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ON UNITARY SUBGROUPS OF GROUP ALGEBRAS

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ABSTRACT. Let FG be the group algebra of a finite *p*-group G over a finite field F of characteristic p and let * be the classical involution of FG. The *-unitary subgroup of FG, denoted by $V_*(FG)$, is defined to be the set of all normalized units u satisfying the property $u^* = u^{-1}$. In this paper we give a recursive method how to compute the order of the *-unitary subgroup for certain non-commutative group algebras. A variant of the modular isomorphism question of group algebras is also considered.

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1. Introduction and results

Let FG be the group algebra of a finite *p*-group *G* over a finite field *F* of positive characteristic *p*. Let V(FG) denotes the group of normalized units in *FG*. The description of the structure of V(FG) is a central problem in the theory of group algebras and it has been investigated by several authors. For an excellent survey on this topic we refer the reader to [8].

An element $u \in V(FG)$ is called *unitary* if $u^* = u^{-1}$, with respect to the classical *-involution of FG (the linear extension of the involution on G which sends each element of G to its inverse). The set of all unitary elements of V(FG) forms a subgroup of V(FG) which is denoted by $V_*(FG)$ and is called *-unitary subgroup. The group $V_*(FG)$ plays an important role of studying the structure of the group of units of group algebras and has been investigated in several papers (see [4], [5], [9], [10], [11], [12], [15], [16], [18] and [19]). Let L be a finite Galois extension of F with Galois group G, where F is a finite field of characteristic two. Serre [21] has showed that there is a relation between the self-dual normal basis of L over Fand the unitary subgroup of FG. This application gives a hard reason to continue studying of the unitary subgroups.

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The order of *-unitary subgroup when G is a p-group and p is an odd prime is given in [13] and [14]. To compute the order of $V_*(FG)$ when G is a 2-group and p = 2 is an open and is a particularly challenging problem. It is to be expected that the order is divisible by $|F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$, where $G\{2\}$ is the set of elements of order two in G with the unit element. In [14] this conjecture was confirmed when G is an abelian 2-group and F is a finite field of characteristic 2. For dihedral and generalized quaternion 2-groups G, where F is a finite field of characteristic 2 it was confirmed in [13]. In our paper we present a recursive method how to compute the order of $V_*(FG)$ and confirm the conjecture above.

The modular isomorphism problem is an old and unanswered problem in the theory of group representation. A stronger variant of the problem is said to be the *isomorphism problem of normalized units* (UIP) is due to Berman [7]. Let F be a finite field of characteristic p, let G and H be finite p-groups such that V(FG) and V(FH) are isomorphic. One may ask whether G and H are isomorphic groups? The studies in [1] and [2] resulted in proving the conjecture for some group classes. The *-unitary group of a group algebra is a small subgroup in V(FG) so it is interesting to ask whether this smaller subgroup determines the basic group G or not. This problem is called the *-unitary isomorphism problem (*-UIP). Recently, the *-unitary isomorphism problem was solved for some classes of non-abelian groups in [3].

Define the following 2-groups: the dihedral $D_{2^{n+1}}$, the generalized quaternion $Q_{2^{n+1}}$, the semidihedral group $D_{2^{n+1}}^-$ the modular group $M_{2^{n+1}}$ and H_{2^n} , respectively:

$D_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1, b^2 = 1, (a, b) = a^{-2} \rangle;$	$(n \ge 2)$	
$Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1, b^2 = a^{2^{n-1}}, (a, b) = a^{-2} \rangle;$	$(n \ge 2)$	
$D^{2^{n+1}} = \langle \ a,b \ \ a^{2^n} = 1, b^2 = 1, (a,b) = a^{-2+2^{n-1}} \ \rangle;$	$(n\geq 3)$	(1)
$M_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1, b^2 = 1, (a, b) = a^{2+2^{n-1}} \rangle;$	$(n\geq 3)$	(1)
$H_{2^n} = \langle a, b, c \mid a^{2^{n-2}} = b^2 = c^2 = 1, (a, b) = c,$		
$(a,c) = (b,c) = 1 \rangle,$	$(n \ge 4)$	

where $(a, b) := a^{-1}b^{-1}ab$.

Let G be a finite 2-group. We denote by $G[2^i]$ the subgroup of G generated by elements of order 2^i . We use the notation G^{2^i} for the subgroup $\langle g^{2^i} | g \in G \rangle$. Set $\Omega\{G\} = \{g^2 | g \in G\}$. Let $\zeta(G)$ and G' be the center and the commutator subgroup of G, respectively.

Let Θ denote the class of all groups with the property that $g^h := h^{-1}gh = g^{\pm 1}$ for all $g \in G \setminus G\{2\}$ and $h \in G$, which does not commute with g. It is clear that each abelian group, the dihedral $D_{2^{n+1}}$ and generalized quaternion $Q_{2^{n+1}}$ groups belong to the class Θ .

Theorem 1.1. Let F be a field with $|F| = 2^m \ge 2$ and let G be a finite 2-group. If $C = \langle c | c \in \zeta(G)[2] \setminus \Omega\{G\} \rangle$ such that $G/C \in \Theta$, then

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G| + |G\{2\}|)} \cdot |V_*(F[G/C])|$$

Corollary 1.2. Let F be a field with $|F| = 2^m \ge 2$ and let $G = H \times E$ be a finite 2-group, in which E is a finite elementary abelian 2-group and $H \in \Theta$. If

$$|V_*(FH)| = n \cdot |F|^{\frac{1}{2}(|H| + |H\{2\}|) - 1}$$

Corollary 1.3. Let $G = H_{2^n}$ with $n \ge 4$. If $|F| = 2^m \ge 2$, then

$$|V_*(FG)| = 2 \cdot |F|^{\frac{1}{2}(|G| + |G\{2\}|) - 1}.$$

Let us denote by $D_8 Y C_4$ the central product of the dihedral group D_8 and the cyclic group C_4 .

Theorem 1.4. Let G be a finite non-abelian 2-group of order $|G| = 2^4$. If F is a field with $|F| = 2^m \ge 2$, then $|V_*(FG)| = n \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$, where

- (i) n = 1 if $G \in \{D_8 \lor C_4, D_{16}, D_8 \times C_2\};$
- (ii) n = 2 if $G \in \{M_{16}, D_{16}^-, H_{16}\};$
- (iii) n = 4 if $G \in \{Q_{16}, C_4 \ltimes C_4, Q_8 \times C_2\}.$

Theorem 1.5. Let G and H be non-abelian 2-groups of order at most 2^4 . If F is a finite field of characteristic two, then the isomorphism $V_*(FG) \cong V_*(FH)$ implies the following isomorphism $G \cong H$.

2. Notations and preliminaries

Let G be a finite p-group. If char(F) = p, then (see [8, Chapters 2-3, p. 194-196])

$$V(FG) = \Big\{ x = \sum_{g \in G} \alpha_g g \in FG \mid \chi(x) = \sum_{g \in G} \alpha_g = 1 \Big\},\$$

where $\chi(x)$ is the augmentation of the element $x \in FG$. Let $\operatorname{supp}(x)$ denote the support of $x \in FG$ and $x^g = g^{-1}xg$, where $g \in G$. We define $\widehat{C} := \sum_{g \in C} g$, where

C is a subset of G. Throughout this paper |S| denotes the cardinality of a finite set S and |g| the order of $g \in G$.

The following two lemmas will be useful.

Lemma 2.1. ([17, Theorem 2]) Let $|F| = 2^m \ge 2$. If G is a finite abelian 2-group, then

$$|V_*(FG)| = |G^2[2]| \cdot |F|^{\frac{1}{2}(|G| + |G[2]|) - 1}.$$

Lemma 2.2. ([13, Corollary 2]) If $|F| = 2^m \ge 2$, then

- (i) $|V_*(FG)| = |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$ if G is a dihedral 2-group;
- (ii) $|V_*(FG)| = 4 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$ if G is a generalized quaternion 2-group.

Let H be a normal subgroup of G and $I(H) := \langle 1+h | h \in H \rangle_{FG}$ be the ideal of FG generated by the set $\{1+h | h \in H\}$. Moreover, for the natural homomorphism $\Psi : FG \to F[G/H]$ we have that $FG/I(H) \cong F[G/H]$ and $\ker(\Psi) = I(H)$. Let us denote by $V_*(F\overline{G})$ the unitary subgroup of the factor algebra FG/I(H), where $\overline{G} = G/H$. It is easy to check that the set

$$N_{\Psi}^* = \{ x \in V(FG) \mid \Psi(x) \in V_*(F\overline{G}) \}$$

forms a subgroup in V(FG). Furthermore, the set $I(H)^+ = \{1 + x \mid x \in I(H)\}$ forms a normal subgroup in V(FG). It is obvious that $S_H := \langle xx^* \mid x \in N_{\Psi}^* \rangle$ is a subgroup of $I(H)^+$, because $xx^* \in 1 + \ker(\Psi) = I(H)^+$ for $x \in N_{\Psi}^*$.

Lemma 2.3. Let H be a normal subgroup of order two in a finite 2-group G and let $|F| = 2^m \ge 2$. If S_H is central in N_{Ψ}^* , then

$$|V_*(FG)| = |F|^{\frac{1}{2}|G|} \cdot \frac{|V_*(F\overline{G})|}{|S_H|}.$$
(2)

Proof. Let $\Phi(x) = xx^*$ for each $x \in V(FG)$. The map $\Phi: V(FG) \to V(FG)$ is not necessary a group homomorphism on V(FG). However, S_H being central in N_{Ψ}^* implies that the restriction $\Phi|_{N_{\Psi}^*}$ is a homomorphism. Since $N_{\Psi}^*/\ker(\Phi|_{N_{\Psi}^*}) \cong S_H$,

$$|\ker(\Phi|_{N_{\Psi}^*})| = |V_*(FG)| = \frac{|N_{\Psi}^*|}{|S_H|} = \frac{|I(H)^+|\cdot|V_*(F\overline{G})|}{|S_H|}.$$

Evidently, I(H) can be considered as a vector space over F with the following basis $\{u(1+h) \mid u \in T(G/H), h \in H\}$, where T(G/H) is a complete set of left coset representatives of H in G. Thus we have that $|I(H)^+| = |F|^{\frac{1}{2}|G|}$ and (2) holds. \Box

Let $|F| = 2^m \ge 2$. Let *C* be a central subgroup of a 2-group *G*. We denote by V_{g_1,\ldots,g_n} the vector space in *FG* over *F* spanned by elements $\alpha_1 g_1 \widehat{C}, \ldots, \alpha_n g_n \widehat{C}$, where $g_1, \ldots, g_n \in G$ and $\alpha_1, \ldots, \alpha_n \in F$. Set the following subgroup of *G*:

$$G_{g_1,\ldots,g_n} := \langle 1 + \alpha_1 g_1 \widehat{C}, \ldots, 1 + \alpha_n g_n \widehat{C} \mid \alpha_i \in F \rangle.$$

Lemma 2.4. The set $1 + V_{g_1,\ldots,g_n}$ coincides with G_{g_1,\ldots,g_n} .

Proof. If $x_1, x_2 \in FG$, then the proof follows from the fact that

$$1 + (x_1 + x_2)\widehat{C} = 1 + x_1\widehat{C} + x_2\widehat{C} = (1 + x_1\widehat{C})(1 + x_2\widehat{C}).$$

Lemma 2.5. Let $|F| = 2^m \ge 2$. If G is a finite 2-group, then

$$supp(xx^*) \cap G\{2\} = \{1\}$$
 $(x \in V(FG)).$

Proof. If $x = \sum_{i=1}^{|G|} \alpha_i g_i \in V(FG)$, then

$$xx^* = 1 + \sum_{1 \le i < j \le |G|} \alpha_i \alpha_j \left(g_i g_j^{-1} + (g_i g_j^{-1})^{-1} \right).$$

Obviously, $g_i g_j^{-1} + (g_i g_j^{-1})^{-1} = 0$ for $g_i g_j^{-1} \in G\{2\}.$

Lemma 2.6. Let $|F| = 2^m \ge 2$. Let $H = \langle c | c^2 = 1 \rangle$ be a central subgroup of a finite 2-group G. If $1 + g\hat{H} \in S_H$ for some $g \in G$, then $g^2 = c$.

Proof. Assume that $1 + g\hat{H} \in S_H$ for some $g \in G$. Since S_H contains only *-symmetric elements, $1 + g\hat{H} = 1 + g^{-1}\hat{H}$, so

$$(g+g^{-1})\widehat{H} = g+gc+g^{-1}+g^{-1}c = 0.$$

The case |g| = 2 is impossible by Lemma 2.5. Thus, $g = g^{-1}c$ and $g^2 = c$.

Lemma 2.7. Let $|F| = 2^m \ge 2$ and let G be a finite 2-group. For each subgroup $H = \langle c \in \zeta(G) | c^2 = 1 \rangle$ of G we define the set $H_c := \{g \in G | g^2 = c\}$. Then

$$S_H = \langle 1 + \alpha_g g \widehat{H}, \quad 1 + \beta_h (h + h^{-1}) \widehat{H} \quad | \quad g \in H_c, h \notin H_c, \quad \alpha_g, \beta_h \in F \rangle.$$

Proof. Obviously, $gh + (gh)^{-1} = gh(1 + (gh)^{-2})$, so $S_H \subseteq I(H)^+$ and S_H contains only *-symmetric elements. Thus each $x \in S_H$ can be expressed (see Lemma 2.5 and Lemma 2.6) in the following form

$$x = 1 + \sum_{g \in H_c} \alpha_g g \widehat{H} + \sum_{h \notin H_c} \beta_h (h + h^{-1}) \widehat{H}.$$

Since $(1 + x_1\hat{H})(1 + x_2\hat{H}) = 1 + (x_1 + x_2)\hat{H}$ for each $x_1, x_2 \in FG$, the proof is done.

Lemma 2.8. Let $|F| = 2^m \ge 2$ and let G be a finite 2-group. If $H \le \zeta(G)$ has order 2, then $1 + \alpha(g + g^{-1})\widehat{H} \in S_H$ for all $g \in G$ such that $g^2 \notin H$.

Proof. Let $g \in G$ such that $g^2 \notin H$. Since $g \neq g^{-1}$ and $1 + \alpha g \widehat{H} \in \ker(\Psi)$,

$$(1 + \alpha g \widehat{H})(1 + \alpha g \widehat{H})^* = 1 + \alpha (g + g^{-1})\widehat{H}, \qquad (\alpha \in F)$$

which proves the lemma.

3. Proofs of Theorems

Proof of Theorem 1.1. If $c \in \zeta(G)[2] \setminus \Omega\{G\}$, then (see Lemmas 2.4, 2.5 and 2.8)

$$S_C = \langle 1 + \alpha (g + g^{-1})\widehat{C} \mid \alpha \in F, \ g \in G \setminus G\{2\} \rangle.$$

Furthermore, $h^{-1}(g+g^{-1})\widehat{C}h = (g+g^{-1})\widehat{C}$ for all $h \in G$, because $\overline{G} \in \Theta$. Now using Lemma 2.3 we have that

$$|V_*(FG)| = |F|^{\frac{1}{2}|G|} \cdot \frac{|V_*(F\overline{G})|}{|F|^{\frac{1}{4}(|G| - |G\{2\}|)}} = |F|^{\frac{1}{4}(|G| + |G\{2\}|)} \cdot |V_*(F\overline{G})|.$$

Proof of Corollary 1.2. Let $G = H \times E$, where $H \in \Theta$, E is an elementary abelian 2-group and $|E| = 2^m \ge 1$. Now, we proceed by induction on the order of E.

For the base case (m = 0) we have that $|V_*(FH)| = n \cdot |F|^{\frac{1}{2}(|H|+|H\{2\}|)-1}$ for some n.

Let $m \ge 1$. If $C = \langle c \mid 1 \neq c \in E \rangle$, then $N := G/C \cong H \times E_1$ in which $|E_1| = 2^{m-1}$. Obviously, $N \in \Theta$ and using Theorem 1.1 we conclude that

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G| + |G\{2\}|)} |V_*(FN)|$$

where $|V_*(FN)| = n \cdot |F|^{\frac{1}{2}(|N|+|N\{2\}|)-1}$. Since $|G| + |G\{2\}| = 2(|N| + |N\{2\}|)$,

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot n \cdot |F|^{\frac{1}{2}(|N|+|N[2]|)-1} = n \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$$

The second sentence follows immediately from the fact that D_{2^s} and Q_{2^s} belong to Θ for every s > 2.

Proof of Corollary 1.3. Let $G' = \langle c \rangle$ (see (1)). Clearly,

$$c \in \zeta(G)[2] \setminus \Omega(G)$$
 and $\overline{G} = G/G' \cong C_{2^{n-2}} \times C_2 \in \Theta$.

Therefore $|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} |V_*(F\overline{G})|$ by Theorem 1.1 and

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot 2 \cdot |F|^{\frac{1}{4}(|G|+|G\{2\}|)} = 2 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$$

by Lemma 2.1 and the fact that $|G\{2\}| = 2|\overline{G}[2]|$.

Lemma 3.1. Let $G = (C_4 \rtimes C_4) \times E$, where E is a finite elementary abelian 2-group. If $|F| = 2^m \ge 2$, then $|V_*(FG)| = 4 \cdot |F|^{\frac{1}{2}(|G| + |G\{2\}|) - 1}$.

Proof. Let $G = \langle a, b \rangle \cong C_4 \rtimes C_4$. If $C = \langle a^2 b^2 \rangle$, then $a^2 b^2 \in \zeta(G)[2] \setminus \Omega\{G\}$ and $\overline{G} = G/C \cong Q_8 \in \Theta$. Now using Theorem 1.1, we obtain that

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} |V_*(FQ_8)|.$$

According to Lemma 2.2 (ii) and the fact that $|G\{2\}| = 4$, we have that

$$|V_*(FG)| = |F|^{\frac{1}{4}(|G|+|G\{2\}|)} \cdot 4 \cdot |F|^{\frac{1}{4}(|G|+|G\{2\}|)-1} = 4 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

Since $(C_4 \rtimes C_4) \in \Theta$, the proof follows from Corollary 1.2.

Lemma 3.2. If $|F| = 2^m \ge 2$, then the map $\tau : F \to F$, such that $\tau(x) = x^2 + x$ is a homomorphism on the additive group of F, in which ker $(\tau) = \{0, 1\}$.

Proof. Obviously,

$$\tau(x+y) = (x+y)^2 + (x+y) = x^2 + y^2 + x + y = \tau(x) + \tau(y) \qquad (x, y \in F)$$

and $x^2 + x = x(x+1) = 0$ if and only if $x \in \{0, 1\}$.

Consider the following system of equations over F with variables w_1, w_2, w_3, w_4 :

$$w_{1} + w_{2} + w_{3} + w_{4} = 1;$$

$$w_{1}w_{4} + w_{2}w_{3} = A;$$

$$(3)$$

$$w_{1}w_{2} + w_{3}w_{4} = 0,$$

$$(A \in F).$$

Lemma 3.3. Let $|F| = 2^m \ge 2$. If \mathbb{S} is a subset of F consisting of all such $A \in F$ for which (3) has a solution in F, then $|\mathbb{S}| = \frac{1}{2}|F|$.

Proof. First, we prove that $\mathbb{S} \subseteq \operatorname{im}(\tau)$ (see Lemma 3.2). Suppose that $A \in \mathbb{S}$ and $w_1, w_2, w_3, w_4 \in F$ satisfy the system (3). Then

$$\tau(w_1 + w_3) = (w_1 + w_3)^2 + (w_1 + w_3)$$
$$= (w_1 + w_3)(1 + w_1 + w_3) = (w_1 + w_3)(w_2 + w_4) = A.$$

Thus for $w = w_1 + w_3$ we have $\tau(w) = A$ so $\mathbb{S} \subseteq \operatorname{im}(\tau)$.

Assume that $\tau(w) = A$ for some $w \in F$. If w = 0, then $\tau(w) = A = 0$ and $w_1 = 0, w_2 = 1, w_3 = 0, w_4 = 0$ is a solution of the equation system 3. Let $w_1 + w_3 = w \neq 0$ for some $w_1, w_3 \in F$. Set $w_2 = (A + w_1 + ww_1)w^{-1}$ and $w_4 = w_2 + w + 1$. It is clear that $w_1 + w_3 + w_2 + w_4 = w + w + 1 = 1$. Furthermore,

 $w_1w_2 + w_3w_4 = w_1w_2 + (w_1 + w)(w_2 + w + 1) = w_1(1 + w) + A + ww_2,$

because $\tau(w) = w^2 + w = A$. Since $w_2 = (A + w_1 + ww_1)w^{-1}$ we can compute that $w_1(w+1) + ww_2 + A = w_1(1+w) + (A + w_1 + ww_1) + A = 0$. Thus we have proved that $w_1w_2 + w_3w_4 = 0$. Finally,

 $A = w(w+1) = (w_1 + w_3)(w_2 + w_4)$

 $= w_1w_2 + w_1w_4 + w_2w_3 + w_3w_4 = w_1w_4 + w_2w_3,$

which shows that $im(\tau) = S$. The proof is complete.

Lemma 3.4. Let $G = \langle a, b \rangle \cong D_{16}^-$ (see (1)). If $|F| = 2^m \ge 2$, then

$$|V_*(FD_{16}^-)| = 2 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

Proof. Clearly, $\zeta(G) = \langle a^4 \rangle$ and $N = \langle a^2 \rangle$ is normal in G. Furthermore, each $x \in FG$ can be written as $x = x_1 + x_2a + x_3b + x_4ab$ with $x_i \in FN$ and

$$xx^* = (x_1x_1^* + x_2x_2^* + x_3x_3^* + x_4x_4^*) + (x_2x_1^* + x_4x_3^*)a + (x_1x_2^* + x_3x_4^*)a^7 + (x_1x_4^* + x_2x_3^*)(a + a^5)b.$$

Set $w_i = \chi(x_i)$. If $xx^* \in S_{\zeta(G)}$, then $w_1 + w_2 + w_3 + w_4 = 1$ and $w_1w_2 + w_3w_4 = 0$ by the previous formula. Therefore if $xx^* \in S_{\zeta(G)}$, then there exist $w_1, w_2, w_3, w_4 \in F$ satisfying the system (3), for some $A \in \mathbb{S}$.

Let $C := \zeta(G)$ and $M = \{g \in G \mid g^2 = a^4\} = \{a^2, a^6, ab, a^3b, a^5b, a^7b\}$. Each *-symmetric element of $I(C)^+$ (see Lemma 2.7) can be written as

$$\begin{split} 1+\alpha_1(a+a^{-1})\widehat{C}+\alpha_2a^2\widehat{C}+\alpha_3ab\widehat{C}\\ &+\alpha_4a^3b\widehat{C}+\alpha_5b+\alpha_6a^2b+\alpha_7a^4b+\alpha_8a^6b, \end{split} \qquad (\alpha_i\in F). \end{split}$$

According to Lemma 2.8, $1 + \alpha(a + a^{-1})\widehat{C} \in S_C$ for any $\alpha \in F$. It follows that $1 + \alpha g \notin S_C$ if $g \in G\{2\}$ by Lemma 2.5.

Since $\delta + \delta a^2 + a \in V(FG)$ for every $\delta \in F$, an easy computation shows that

$$(\delta + \delta a^2 + a) (\delta + \delta a^2 + a)^* = 1 + \delta^2 (a^2 + a^{-2}) = 1 + \delta^2 a^2 \widehat{C},$$

which confirm that $\delta + \delta a^2 + a \in N_{\Psi}^*$. Obviously, $\eta(\alpha) = \alpha^2$ is an automorphism of U(F), so we can pick δ such that $\alpha_2 = \delta^2$. Therefore $1 + \alpha_2 a^2 \widehat{C} \in S_C$ for every $\alpha_2 \in F$.

A straightforward computation shows that

$$\left(\alpha(a+a^7)+b\right)\left(\alpha(a+a^7)+b\right)^* = 1 + \alpha^2 a^2 \widehat{C} + \alpha(ab+a^3b)\widehat{C}$$

for every $\alpha \in F$ so $\alpha(a + a^7) + b \in N_{\Psi}^*$. Using Lemma 2.4 and the fact that $1 + \alpha_2 a^2 \widehat{C} \in S_C$, we have that $1 + \alpha(ab + a^3b)\widehat{C} \in S_C$ for every $\alpha \in F$.

We have proved that the group N_1 generated by the set

$$\{1 + \alpha_1(a + a^{-1})\widehat{C}\} \cup \{1 + \alpha_2 a^2 \widehat{C}\} \cup \{1 + \alpha_3(ab + a^3b)\widehat{C}\}, \qquad (\alpha_i \in F)$$

is a subgroup of S_C by Lemma 2.4 and $|N_1| = |F|^3$.

Let $u = w_1 + w_2a + w_3b + w_4ab \in FD_{16}^-$, such that $w_1, w_2, w_3, w_4 \in F$ satisfy the system (3). It is easy to check that

$$uu^* = 1 + (w_1w_4 + w_2w_3)ab\hat{C}$$

and $N_2 = \langle 1 + \alpha ab\widehat{C} \mid \alpha \in \mathbb{S} \rangle$ is a subgroup of S_C with order $|N_2| = \frac{1}{2}|F|$ by Lemma 3.3. Using a similar argument, $1 + \alpha a^3 b\widehat{C} \in S_C$ if $\alpha \in \mathbb{S}$. It follows that $S_C = N_1 \times N_2$ and $|S_C| = \frac{1}{2}|F|^4$.

Since $\overline{G} = G/\zeta(G) \cong D_8$, the order $|V_*(F\overline{G})| = |F|^{\frac{3}{8}|G|}$ by Lemma 2.2 (i). It is clear that $\frac{3}{8}|G| - 3 = \frac{1}{2}|G\{2\}|$. According to Lemma 2.3

$$|V_*(FG)| = 2 \cdot |F|^{\frac{1}{2}|G|} |F|^{(\frac{3}{8}|G|-3)-1} = 2 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

Lemma 3.5. Let $G = \langle a, b \rangle \cong M_{16}$ (see (1)). If $|F| = 2^m \ge 2$, then

$$|V_*(FG)| = 2 \cdot |F|^{\frac{1}{2}(|G| + |G\{2\}|) - 1}$$

Proof. If $y \in S_{G'}$, then

$$y = 1 + \beta_1(a+a^3)\widehat{G'} + \beta_2a^2\widehat{G'} + \beta_3\widehat{G'} + \beta_4b\widehat{G'} + \beta_5(a+a^3)b\widehat{G'} + \beta_6a^2b\widehat{G'},$$

in which $\beta_1, \ldots, \beta_6 \in F$. Moreover,

$$1 + \beta_3 \widehat{G'}, \quad 1 + \beta_4 b \widehat{G'} \notin S_{G'} \quad \text{and} \quad \beta_1 (a + a^3) \widehat{G'}, \beta_5 (a + a^3) b \widehat{G'} \in S_{G'}$$

by Lemma 2.5 and Lemma 2.8, respectively. Since $\eta(\alpha) = \alpha^2$ is an automorphism of U(F) we can pick α such that $\beta_2 = \alpha^2$. Then $u = \alpha^2 + a + \alpha^2 a^2 \in V(FG)$ and

$$uu^* = 1 + \beta_2 a^2 \widehat{G'}.$$

which proves that $u \in N_{\Psi}^*$ and $1 + \beta_2 a^2 \widehat{G'} \in S_{G'}$ for every $\beta_2 \in F$. The identity

$$\left(\alpha a^2 + (1 + \alpha a^2)b\right)\left(\alpha a^2 + (1 + \alpha a^2)b\right)^* = 1 + \alpha a^2 \widehat{G'} + \alpha a^2 \widehat{G'} b^2$$

shows that $\alpha a^2 + (1 + \alpha a^2)b \in N_{\Psi}^*$. Therefore $1 + \alpha a^2 \widehat{G'} + \alpha a^2 b \widehat{G'} \in S_{G'}$ for every $\alpha \in F$. From $1 + \alpha a^2 \widehat{G'} \in S_{G'}$ we get $1 + \alpha a^2 b \widehat{G'} \in S_{G'}$ by Lemma 2.4.

We have proved that

$$\begin{split} S_{G'} &= \langle 1 + \alpha_1 (a + a^3) \widehat{G'}, \quad 1 + \alpha_2 a^2 \widehat{G'}, \quad 1 + \alpha_3 (a + a^3) b \widehat{G'}, \\ &\qquad 1 + \alpha_4 a^2 b \widehat{G'} \mid \alpha_i \in F \rangle \subseteq \zeta(V(FG)). \end{split}$$

Consequently, $|S_{G'}| = |F|^4$ and $|V_*(F\overline{G})| = 2 \cdot |F|^5$, by Lemma 2.1 and the fact that $\overline{G} = G/G' \cong C_4 \times C_2$.

Finally, using that $|G\{2\}| = 4$ Lemma 2.3 shows that

$$|V_*(FG)| = 2 \cdot |F|^{\frac{1}{2}|G|+1} = 2 \cdot |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

Lemma 3.6. Let $G = \langle a, b \rangle \mathsf{Y} \langle c \rangle \cong D_8 \mathsf{Y} C_4$ (see (1)). If $|F| = 2^m \ge 2$, then $|V_*(FG)| = |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}$. **Proof.** Clearly, $G' = \langle a^2 \rangle$ and $\{g \in G \mid g^2 = a^2\} = \{a, a^3, c, a^2c, bc, abc, a^2bc, a^3bc\}$. Let us prove that $S_{G'} = \langle 1 + \alpha g \widehat{G'} \mid g \in G \setminus G\{2\}, \alpha \in F \rangle$. Indeed, each $x \in S_{G'}$ can be written as $x = 1 + \alpha_1 a \widehat{G'} + \alpha_2 bc \widehat{G'} + \alpha_3 c \widehat{G'} + \alpha_4 abc \widehat{G'}$ by Lemma 2.7 and 2.8. Using the following computation

$$(1 + \alpha b + \alpha a)(1 + \alpha b + \alpha a)^* = 1 + \alpha a \widehat{G'},$$

$$(1 + \alpha c + \alpha a)(1 + \alpha c + \alpha a)^* = 1 + \alpha c \widehat{G'} + \alpha a \widehat{G'},$$

$$(1 + \alpha c + \alpha b)(1 + \alpha a^2 c + \alpha b)^* = 1 + \alpha c \widehat{G'} + \alpha^2 b c \widehat{G'},$$

$$(a^2 c + \alpha a b + \alpha a c)(a^2 c + \alpha a b + \alpha a c)^* = 1 + (\alpha a b c + \alpha a + \alpha^2 b c) \widehat{G'},$$

it is easy to check that $S_{G'} = \langle 1 + \alpha_1 a \widehat{G'}, 1 + \alpha_2 c \widehat{G'}, 1 + \alpha_3 b c \widehat{G'}, 1 + \alpha_4 a b c \widehat{G'} \mid \alpha_i \in F \rangle$ by Lemma 2.4 so $|S_{G'}| = |F|^4$.

Since $\overline{G} = G/G' \cong C_2 \times C_2 \times C_2$, Lemma 2.1 shows that $|V_*(F\overline{G})| = |F|^7$. It is obvious that $|G\{2\}| = 4$, so $\frac{|V_*(F\overline{G})|}{|S_{G'}|} = |F|^{\frac{1}{2}|G\{2\}|-1}$. According to Lemma 2.3 we get

$$|V_*(FG)| = |F|^{\frac{1}{2}|G|} |F|^{\frac{1}{2}|G\{2\}|-1} = |F|^{\frac{1}{2}(|G|+|G\{2\}|)-1}.$$

Proof of Theorem 1.4. It follows immediately from Corollaries 1.2, 1.3 and Lemmas 3.1, 3.4 – 3.6.

Proof of Theorem 1.5. Our statement holds if G is a non-abelian group of order $|G| = 2^3$ by [18] and [19]. Moreover, it is also true if $|G| = 2^4$ and |F| = 2 by [3] and [9].

Let |F| > 2 and $|G| = 2^4$. Theorem 1.4 yields that $|V_*(FG)| = |V_*(FH)|$ if and only if $G \in \{C_4 \ltimes C_4, Q_8 \times C_2\}$. Without loss of generality we can assume that $G \cong Q_8 \times C_2 = \langle a, b \rangle \times \langle c \rangle$ and $H \cong C_4 \ltimes C_4$. If $M = \langle a, c \rangle < G$, then each $x \in V(FG)$ can be written as $x = x_1 + x_2 b$, where $x_1, x_2 \in FM$. Obviously, $xx^* = x_1x_1^* + x_2x_2^* + (x_1x_2 + x_1x_2a^2)b$ and

$$x^{2} = x_{1}^{2} + x_{2}x_{2}^{*}a^{2} + (x_{1}x_{2} + x_{1}^{*}x_{2})b.$$
(4)

Furthermore, $x \in V_*(FG)$ if and only if $x_1x_1^* = x_2x_2^* + 1$ and $x_1x_2 = x_1x_2a^2$. Since x is a unit, $\chi(x_1) + \chi(x_2) = 1$, so consider the following cases.

Case 1. Let $\chi(x_1) = 1$ and $\chi(x_2) = 0$. From the equality $x_1x_2 = x_1x_2a^2$ we conclude that $x_2(1+a^2) = 0$ and (see [20, Theorem 11]) we can write

 $x_2 = \alpha_0(1+a^2) + \alpha_1(1+a^2)a + \alpha_2(1+a^2)c + \alpha_3(1+a^2)ac, \qquad (\alpha_i \in F).$

By (4) and the fact that

$$x_1 + x_1^* = \beta_0(1+a^2) + \beta_1(1+a^2)a + \beta_2(1+a^2)c + \beta_3(1+a^2)ac, \qquad (\beta_i \in F)$$

we conclude that $x^2 = x_1^2$ and $x_1^{-1} = x_1^*$. According to [14, Theorem 2(ii)] $V_*(FM) \cong M \times N$ in which N is an elementary abelian group. Consequently, $x^2 \in \{1, a^2\}$.

Case 2. Let $\chi(x_1) = 0$ and $\chi(x_2) = 1$. From the equation $x_1x_2 = x_1x_2a^2$ we conclude that $x_1(1+a^2) = 0$, so (see [20, Theorem 11])

$$x_1 = \alpha_0(1+a^2) + \alpha_1(1+a^2)a + \alpha_2(1+a^2)c + \alpha_3(1+a^2)ac, \qquad (\alpha_i \in F).$$

Equations (4), $x_2x_2^* = x_1x_1^* + 1 = 1$ and $x_1 + x_1^* = 0$ imply that $x^2 = x_1^2 + 1 = 1$. Consequently, if $x \in V_*(FG)$, then $x^2 \in \{1, a^2\}$, so $|V_*^2(FG)| = |\langle a^2 \rangle| = 2$.

Let $H \cong C_4 \ltimes C_4$. Clearly, $|V_*(FH)| > 2$ because $H^2 \subseteq V^2_*(FH)$. This proofs that $V_*(FG)$ and $V_*(FH)$ are not isomorphic groups.

Note that Theorem 1.4 was verified by GAP package RAMEGA [6].

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