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ACHROMATIC COLORING OF QUADRILATERAL SNAKES

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ABSTRACT

The main objective of this article is to discuss achromatic coloring and to investigate the achromatic number of the central graph of k-quadrilateral and k-alternate quadrilateral snakes that is $\chi_a\,(C(kQ_n)) = 2k(n-1)+1$ and $\chi_a\,(C(k(AQ_n))) = \frac{n(4k-1)}{2}$.

Keywords: Achromatic coloring; Achromatic number; Central graph; Quadrilateral and alternate quadrilateral snakes.

ÖZET

Bu makalenin temel amacı, akromatik renklendirmeyi tartışmak ve k-dörtgen ve k-alternatif dörtgen yılanların merkez grafiğinin akromatik sayısını yani $\chi_a\,(C(kQ_n)) = 2k(n-1)+1$ ve $\chi_a\,(C(k(AQ_n))) = \frac{n(4k-1)}{2}$. araştırmaktır.

Anahtar Kelimeler: Akromatik renklendirme, Akromatik sayı, Merkezi grafik, Dörtgen ve alternatif dörtgen yılanlar.

1. Introduction

The achromatic coloring [1, 4, 8, 9, 14, 15] is kind of proper vertex coloring of a graph G in which every pair of different colors are adjacent by at least one edge and the largest number of colors are required for achromatic coloring is called achromatic number, denoted by $\chi_a(G)$. For a given graph $G = (V, E)$ by subdividing each edge exactly once and joining all the non-adjacent vertices of G, obtained graph is called central graph [1, 4, 15] of G denoted by $C(G)$. A quadrilateral snake Q_n [5, 10, 11, 12, 13] is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and $u_{i+1}$to new vertices $v_i$and $w_i$respectively and adding edges $v_i w_i$ for $(1 \le i \le n-1)$. That is every edge of a path is replaced by a cycle C_4. In this article we investigate the achromatic number of the central graph of quadrilateral snake, double quadrilateral snake, triple quadrilateral snake, k −quadrilateral snake (k −quadrilateral snake graph $k(Q_n)$ consists of k quadrilateral snakes with a common path), alternate quadrilateral snake, double alternate quadrilateral snake, triple alternate quadrilateral snake and k −alternate quadrilateral snake (k −alternate quadrilateral snake graph $k(AQ_n)$ consists of k alternate quadrilateral snakes with a common path), denoted by $\chi_a(C(Q_n))$, $\chi_a(C(DQ_n))$, $\chi_a(C(TQ_n))$, $\chi_a(C(kQ_n))$, $\chi_a(C(AQ_n))$, $\chi_a(C(D(AQ_n)))$, $\chi_a(C(T(AQ_n)))$, $\chi_a(C(k(AQ_n)))$ respectively.

Throughout the paper we consider n as the number of vertices of the path P_n.

2. Definitions

Definition 2.1. A quadrilateral snake Q_n [5, 10, 11, 12, 13] is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and $u_{i+1}$to new vertices v_i and w_i respectively and adding edges $v_i w_i$ for $(1 \le i \le n - 1)$. That is every edge of a path is replaced by a cycle C_4.

Definition 2.2. A double quadrilateral snake $D(Q_n)$ [5, 10, 11, 12, 13] consists of two quadrilateral snakes that have a common path. That is, a double quadrilateral snake is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and u_{i+1} to new vertices v_i, x_i and w_i, y_i and then joining v_i and w_i, x_i and y_i for $(1 \le i \le n-1)$.

Definition 2.3. A triple quadrilateral snake $T(Q_n)$ [5, 11, 12, 13] is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i, x_i, p_i and w_i, y_i, q_i and then joining v_i and w_i, x_i and y_i, p_i and q_i for $(1 \le i \le n-1)$.

Definition 2.4. An alternate quadrilateral snake AQ_n [5, 12, 13] is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i and w_i respectively and adding edges $v_i w_i$ for $(1 \le i \le n-1)$. That is every alternate edge of a path is replaced by a cycle C_4.

Definition 2.5. A double alternate quadrilateral snake $D(AQ_n)$ [5, 11, 12, 13] is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to new vertices v_i, x_i and w_i, y_i and then joining v_i and w_i, x_i and y_i for $(1 \le i \le n-1)$.

Definition 2.6. A triple alternate quadrilateral snake $T(AQ_n)$ [5, 11, 12, 13] is obtained from a path $u_{1,} u_{2,} \dots, u_n$ by joining u_i and u_{i+1} (alternatively) to a new vertex v_i, x_i, p_i and w_i, y_i, q_i and then joining v_i and w_i, x_i and y_i, p_i and q_i for $(1 \le i \le n-1)$.

3. Achromatic number of $C(Q_n)$, $D(Q_n)$, $T(Q_n)$

Theorem 3.1. For quadrilateral snake Q_n, the achromatic number, $\chi_a\left(C(Q_n)\right) = 2n,\ n \ge 2$.

Proof. Let P_n be the path with n vertices $u_{1,} u_{2,} \dots, u_n$ and Q_n be the quadrilateral snake. To obtain central graph, let each edge $u_i u_{i+1}$, $u_i v_i$, $u_i w_i$ and $v_i w_i$ $(1 \le i \le n-1)$ of Q_n be subdivided by the vertices $e_{i,} e_i'$, l_i' and l_i'' $(1 \le i \le n-1)$. $V(C(Q_n)) = \{u_i\colon 1 \le i \le n\} \cup \{v_i, w_i\colon 1 \le i \le n-1\} \cup \{e_{i,} e_i'\colon 1 \le i \le n-1\} \cup \{l_i', l_i''\colon 1 \le i \le n-1\}$. Now coloring the vertices of $C(Q_n)$ as follows: define $c\colon V(C(Q_n)) \to \{1, 2, 3, \dots, 2n\}$ for $n \ge 2$ by $c(u_i) = 2i-1$ for $(1 \le i \le n)$ and $c(v_i) = 2i-1$, $c(w_i) = 2i$, $c\ (e_i') = 2n - 2$, $c(e_i) = = 2n$, $c\ (l_i') = 2n$, $c\ (l_i'') = 2n$ for $(1 \le i \le n)$.

Claim 1: c is proper; from above each $c(u_i), c(v_i), c(w_i)$ and its neighbors are assigned by different colors. Hence it is proper coloring.

Claim 2: c is achromatic; it is clear that every pair of different colors is assigned by at least one edge, so achromatic. Figure 1 shows the achromatic coloring for $C(Q_3)$.

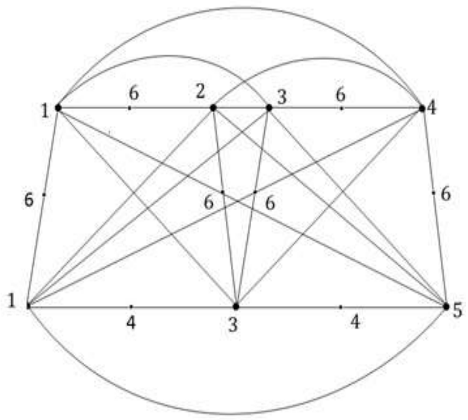

Figure 1. $C(Q_n)$ with coloring, $\chi_a\left(C(Q_3)\right) = 6$.

Claim 3: c is maximum. **Case (i)**: all the vertices are colored by $2n$ colors. Now if we assign $(2n+1)^{th}$ color on any vertex, then we lead to contradict the achromatic coloring. Therefore, it is maximum. **Case (ii):** Assume that the adjacent vertices of u_i, v_i and w_i are assigned by the $(2n+1)^{th}$ color, again we get a contradiction. Therefore, the maximum number of colors are required for this coloring is $2n$. Therefore, c is maximum. Hence $\chi_a\left(C(Q_n)\right) = 2n$.

Theorem 3.2. For double quadrilateral snake DQ_n, achromatic number, χ_a $(C(DQ_n))$= 4 n - 3, $n \geq$ 2.

Proof. Let P_n be the path with n vertices $u_{1,} u_{2,} \dots, u_n$ and DQ_n be the double quadrilateral snake. Now we obtain the central graph as described in theorem 3.1, therefore $V(C(DQ_n))$= = $\{u_i\colon 1 \leq i \leq n\} \cup \{v_i, w_i\colon 1 \leq i \leq n-1\} \cup \{\{x_i, y_i : 1 \leq i \leq n-1\} \cup \{e_i', e_i'', e_{i,}\colon 1 \leq i \leq n-1\} \cup \{l_i', l_i''\colon 1 \leq i \leq n-1\} \cup \{m_i', m_i''\colon 1 \leq i \leq n-1\}$. Now coloring the vertices of $C(DQ_n)$ as follows: define $c : V(C(DQ_n)) \rightarrow \{1, 2, 3, \dots, 4n - 3\}$ for $n \geq 2$ by $c(u_i) = 1$, $c(u_n)= n$, $c(v_i) = 2i-1$, $c(w_i) = 2i$, $c(x_i) = 2n+2i-3$, $c(y_i) = 2n+2i-2$, $c(e_i)$= $c(e_i')$ = $c(e_i'')$ = $4n - 3$, $c(l_i'') = c(v_i)$, $c(l_i') = c(w_i)$, $c(m_i'') = c(x_i)$, $c(e_i') = c(y_i)$ and at last $c(u_{i+1}) = c(w_i)$ for $(1 \leq i \leq n-1)$. Figure 2 shows the achromatic coloring for $C(DQ_3)$. To prove c is achromatic and maximum, follow theorem 3.1.

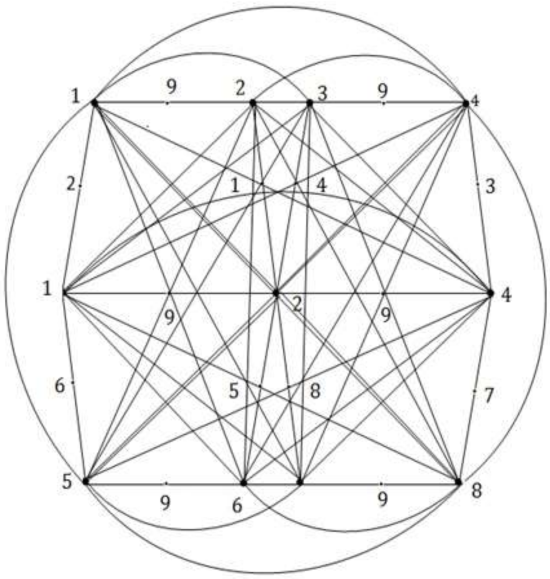

Figure 2. $C(DQ_3)$. with coloring, $\chi_a\,(C(DQ_3))$= 49.

Theorem 3.3. For triple quadrilateral snake TQ_n, the achromatic number, $\chi_a\,(C(TQ_n))$= 6 n - 5, $n \geq 2$.

Proof. Let P_n be the path with n vertices $u_{1,}\,u_{2,}\,\ldots,u_n$ and TQ_n be the triple quadrilateral snake. Now we obtain the central graph as described in theorem 3.1, therefore $V(C(TQ_n))$= $\{u_i\colon 1 \leq i \leq n\}$ $\cup$ $\{v_i, w_i\colon\; 1 \leq i \leq n-1\}$ $\cup$ $\{x_i, y_i\;\;:\;\; 1 \leq i \leq n-1\}$ $\cup$ $\{p_i, q_i\;\;:\;\; 1 \leq i \leq n-1\}$ $\cup\{e_{i,}\, e_i', e_i'', e_i'''\colon\; 1 \leq i \leq n-1\}$ $\cup$ $\{l_i', l_i''\colon\; 1 \leq i \leq n-1\}$ $\cup$ $\{m_i', m_i''\colon\; 1 \leq i \leq n-1\}$ $\cup$ $\{z_i', z_i''\colon 1 \leq i \leq n-1\}$. Now coloring the vertices of $C(TQ_n)$ as follows: define c : $V(C(TQ_n)) \rightarrow$ {1, 2, 3, ..., 6 n - 5} for $n \geq$ 2by $c(u_1)$ = 1, $c(u_n) = n$, $c(v_i) = 2i-1$, $c(w_i) = 2i$, $c(x_i) = 2n + 2i - 3$, $c(y_i) = 2n + 2i - 2$, $c(p_i) = 4n + 2i - 5$, $c(q_i) = 4n + 2i - 4$, $c(e_i)$= $c(e_i')$ = $c(e_i'')$ = $6n - 5$, $c(l_i'')$ = $c(v_i)$, $c(l_i')$ = $c(w_i)$, $c(\,m_i'')$ = $c(x_i)$, $c(e_i')$ = $c(y_i)$, c(z_i'') = $c(p_i)$, c(z_i') = $c(q_i)$ and at last $c(u_{i+1})$ = $c(w_i)$ for ($1 \leq i \leq n-1$). Figure 3 shows the achromatic coloring for $C(TQ_3)$. To prove c is achromatic and maximum, follow theorem 3.1.

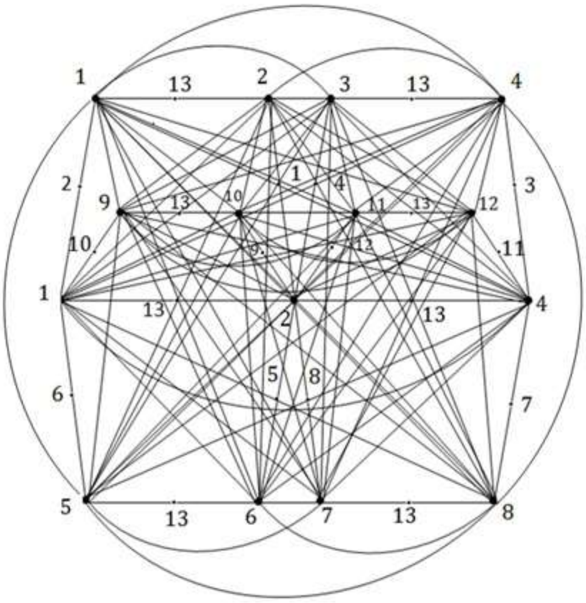

Figure 3. $C(TQ_3)$ with coloring, $\chi_a\,(C(TQ_3))$=13.

4. Achromatic Number of k-Quadrilateral Snake

Theorem 4.1. For k −quadrilateral snake kQ_n, the achromatic number, $\chi_a\,(C(kQ_n)) = 2k(n-1)+1$ for n, k ≥2.

Proof. By continuing in the same manner as discussed in theorem 3.1, 3.2 and 3.3, it is easy to conclude that the achromatic number of the central graph of k −quadrilateral snake is $2k(n-1)+1$ for $k \geq 2$, where k denotes the quadrilateral snakes like double, triple etc.

5. Achromatic Number of $C(AQ_n)$, $D(AQ_n)$, $T(AQ_n)$

Theorem 5.1. For alternate quadrilateral snake AQ_n, the achromatic number, $\chi_a\left(C(AQ_n)\right) = \frac{3n}{2}$, where n is even and $n \geq 4$.

Proof. Let P_n be the path with n vertices $u_1, u_2, \ldots, u_n$ and AQ_n be an alternate quadrilateral snake. Now we obtain the central graph as described in theorem 3.1, therefore $V(C(AQ_n))= \{u_i: 1 \leq i \leq n\} \cup \{v_i, w_i: \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{e_i: (1 \leq i \leq n-1)\} \cup \{e_i': \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{l_i', l_1'': \left(1 \leq i \leq \frac{n}{2}\right)\}$.

Now coloring the vertices of $C(AQ_n)$ as follows : define $c: V(C(AQ_n)) \rightarrow \{1, 2, 3, \ldots, \frac{3n}{2}\}$ for $n \geq 4$ by $c(u_1) = 1$, $c(u_n) = n$, $c(v_i) = 2i-1$, $c(w_i) = 2i$, $c(e_i) = n+1$, $c(e_i') = n+1$ for $(1 \leq i \leq \frac{n}{2})$, $c(u_i) = n+1+\frac{i}{2}$ (i = 2,4, 6,..., n - 2) and $c(u_i) = n+1+\frac{i-1}{2}$ (i = 3, 5, 7, ..., n - 1) and $c(l_i'')$ =

$c(v_i)$, $c(l_i')$ = $c(w_i)$ for $\left(1 \leq i \leq \frac{n}{2}\right)$. Figure 4 shows the coloring of $C(AQ_4)$. To prove c is achromatic and maximum, follow theorem 3.1.

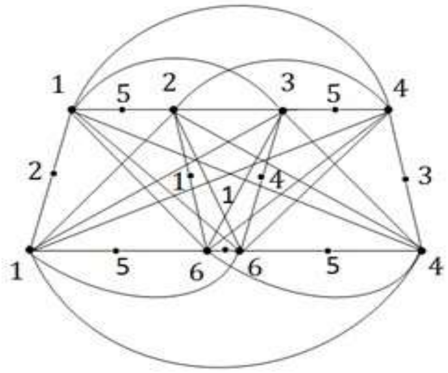

Figure 4. $C(AQ_4)$ with coloring, $\chi_a\left(C(AQ_n)\right) = 6$.

Theorem 5.2. For double alternate quadrilateral snake $D(AQ_n)$, the achromatic number, $\chi_a\left(C(D(AQ_n))\right) = \frac{5n}{2}$, where n is even and $n \geq 4$.

Proof. Let P_n be the path with n vertices $u_1, u_2, \ldots, u_n$ and $D(AQ_n)$ be the double alternate quadrilateral snake. Now we obtain the central graph as described in theorem 3.1, therefore $V(C(D(AQ_n)))$= $\{u_i\colon 1 \leq i \leq n\} \cup \{v_i, w_i\colon \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{x_i, y_i\colon \left(1 \leq i \leq \frac{n}{2}\right)\}$ $\{e_i\colon (1 \leq i \leq n-1)\} \cup \{e_i', e_i''\colon \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{l_i', l_1''\colon \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{m_i', m_1''\colon \left(1 \leq i \leq \frac{n}{2}\right)\}$. Now coloring the vertices of $C\left(D(AQ_n)\right)$ as follows: define $c\colon V(C(D(AQ_n))) \to \{1, 2, 3, \ldots, \frac{5n}{2}\}$ for $n \geq 4$ by $c(u_1) = 1$, $c(u_n) = n$, $c(v_i) = 2i - 1$, $c(w_i) = 2i$, $c(x_i) = n + 2i - 1$, $c(y_i) = n + 2i$ for $\left(1 \leq i \leq \frac{n}{2}\right)$, $c(e_i) = 2n + 1$ $(i = 1, 3, 5, \ldots)$, $c(e_i) = i$ $(i = 2, 4, 6, \ldots, \frac{n}{2} - 1)$, $c(e_i') = c(e_i'') = 2n + 1$ for $\left(1 \leq i \leq \frac{n}{2}\right)$, $c(u_i) = 2n + 1 + \frac{i}{2}$ (i = 2,4, 6,..., n - 2) and $c(u_i) = 2n + 1 + \frac{i-1}{2}$ (i = 3, 5, 7, ..., n - 1) and at last $c(l_i'') = c(v_i)$, $c(l_i') = c(w_i)$, $c(m_i'') = c(x_i)$, $c(m_i') = c(y_i)$ for $\left(1 \leq i \leq \frac{n}{2}\right)$. Figure 5 shows the coloring of $C(D(AQ_4))$. To prove c is achromatic and maximum, follow theorem 3.1.

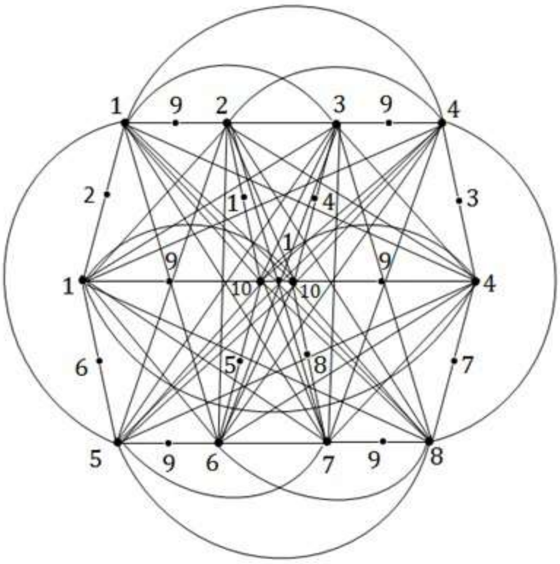

Figure 5. $C(D(AQ_4))$ with coloring, $\chi_a\,(C(D(AQ_4))) = 10$

Theorem 5.3. For triple alternate quadrilateral snake $T(AQ_n)$, the achromatic number, $\chi_a\,(C(T(AQ_n))) = \frac{7n}{2}$, where n is even and $n \geq 4$.

Proof. Let P_n be the path with n vertices $u_{1,}\, u_{2,} \dots, u_n$ and $T(AQ_n)$ be the triple alternate quadrilateral snake. Now we obtain the central graph as described in theorem 3.1, therefore $V(C(T(AQ_n)))= \{u_i\colon 1 \leq i \leq n\} \cup \{v_i, w_i,\ x_i,\ y_i, p_i,\ q_i : \left(1 \leq i \leq \frac{n}{2}\right)\}\ \ \{e_i\colon (1 \leq i \leq n-1)\} \cup \{e_i', e_i'', e_i''' \colon \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{l_i, l_i', l_1''\colon \left(1 \leq i \leq \frac{n}{2}\right)\} \cup \{m_i', m_1'', m_1'''\colon \left(1 \leq i \leq \frac{n}{2}\right)\}$. Now coloring the vertices of $C(T(AQ_n))$ as follows; define $c\colon V(C(T(AQ_n))) \to \{1, 2, 3, \dots, \frac{7n}{2}\}$ for $n \geq 4$ by $c(u_1) = 1$, $c(u_n) = n$, $c(v_i) = 2i-1$, $c(w_i) = 2i$, $c(x_i) = n+2i-1$, $c(y_i) = n+2i$, $c(p_i) = 2n+2i-1$, $c(q_i) = 2n+2i$ for $\left(1 \leq i \leq \frac{n}{2}\right)$, $c(e_i) = 3n+1\ (i = 1, 3, 5, \dots)$, $c(e_i) = i\ (i = 2, 4, 6, \dots, \frac{n}{2}-1)$, $c(e_i') = c(e_i'') = c(e_i''') = 3n+1$ for $\left(1 \leq i \leq \frac{n}{2}\right)$, $c(u_i) = 3n+1+\frac{i-1}{2}$ $(i = 2, 4, 6, \dots, n-2)$ and $c(u_i) = 3n+1+\frac{i-1}{2}$ $(i = 3, 5, 7, \dots, n-1)$ and at last $c(l_i'') = c(v_i)$, $c(l_i') = c(w_i)$, $c\,(m_i'') = c(x_i)$, $c(m_i') = c(y_i)$, $c(m_i) = c(p_i)$, $c(l_i) = c(q_i)$ for $\left(1 \leq i \leq \frac{n}{2}\right)$. To prove c is achromatic and maximum, follow theorem 3.1. Figure 6 shows the coloring of $C(T(AQ_4))$.

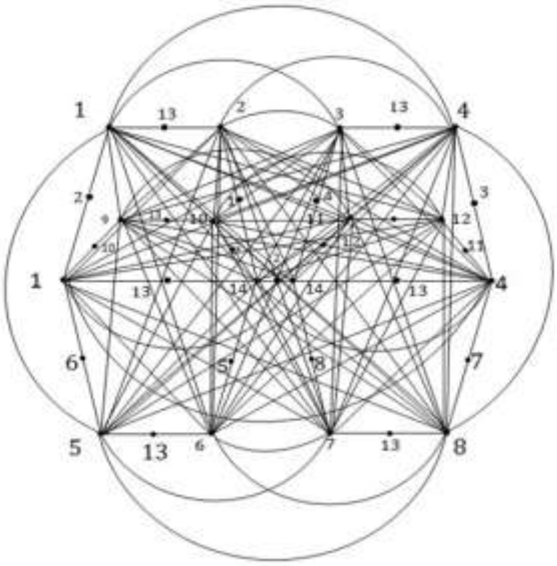

Figure 6. $C(T(AQ_4))$with coloring, $\chi_a\,(C(T(AQ_n))) = 14$.

6. Achromatic Number of k-Alternate Quadrilateral Snake

Theorem 6.1. For $k-$ quadrilateral snake kQ_n, the achromatic number, $\chi_a\, C((kAQ_n)) = \frac{n(4k-1)}{2}$, where n is even and $n \geq 4$.

Proof. By continuing in the same manner as discussed in theorems 5.1, 5.2 and 5.3, it is easy to conclude that the achromatic number of the central graph of $k-$alternate quadrilateral snake is $\frac{n(4k-1)}{2}$.

7. Conclusion

We obtain the achromatic number of the central graph of $k-$quadrilateral and $k-$alternate quadrilateral snakes that is $\chi_a\, C((kQ_n)) = 2k(n-1)+1$ and $\chi_a\, C((kAQ_n)) = \frac{n(4k-1)}{2}$. For motivation and future scope, we can examine the different type of colorings for these quadrilateral snakes.

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