = 2.
- (2) $diam(ET_U(M \setminus U)) = 1$ if and only if either $|M \setminus U| = 1$ and $r+s \notin (U:M)$ for some $r, s \in R \setminus (U:M)$ or $|M \setminus U| = 2$, $r+s \in (U:M)$ for every $r, s \in R \setminus (U:M)$ and $x+y \in U$ for some distinct elements $x, y \in M \setminus U$.
- (3) $diam(ET_U(M \setminus U)) = 2$ if and only if $|M \setminus U| = 2$, $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$, $x + y \in U$ for some distinct elements $x, y \in M \setminus U$ and $|m + U| \ge 2$ for some $m \in M \setminus U$.
- (4) Otherwise, $diam(ET_U(M \setminus U)) = \infty$.

Proof. (1) If $diam(ET_U(M \setminus U)) = 0$, then $ET_U(M \setminus U)$ is a complete graph K^1 , and so |U| = |M/U| = 1 by Theorem 2.4. Hence $U = \{0\}$ and |M| = 2. Now, let $U = \{0\}$ and $M = \{0, m\}$. Then m + U is a single graph K^1 . So $diam(ET_U(M \setminus U)) = 0$.

(2) It is clear that $ET_U(M \setminus U)$ is a complete graph if and only if $diam(ET_U(M \setminus U)) = 1$. So the proof is clear by Theorem 2.4.

(3) If $diam(ET_U(M \setminus U)) = 2$, then $ET_U(M \setminus U)$ is a complete bipartite graph $K^{m,n}$ such that $m \ge 2$ or $n \ge 2$. Thus $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ by Theorem 2.4. Therefore $|M \setminus U| = 2$ and $x + y \in U$ for some $x, y \in M \setminus U$. Since $m \ge 2$ or $n \ge 2$, we have $|x + U| \ge 2$ or $|y + U| \ge 2$. Conversely, let $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $|M \setminus U| = 2$. Then $M = U \cup (x + U) \cup (y + U)$ and $ET_U(M \setminus U)$ is a complete bipartite graph since $x + y \in U$. Hence $diam(ET_U(M \setminus U)) = 2$, since $|x + U| \ge 2$ or $|y + U| \ge 2$.

Theorem 2.6. Let M be a module over a commutative ring R such that U is a prime submodule of M. Then $gr(ET_U(M \setminus U)) = 3, 4$, or ∞ . In particular, $gr(ET_U(M \setminus U)) \leq 4$ if $ET_U(M \setminus U)$ contains a cycle.

Proof. Assume that $ET_U(M \setminus U)$ contains a cycle. Then $ET_U(M \setminus U)$ is not a totally disconnected graph; so by the proof of Theorem 2.4, $ET_U(M \setminus U)$ has either a complete or a complete bipartite subgraph. Therefore it must contain either a 3-cycle or 4-cycle. Thus $gr(ET_U(M \setminus U)) \leq 4$.

Theorem 2.7. Let M be a module over a commutative ring R such that U is a prime submodule of M.

- (1) $gr(ET_U(M \setminus U)) = 3$ if and only if $r + s \in (U : M)$ and $|y + U| \ge 3$ for some $r, s \in R \setminus (U : M)$ and $y \in M \setminus U$.
- (2) $gr(ET_U(M \setminus U)) = 4$ if and only if $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$.
- (3) Otherwise, $gr(ET_U(M \setminus U)) = \infty$.

Proof. (1) Assume that $gr(ET_U(M \setminus U)) = 3$. Then by Theorem 2.4, $ET_U(M \setminus U)$ is a complete graph K^{λ} , where $\lambda \geq 3$. Then $r + s \in (U : M)$ for some $r, s \in R \setminus (U : M)$ and $|y + U| \geq 3$ for some $y \in M \setminus U$ by Theorem 2.4.

(2) If $gr(ET_U(M)) = 4$, then $ET_U(M \setminus U)$ has a complete bipartite subgraph. So $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$ by Theorem 2.4.

The other implications of (1) and (2) follows directly from Theorem 2.4.

We end this section with the following theorem.

Theorem 2.8. Let M be a module over a commutative ring R such that U is a prime submodule of M.

- (1) $gr(ET_U(M)) = 3$ if and only if $|U| \ge 3$.
- (2) $gr(ET_U(M)) = 4$ if and only if $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$, |U| < 3, and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$.
- (3) Otherwise, $gr(ET_U(M)) = \infty$.

Proof. (1) This follows from Theorem 2.2.

(2) Assume that $gr(ET_U(M)) = 4$. Since $gr(ET_U(U)) = 3$ or ∞ , then $gr(ET_U(M \setminus U)) = 4$. Therefore $r + s \notin (U : M)$ for every $r, s \in R \setminus (U : M)$ and $pm + qm' \in U$ for some $m, m' \in M \setminus U$ and $p, q \in R \setminus (U : M)$ by Theorem 2.7. On the other hand, $gr(ET_U(M)) \neq 3$; so |U| < 3. The other implication follows from Theorem 2.4.

3. The case when U is not a submodule of M

In this section, we study $ET_U(M)$ when the multiplicative-prime subset U is not a submodule of M. Since U is always closed under multiplication by elements of R, this just means that $0 \in U$ and there are distinct $x, y \in U$ such that $x + y \in M \setminus U$.

First, we begin with the following example that shows we may have $ET_U(M) \neq GT_U(M)$.

Example 3.1. Let $R = M = \mathbb{Z}$. Set $U = 4\mathbb{Z} \cup 6\mathbb{Z}$. It is clear that (U : M) = Uand U is not a submodule of M since $4, 6 \in U$, but $4 + 6 = 10 \notin U$. So 4 - 6 is not an edge in $GT_U(M)$. But $2(4) + 2(6) = 20 \in U$ and $2 \in R \setminus U$. Thus 4 - 6 is an edge in $ET_U(M)$. Hence $ET_U(M) \neq GT_U(M)$.

Now, we have the following theorem that shows $ET_U(U)$ is always connected (but never complete), $ET_U(U)$ and $ET_U(M \setminus U)$ are never disjoint subgraphs of $ET_U(M)$, and $ET_U(M)$ is connected when $ET_U(M \setminus U)$ is connected.

Theorem 3.2. Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then the following hold:

- (1) $ET_U(U)$ is connected with $diam(ET_U(U)) = 2$.
- (2) Some vertex of $ET_U(U)$ is adjacent to a vertex of $ET_U(M \setminus U)$. In particular, the subgraphs $ET_U(U)$ and $ET_U(M \setminus U)$ are not disjoint.
- (3) If $ET_U(M \setminus U)$ is connected, then $ET_U(M)$ is connected.

Proof. (1) Let $u \in U^* = U \setminus \{0\}$. Then u is adjacent to 0. Thus u - 0 - u' is a path in $ET_U(U)$ of length two between any two distinct $u, u' \in U^*$. Moreover, there exist nonadjacent $u, u' \in U^*$ since U is not a submodule of M; thus $diam(ET_U(U)) = 2$. (2) Since U is not a submodule of M, there exist distinct $m, n \in U^*$ such that $m + n \notin U$. Then $-m \in U$ and $m + n \notin U$ are adjacent vertices in $ET_U(M)$. Finally, the "in particular" statement is clear.

(3) $ET_U(U)$ and $ET_U(M \setminus U)$ are connected, and there is an edge between $ET_U(U)$ and $ET_U(M \setminus U)$; thus $ET_U(M)$ is connected.

We determine when $ET_U(M)$ is connected and compute $diam(ET_U(M))$ with the following theorem. Compare the next theorem with [10, Theorem 4.2].

Theorem 3.3. Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then $ET_U(M)$ is connected if and only if for every $m \in M$ there exists $r \in R \setminus (U : M)$ such that $rm \in \langle U \rangle$.

Proof. Suppose that $ET_U(M)$ is connected and $m \in M$. Then there exists a path $0 - x_1 - x_2 - \cdots - x_n - m$ from 0 to m in $ET_U(M)$. Thus

 $r_1x_1, r_2x_1 + r_3x_2, \dots, r_{2n-2}x_{n-1} + r_{2n-1}x_n, r_{2n}x_n + sm \in U$

for some $r_1, r_2, \ldots, r_{2n}, s \in R \setminus (U:M)$. Then

 $sr_1r_3r_5\cdots r_{2n-1}m = (r_1r_3r_5\cdots r_{2n-1})(sm + r_{2n}x_n) -$

$$(r_1r_3r_5\cdots r_{2n-3}r_{2n})(r_{2n-2}x_{n-1}+r_{2n-1}x_n)+\cdots$$

$$-(r_1r_3\cdots r_{2n-2k-5}r_{2n-2k-3}r_{2n-2k-3}r_{2n-2k-2}\cdots r_{2n})(r_{2n-2k-1}x_{n-k}+r_{2n-2k-2}x_{n-(k+1)})$$

 $+(r_1r_3\cdots r_{2n-2k-5}r_{2n-2k-2}r_{2n-2k}r_{2n-2k-2}\cdots r_{2n})(r_{2n-2k-3}x_{n-(k+1)}+r_{2n-2k-4}x_{n-(k+2)})$

 $\cdots - (r_2 r_4 r_6 \cdots r_{2n})(r_1 x_1) \in \langle U \rangle$

Since (U:M) is a multiplicative-prime subset of R, we have $r = sr_1r_3r_5\cdots r_{2n-1} \in R \setminus (U:M)$ and $rm \in \langle U \rangle$. Conversely, suppose that for every $m \in M$ there exists $r \in R \setminus (U:M)$ such that $rm \in \langle U \rangle$. We show that for each $0 \neq m \in M$, there exists a path in $ET_U(M)$ from 0 to m. By assumption, there are elements $u_1, u_2, \ldots, u_n \in U$ such that $rm = u_1 + u_2 + \cdots + u_n$. Set $y_0 = 0$ and $y_k = (-1)^{n+k}(u_1 + u_2 + \cdots + u_k)$ for each integer k with $1 \leq k \leq n$. Then $y_k + y_{k+1} = (-1)^{n+k+1}u_{k+1} \in U$ for each integer $1 \leq k \leq n-1$. Also, $y_{n-1} + rm = y_{n-1} + y_n = u_n \in U$. Thus $0-y_1-y_2-\cdots-y_{n-1}-m$ is a path from 0 to m in $ET_U(M)$. Now, let $0 \neq x, y \in M$. Then by the preceding argument, there are paths from x to 0 and 0 to y in $ET_U(M)$. Hence there is a path from x to y in $ET_U(M)$. So $ET_U(M)$ is connected.

Theorem 3.4. Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M and for every $m \in M$ there exists $r \in R \setminus (U : M)$ such that $rm \in \langle U \rangle$. Let $n \ge 2$ be the least integer such that $\langle U \rangle = \langle m_1, m_2, ..., m_n \rangle$ for some $m_1, m_2, ..., m_n \in U$. Then $diam(ET_U(M)) \le n$.

Proof. Let m and m' be distinct elements in M. We show that there exists a path from m to m' in $ET_U(M)$ with length at most n. By hypothesis, $rm, r'm' \in \langle U \rangle$ for some $r, r' \in R \setminus (U:M)$; so we can write $rm = \sum_{i=1}^{n} r_i m_i$ and $r'm' = \sum_{i=1}^{n} s_i m_i$ for some $r_i, s_i \in R$. Define $x_0 = m$ and $x_k = (-1)^k (\sum_{i=k+1}^{n} r_i m_i + \sum_{i=1}^{k} s_i m_i)$; so $x_k + x_{k+1} = (-1)^k (r_{k+1} - s_{k+1}) m_{k+1} \in U$ for each integer k with $1 \leq k \leq n-1$. On the other hand, $rm + x_1 = (r_1 - s_1)m_1 \in U$ and $r'm' + (-1)^n x_{n-1} = (s_n - r_n)m_n \in U$. So $m - x_1 - x_2 - \cdots - x_{n-1} - m'$ is a path from m to m' in $ET_U(M)$ with length at most n since $1, (-1)^n \notin (U:M)$.

We end the paper with the following theorem.

Theorem 3.5. Let M be a module over a commutative ring R such that U is a multiplicative-prime subset of M that is not a submodule of M. Then the following hold:

- (1) Either $gr(ET_U(U)) = 3$ or $gr(ET_U(U)) = \infty$.
- (2) If $gr(ET_U(M)) = 4$, then $gr(ET_U(U)) = \infty$.

Proof. (1) If $rm + sm' \in U$ for some distinct $m, m' \in U \setminus \{0\}$ and $r, s \in R \setminus (U : M)$, then 0 - m - m' - 0 is a cycle of length 3 in $ET_U(U)$; so $gr(ET_U(U)) = 3$. Otherwise, $rm + sm' \in M \setminus U$ for all distinct $m, m' \in U \setminus \{0\}$ and all elements $r, s \in R \setminus (U : M)$. Therefore in this case, each nonzero element $m \in U$ is adjacent to 0, and no two distinct $m, m' \in U \setminus \{0\}$ are adjacent. Thus $gr(ET_U(U)) = \infty$. (2) If $gr(ET_U(M)) = 4$, then it is clear $gr(ET_U(U)) \neq 3$. So $gr(ET_U(U)) = \infty$ by

part (1) above.

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