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Approximate Solutions of the Fourth-Order Eigenvalue Problem

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Abstract − In this paper, the differential transformation (DTM) and the Adomian decomposition (ADM) methods are proposed for solving fourth order eigenvalue problem. This fourth order eigenvalue problem has nonstrongly regular boundary conditions. This the fourth order problem has been examined for $p(t) = t$, $B = 0$, $a = 0,01$ where $p(t) \neq 0$ is a complex valued and $a \neq 0$. The differential transformation and the Adomian decomposition methods are briefly described. An approximate solution is obtained by performing seven iterations with the Adomian decomposition method. The same number of iterations have been made in the differential transformation method. The approximation results obtained by both methods have been compared with each other. These data have been presented in table. The ADM and the DTM approximation solutions have been shown by plotting in Figure 1. Here, the approaches obtained by using the two methods are found to be in high agreement. Consequently, highly accurate approximate solutions of fourth order eigenvalue problem are obtained. Such good results also revealed that the Adomian decomposition and the differential transformation methods are fast, economical and motivating. The exact solution of the fourth order eigenvalue problem for nonstrongly regular can not be found in the literature. Therefore, this study will give an important idea to determine approximate solution behavior of this fourth order problem.

1. Introduction

We examine the problem with nonstrongly regular boundary conditions [1] as

$$u^{(4)} + p(t)u = \lambda u, \ 0<t<1, \tag{1}$$

$$u(1)-(-1)^{\beta} u(0) = 0, \ \ u(1)-(-1)^{\beta} u'(0) = 0, \tag{2}$$

$$u''(1)-(-1)^{\beta} u''(0) = 0, \ \ u'''(1)-(-1)^{\beta} u'''(0)+\alpha u(0) = 0, \tag{3}$$

where λ is spectral parameter; $p(t) \neq 0$ is a complex valued function; $a \neq 0$ and $b = 0,1$. We deal with DTM and ADM to solve the above problem at $b = 0$.

Many numerical methods, such as asymptotic formula for eigenfunctions of the considered boundary value problem have been obtained in (Kaya, 2020), the regularized sampling method (Chanane, 2010), the extended sampling method (Chanane, 2010), the α-parameterized differential transform method (Mukhtarov, Yucel, & Aydemir, 2020) variational iteration methods (Syam, & Siyyam, 2009), Sinc-Galerkin method (Alquran, Al-Khaled , 2010), fourth order sturm liouville problem via decomposition method for $p(t) \neq 0$ (Attili, & Lesnic, 2006), Magnus Method (Alalyani, 2019), differential transform method for high order

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Sturm-Liouville problems (Biazar, Dehghan, & Houlari, 2020), lie group method, FDM and the asymptotic iteration method etc. are implemented to solve this eigenvalue BVP numerically.

Some different authors have worked on the development of numerical methods for solving these differential equations (Gao, Ismael, & Husien, 2020; Baskonus, Sulaiman, & Bulut, 2018). We use ADM and DTM to compare the approximation solutions of the problem that we have suggested as a contribution to the literature.

This paper will continue as follows: In part 2 we give the basic process of the ADM and DTM. In part 3 we present the implementation of the ADM and DTM for computing and comparing the solutions of this eigenvalue problem. The figure of eigenfunctions (approximate solutions) for a found λ (eigenvalue) is plotted. Numerical results are shown in the table.

2. Basic Process of ADM and DTM

Here we will briefly introduce the ADM and DTM as follows:

In the beginning of the 1980s, ADM has been developed by Adomian (Adomian, & Rach, 1993). In these years, the Adomian decomposition method has been implemented for problems arising from physics, biology and engineering. Until now, there has been great interest in DTM and ADM applications to solve various scientific models, you can refer to the references (Adomian et al., 1993; Zhou, 1986; Ayaz, 2004; Abdel-Halim Hassan, 2002; Li et al., 2020; Chakraverty et al., 2019; Adebısı et al., 2021; *Çakır et al., 2019*; Arslan, 2019a; Arslan, 2018b; Peker et al., 2011; Gubes et al., 2015; Peker et al., 2010).

The equation (1) is rewritten as

$$Fu = g \Rightarrow Lu + Ru + Nu = g(t). \tag{4}$$

F and L are differential operator and fourth order derivative in Equation (4), respectively.

R and N are linear and nonlinear terms in Equation (4), respectively. If the integral operator is applied to each term of Equation (4), we get

$$u(t) = \left(L^{-1}R\right)u - \left(L^{-1}N\right)u + L^{-1}\left(g(t)\right), \tag{5}$$

where $L(...) = \dfrac{d^{(4)}}{dt^{(4)}}(...)$ is the differential operator and $L^{-1}(...) = \int_0^t\int_0^t\int_0^t\int_0^t(...)\,dtdtdtdt$ is integral operator

(inverse operator) of L. If we operate on both sides of Equation (5) with the inverse operator of L^{-1}, we obtain

$$u(t) = u_0(t) - \left(L^{-1}R\right)u - \left(L^{-1}N\right)u. \tag{6}$$

After some calculations, the following iteration system is written:

$$u_{k+1}(t) = u_0(t) - \left(L^{-1}R\right)u_k - \left(L^{-1}N\right)u_k, \quad k = 0,1,2, \tag{7}$$

$$u(t) = \sum_{k=0}^{\infty} u_k(t) = u_0 + L^{-1}(Ru) + L^{-1}(Nu), \tag{8}$$

where

$$Ru = \sum_{k=0}^{\infty} u_k(t), \quad Nu = \sum_{k=0}^{\infty} A_k(u_0 + u_1 + ... + u_k).$$

The first approximation $u_0(t)$ can be obtained by using boundary conditions. We have recurrence formula Equation (7) for obtaining other components $u_1(t), u_2(t)$ of the Adomian decomposition Equation (8). Finally, we have the serial solution of problem Equation (1).

$$u(t) = u_0(t) + u_1(t) + ...$$
(9)

The DTM is effective in solving most differential equations. The DTM is derived based on the Taylor expansion and was proposed by Zhou for electrical circuits (Zhou, 1986).

The differential transformation $Y(k)$ of function $u(t)$ is defined as (Ayaz, 2004),

$$Y(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0},$$
(10)

where $u(t)$ is original function and $Y(k)$ is the transformed function.

Differential inverse transform $u(t)$ of $Y(k)$ is defined as (Ayaz, 2004),

$$u(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}.$$
(11)

If the expansion Equation (11) with Equation (10) is written as follows:

$$u(t) = \sum_{k=0}^{\infty} Y(k) t^k,$$
(12)

then it is called series solution of the differential transformation method (Ayaz, 2004).

The following theorems will be used in this study, where $Y(k)$ is differential transformation of $u(t)$ (Ayaz, 2004):

Teorem 1. $u(m) = \dfrac{d^4 w(m)}{dm^4}, \quad U(h) = \dfrac{(h+4)!}{h!} W(h+4).$

Theorem 2. $u(m) = \alpha w(m), \quad U(h) = \alpha W(h),$ where α is a reel constant.

Theorem 3. $u(m) = m w(m), \quad U(h) = \sum_{s=0}^{h} \delta(h-1) W(h-s).$

3. Approximation Solutions by ADM and DTM

By applying ADM and DTM, we will find approximation solutions of eigenvalue problem in series form. The advantages and benefits of the proposed methods on an experiment will be presented.

$u^{(4)} + tu = \lambda u, \ 0<t<1,$

$u(1) - u(0) = 0, \ u'(1) - u'(0) = 0,$ \hfill (13)

$u''(1) - u''(0) = 0, \ u'''(1) - u'''(0) + 0.01u(0) = 0,$

it is taken as *b = 0, a = 0,01* in this problem.

ADM solution as follows:

$Lu^{(4)} = \lambda u - .$

According to Equation (6), $\ L^{-1}\left(u^{(4)}\right) = L^{-1}\left(\lambda u\right) - L^{-1}\left(tu\right),$

$u(t) = A + Bt + \dfrac{Ct^2}{2} + (D + \alpha A)\dfrac{t^3}{6} + L^{-1}\left(\lambda u\right) - L^{-1}\left(tu\right),$

$u_{k+1} = A + Bt + \dfrac{Ct^2}{2} + D\dfrac{t^3}{6} + \alpha A\dfrac{t^3}{6} + L^{-1}\left(\lambda u_k - tu_k\right), \ k = 0,1,2,...,$ \hfill (14)

$u_0 = A + Bt + \dfrac{Ct^2}{2} + (D + \alpha A)\dfrac{t^3}{6}, \ k = 0,$

u_0 is obtained with the aid of the following boundary conditions *A, B, C, D.*

$u(0) = A, \ u(1) = A, \ u'(0) = B, \ u'(1) = B,$

$u''(0) = C, \ u''(1) = C, \ u'''(0) = u'''(1) + 0.01u(0) = D.$

From the recursive relation Equation (14) for *k = 0, 1, 2…*, we get

$u_0 = 1.000003877 + 0.001412025487t - 0.0004168111954t^2,$

$u_1 = L^{-1}\left(\lambda u_0 - tu_0\right),$

$u_2 = L^{-1}\left(\lambda u_1 - tu_1\right),$

$u_3 = L^{-1}\left(\lambda u_2 - tu_2\right),$

The solution *u(t)* found by the Adomian decomposition method with seven iterations is obtained as a series and the formula used to normalize the solution (normalized eigenfunction) *u(t)* is as:

$u(t) = \left(\displaystyle\int_0^1 |u(t)|\,dt\right)^{-1} u(t),$

with this formula and the ADM method, λ and u_{ADM} are as

$\lambda = 0.4899667963,$

$u_{ADM}(t) = \displaystyle\sum_{k=0}^{\infty} u_k(t)$

$\approx 1.000003877 + 0.001412025487t - 0.0004168111954t^2$ \hfill (15)

$- 0.0130556804t^3 + 0.02040194953t^4 - 0.008318308894t^5$

$- 0.000007085980934t^6 - 0.000007119093639t^7.$

DTM solution as follows:

Applying DTM on Equation (14), we reach the following iteration system

$$\delta(s-1) = \begin{cases} 1, & s = 1, \\ 0, & s \neq 1, \end{cases}$$

$$Y(k+4) = \frac{\lambda Y(k) - \sum_{r=0}^{h} \delta(s-1)Y(k-s)}{(k+1)(k+2)(k+3)(k+4)}, \quad k = 0, 1, 2, ..., 10.$$

Using above recurrence relation and boundary conditions, the following series coefficients *Y(k)* is obtained

$$\lambda = 0.4899666546,$$

$$Y(0) = a, \quad a \text{ is real constant,}$$

$$Y(1) = 0.001388866321a,$$

$$Y(2) = \frac{-0.0008336125306a}{2},$$

$$Y(3) = \frac{-0.08833244692a + 0.01a}{6},$$

$$Y(4) = \frac{\lambda a}{24}.$$

Utilizing above calculations *Y(k)* and using Equation (12), we obtain the approximation solutions of the problem Equation (13) with seven iterations.

The formula used to normalize the solution (normalized eigenfunction) is as follows,

$$u(t) = u(t) \left(\int_0^1 |u(t)| \, dt \right)^{-1}.$$

By above formula and DTM, we find the following λ and normalized function u_{DMT}

$$\lambda = 0.4899666546,$$

$$
\begin{aligned}
u_{DTM}(t) = {} & 1.000013900 + 0.001388885626t - 0.0004168120589t^2 \\
& - 0.01305558929t^3 + 0.02041556105t^4 - 0.008327778270t^5 \\
& - 0.000004425304544t^6 - 0.000007119037321t^7.
\end{aligned}
\tag{16}
$$

Table 1

Calculated results of the normalized eigenfunctions (u_{ADM} and u_{DMT}) Equation (15) and (16)

t	u_{ADM}	u_{DMT}	$\lvert u_{ADM} - u_{DMT} \rvert$
0.0	1.0000139000	1.0000038770	0.0000100230
0.1	1.0001375240	1.0001298130	0.0000077110
0.2	1.0002005600	1.0001951460	0.0000054140
0.3	1.0001856760	1.0001825050	0.0000031710
0.4	1.0001045390	1.0001035030	0.0000010360
0.5	0.9999877962	0.9999887352	0.0000000939
0.6	0.9998750542	0.9998777441	0.0000026899
0.7	0.9998048344	0.9998089895	0.0000041551
0.8	0.9998045015	0.9998097741	0.0000052726
0.9	0.9998801774	0.9998861621	0.0000059847
1.0	1.0000066240	1.0000128460	0.0000062220

The Table 1 shows the comparison of Equation (15) with Equation (16) for different values of *t*. Next, we plot these results in the Figure 1 to compare the ADM and DTM solutions. In conclusion, it was found that the results obtained by the two methods were in full agreement.
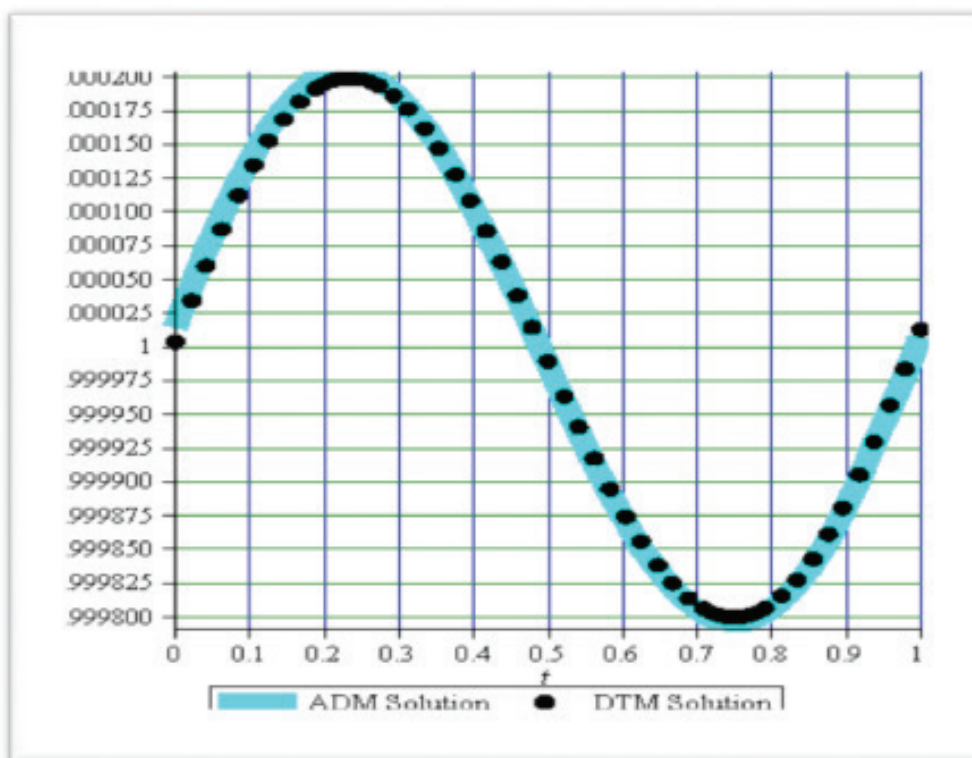

Figure 1. Comparison of ADM and DTM approximate solutions

4. Conclusion

We studied efficient and high accuracy methods for solving fourth order eigenvalue problem with nonstrongly regular boundary conditions. The solutions are very rapidly convergent by utilizing these methods. The numerical results are obtained by mathematics computer programe and are shown in table and figure. The numerical values in all tables and figures prove that we achieved an effective approximation.

Author Contributions

Author has all contributions to this article.

Conflicts of Interest

The author declares no conflict of interest.

References

Abdel-Halim Hassan, I.H. (2002). On solving some eigenvalue problems by using a differential transformation. *Applied Mathematics and Computation*, *127,* 1-22. https://doi.org/10.1016/S0096-3003(00)00123-5

Adebısı, A.F., Uwaheren, O.A., Aboların, O.E., Rajı, M.T., Adedejı, J.A., & Peter, O.J., (2021). Solution of typhoid fever model by Adomian decomposition method. *J. Math. Comput. Sci., 11*(2), 1242-1255. https://doi.org/10.28919/jmcs/5288

Adomian, G., & Rach, R. (1993). Analytic solution of nonlinear boundary-value problems in several dimensions by decomposition. *Journal of Mathematical Analysis and Applications*, *174*, 118-137. https://doi.org/10.1006/jmaa.1993.1105

Alalyani, A. (2019). Eigenvalue computation of regular 4th order Sturm-Liouville Problems. *Applied Mathematics*, *10*, 784-803. https://doi.org/10.4236/am.2019.109056

Alquran, M.T., & Al-Khaled, K. (2010). Approximations of Sturm-Liouville eigenvalues using sinc-Galerkin and differential transform methods. *Applications and Applied Mathematics: An International Journal*, *5*, 128 – 147.

Arslan, D. (2018). Differential transform method for singularly perturbed singular differential equations. *Journal of the Institute of Natural Applied Sciences*, *23,* 254-260.

Arslan, D. (2019). A novel hybrid method for singularly perturbed delay differential equations. *Gazi University Journal of Science*, *32,* 217-223.

Attili, B.S., & Lesnic, D. (2006). An efficient method for computing eigen elements of Sturm-Liouville fourth-order boundary value problems. *Applied Mathematics and Computation*, *182*, 1247–1254. https://doi.org/10.1016/j.amc.2006.05.011

Ayaz, F. (2004). Applications of differential transform method to differential-algebraic equations. *Applied Mathematics and Computation*, *152*, 649-657. http://dx.doi.org/10.1016/S0096-3003(03)00581-2

Baskonus, H.M., Sulaiman, T.A., & Bulut, H. (2018). Dark, bright and other optical solitons to the decoupled nonlinear Schrödinger equation arising in dual-core optical fibers. *Opt Quant Electron*, 50, 165. https://doi.org/10.1007/s11082-018-1433-0

Biazar, J., Dehghan, M., & Houlari, T. (2020). An efficient method to approximate eigenvalues and eigenfunctions of high order Sturm-Liouville problems. *Computational Methods for Differential Equations*, *8*, 389-400. https://doi.org/10.22034/CMDE.2020.29144.1417

Chakraverty, S., Mahato, N.R., Karunakar, P., & Rao, T.D. (2019). Advanced numerical and semi-analytical methods for differential equations. Wiley Online Library, chapter 11, 2019.

Chanane, B. (2010). Accurate solutions of fourth order Sturm-Liouville problems. *Journal of Computational and Applied Mathematics*, *234*(2010), 3064-3071. https://doi.org/10.1016/j.cam.2010.04.023

Çakır, M., & Arslan, D. (2015). The Adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay differential equations. *Applied Mathematics*, *6*, 1332-1343. https://doi.org/10.4236/am.2015.68126

Gao, W., Ismael, H.F., Husien, A.M., Bulut, H., & Baskonus, H.M. (2020). Optical soliton solutions of the cubic-quartic nonlinear Schrödinger and Resonant Nonlinear Schrödinger equation with the parabolic Law. *Applied Sciences*, *10*(1), 219. https://doi.org/10.3390/app10010219

Gubes, M., Peker, H.A., & Oturanç, G. (2015). Application of differential transform method for El Nino Southern oscillation (ENSO) model with compared Adomian Decomposition and variational iteration methods. *Journal of Mathematics and Computer Science, 15*(3), 167-178. www.tjmcs.com

Kaya, U. (2020). Basis properties of root functions of a regular fourth order boundary value problem. *Hacet. J. Math. Stat*., *49*(1), 338-351. https://doi.org/10.15672/hujms.552213.

Li, W., Pang, Y. (2020). Application of Adomian decomposition method to nonlinear systems. *Adv. Differ. Equ., 67*. https://doi.org/10.1186/s13662-020-2529-y

Mukhtarov, O. Sh., Yucel, M., & Aydemir, K. (2020). Treatment a new approximation method and ıts justification for Sturm–Liouville problems. *Hindawi Complexity*, *2020*, Article ID 8019460, 8 pages. https://doi.org/10.1155/2020/8019460

Peker, H.A., & Karaoğlu, O. (2010). Solution of a kind of evolution equation by the differential transformation and Adomian decomposition methods. *Selcuk Journal of Applied Mathematics, Special Isuue,* 19-25. http://hdl.handle.net/123456789/10484

Peker, H.A., Karaoğlu, O., & Oturanç, G. (2011). The differential transformation method and Pade approximant for a form of Blasius equation. *Mathematical and Computational Applications, 16* (2)*,* 507-513. https://doi.org/10.3390/mca16020507

Syam, M.I., & Siyyam, H.I. (2009). An efficient technique for finding the eigenvalues of fourth-order Sturm–Liouville problems. *Chaos, Solitons and Fractals*, *39*, 659–665. https://doi.org/10.1016/j.chaos.2007.01.105

Zhou, J.K. (1986). Differential transform and its application for electrical circuits. Huazhong University Press: Wuhan.