Journal of Engineering Technology and Applied Sciences, 2022 *e-ISSN:* 2548-0391
Received 7 October 2022, Accepted 30 November 2022 *Vol. 7, No. 3, 207-218*
Published online 22 December 2022, Research Article *doi: 10.30931/jetas.1185721*

Citation: Beyendi, S., "Some Results on α-cosymplectic Manifolds Admitting a Non-symmetric Non-metric Connection". Journal of Engineering Technology and Applied Sciences 7 (3) 2022 : 207-218.

SOME RESULTS ON α-COSYMPLECTIC MANIFOLDS ADMITTING A NON-SYMMETRIC NON-METRIC CONNECTION

Selahattin Beyendi

Department of Mathematics, Faculty of Education, Inonu University, 44280, Malatya, Türkiye
selahattin.beyendi@inonu.edu.tr

Abstract

In this study, we study α-cosymplectic manifolds admitting a non-symmetric non-metric connection. Moreover, several results about Ricci semi-symmetric and semi-symmetric α-cosymplectic manifolds admitting the non-symmetric non-metric connection are going to be obtained.

Keywords: α-cosymplectic manifold, non-symmetric non-metric connection, semi-symmetric, Ricci semi-symmetric, Ricci soliton

1. Introduction

One may consider the contact geometry as a basis for several phenomena and also relate it to many structures widely seen in mathematical and physical sciences. Contact structures first appeared in the work of Sophus Lie [20] on partial differantial equations. The geometry of contact Riemmanian manifolds and related issues have also received great attention in recent years. In fact, the most outstanding one of those can be considered as the almost cosymplectic manifolds presented by Goldberg and Yano [15] in 1969. In 1972, K. Kenmotsu introduced and studied a new class of almost contact manifolds called Kenmotsu manifolds [18]. Then, Kim and Pak in [19] described a new class of manifolds known as almost α-cosymplectic manifolds when they combined almost cosymplectic and almost α-Kenmotsu manifolds, in which α is a real number. Note that almost α-Kenmotsu structures are related to some special conformal deformations of almost cosymplectic structures [28]. One can encounter many studies in the literature about almost cosymplectic manifolds ([2], [5], [6], [13], [21], [22]) and many others.

On the other side, the concept of semi-symmetric linear connection defined on a differentiable manifold has been put forward by Friedmann and Schouten ([14], [26]). Then the concept of symmetric linear connection defined on a differentiable manifold was introduced by Hayden [16] and later further investigated by Yano [29]. The main invariants of an affine connection are its torsion and curvature. A torsion tensor of a connection ∇ is a mapping $T: \chi(M) \times \chi(M) \longrightarrow \chi(M)$ defined as

$$T(U,V) = \nabla_U V - \nabla_V U - [U,V],$$

for arbitrary vector fields $U, V \in \chi(M)$. It is said that the connection ∇ is symmetric when the torsion tensor vanishes, otherwise it is known as non-symmetric. A semi-symmetric connection ∇ is said to be a semi-symmetric metric connection if $\nabla g = 0$, else it is said to be a semi-symmetric non-metric connection. The semi-symmetric non-metric connection in a Riemannian manifold have been studied by ([1], [4], [8], [9], [11], [12], [17]) and many others. Recently, Pankaj et al. and Singh et al. ([23], [24], [27]) have been studied the non-symmetric non-metric (abbr. NSNM) connection. In this work, we study an α-cosymplectic manifolds with respect to a (NSNM) connection.

The outline of the present study is as follows: In the second section, some basic definitions and results for α-cosymplectic manifolds and (NSNM) connection are presented. In the third section, some theorems and lemmas on α-cosymplectic manifolds with respect to a (NSNM) connection are given. In the fourth section, we show that a semi-symmetric α-cosymplectic manifold with respect to (NSNM) connection is a cosymplectic manifold. In the fifth section, Ricci semi-symmetric α-cosymplectic manifolds with respect to a (NSNM) connection have been studied and it is also proved that on a Ricci semi-symmetric α-cosymplectic manifold with respect to a (NSNM) connection, the Ricci soliton of data (g, ξ, λ) is steady, expanding as $\alpha = 0, \alpha \neq 0$, respectively.

2. Some basic facts

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold, in which φ is a (1,1)-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It has been a well established fact that the (φ, ξ, η, g) structure satisfies the following the conditions [7].

$$\varphi\xi = 0, \;\; \eta(\varphi\xi) = 0, \;\;\; \eta(\xi) = 1, \tag{2.1}$$

$$\varphi^2 U = -U + \eta(U)\xi, \;\; g(U,\xi) = \eta(U), \tag{2.2}$$

$$g(\varphi U, \varphi V) = g(U,V) - \eta(U)\eta(V), \tag{2.3}$$

for all $U, V \in \chi(M)$; in which $\chi(M)$ denotes the collection of all smooth vector fields of M. If

$$\nabla_U \xi = -\alpha\varphi^2 U, \tag{2.4}$$

$$(\nabla_U \varphi)V = \alpha[g(\varphi U, V)\xi - \eta(V)\varphi U], \tag{2.5}$$

$$(\nabla_U \eta)V = \alpha g(U,V) - \eta(U)\eta(V)], \tag{2.6}$$

in which ∇ indicates the Riemannian connection of hold and α is a real number, then $(M, \varphi, \xi, \eta, g)$ is called an α-cosymplectic manifold [19]. Under this assumption, it is well known that [22]

$$\eta(R(U,V)Z) = \alpha^2[g(U,Z)\eta(V) - g(V,Z)\eta(U)], \tag{2.7}$$

$$R(U,V)\xi = \alpha^2[\eta(U)V - \eta(V)U], \tag{2.8}$$

$$R(\xi,U)V = \alpha^2[\eta(V)U - g(U,V)\xi], \tag{2.9}$$

$$R(U,\xi)\xi = \alpha^2[\eta(U)\xi - U], \tag{2.10}$$

$$S(U,\xi) = -2n\alpha^2\eta(U), \tag{2.11}$$

$$Q\xi = -2n\alpha^2\xi, \tag{2.12}$$

for all U, V and $Z \in \chi(M)$, in which R is the curvature tensor, S is the Ricci-curvature and Q is the Ricci operator of α-cosymplectic manifold. S and Q are related to each other by

$$g(QU,V) = S(U,V). \tag{2.13}$$

When the Ricci tensor denoted by S satisfies the following condition, the α-cosymplectic manifold is called an η-Einsten manifold.

$$S(U,V) = m_1 g(U,V) + m_2\eta(U)\eta(V), \tag{2.14}$$

in which m_1, m_2 are certain scalars. In Eq. (2.14), if $m_2 = 0$, the manifold becomes Einstein.

Definition 2.1 A Ricci soliton $(\mathrm{g}, \mathrm{V}, \lambda)$ on a Riemannian manifold is defined by

$$\mathfrak{L}_{\mathrm{V}}\mathrm{g} + 2\mathrm{S} + 2\lambda\mathrm{g} = 0, \tag{2.15}$$

in which $\mathfrak{L}_{\mathrm{V}}\mathrm{g}$ is a Lie-derivative of Riemannian metric g with respect to vector field V and λ is a real constant. It is said to be shrinking, steady, or expanding according to $\lambda < 0, \lambda = 0$ and $\lambda > 0$ [3].

The Nijenhuis tensor $\mathcal{N}(\mathrm{U},\mathrm{V})$ of φ in (M,g) is a vector valued bilinear function such that

$$\mathcal{N}(\mathrm{U},\mathrm{V}) = (\nabla_{\varphi\mathrm{U}}\varphi)(\mathrm{V}) - (\nabla_{\varphi\mathrm{V}}\varphi)(\mathrm{U}) - \varphi((\nabla_{\mathrm{U}}\varphi)(\mathrm{V})) + \varphi((\nabla_{\mathrm{V}}\varphi)(\mathrm{U})). \tag{2.16}$$

When the following definition is given

$$'\mathcal{N}(\mathrm{U},\mathrm{V},\mathrm{Z}) = \mathrm{g}(\mathcal{N}(U,V),Z), \tag{2.17}$$

then

$$\begin{aligned}'\mathcal{N}(U,V,Z) = {}& g((\nabla_{\varphi U}\varphi)(V),Z) - g((\nabla_{\varphi V}\varphi)(U),Z) - g(\varphi((\nabla_U\varphi)(V)),Z) \\ &+ g(\varphi((\nabla_V\varphi)(U)),Z).\end{aligned} \tag{2.18}$$

Now, let's define a linear connection $\overset{*}{\nabla}$ [24] as

$$\overset{*}{\nabla}_U V = \nabla_U V + g(\varphi U, V)\xi \tag{2.19}$$

satisfying

$$\overset{*}{T}(U, V) = 2g(\varphi U, V)\xi \tag{2.20}$$

and

$$(\overset{*}{\nabla}_U g)(V, Z) = -\eta(Z)g(\varphi U, V) - \eta(V)g(\varphi U, Z), \tag{2.21}$$

for any $U, V,$ and $Z \in \chi(M)$, and call it a (NSNM) connection. It is also known [8]

$$(\overset{*}{\nabla}_U \varphi)(V) = (\nabla_U \varphi)(V) + g(\varphi U, \varphi V)\xi, \tag{2.22}$$

$$(\overset{*}{\nabla}_U \eta)(V) = (\nabla_U \eta)(V) - g(\varphi U, V), \tag{2.23}$$

$$\left(\overset{*}{\nabla}_U g\right)(\varphi V, Z) = -\eta(Z)g(\varphi U, \varphi V). \tag{2.24}$$

By using V by ξ in (2.19), one has

$$\overset{*}{\nabla}_U \xi = \nabla_U \xi. \tag{2.25}$$

This has been stated by the following result in [23].

Proposition 2.2 The vector field ξ is invariant with respect to Levi-Civita connection ∇ and a (NSNM) connection $\overset{*}{\nabla}$.
Putting $U = \xi$ in the Eq. (2.21), we obtain

$$\left(\overset{*}{\nabla}_\xi g\right)(V, Z) = 0 \tag{2.26}$$

This has been stated by the following result in [23].

Proposition 2.3 Covariant differentiation of Riemannian metric g with respect to contravariant vector field ξ vanish identically in a contact metric manifold admitting a (NSNM) connection $\overset{*}{\nabla}$.
The curvature tensor $\overset{*}{R}$ of $\overset{*}{\nabla}$ is described as follows

$$\overset{*}{R}(U, V)Z = \overset{*}{\nabla}_U \overset{*}{\nabla}_V Z - \overset{*}{\nabla}_V \overset{*}{\nabla}_U Z - \overset{*}{\nabla}_{[U,V]} Z, \tag{2.27}$$

in which U, V and $Z \in \chi(M)$. Using Eqs. (2.19), (2.22), (2.23), (2.24) and (2.25), we get

$$\begin{aligned}\overset{*}{R}(U, V)Z = {} & R(U, V)Z + g\big((\nabla_U \varphi)V, Z\big)\xi - g\big((\nabla_V \varphi)U, Z\big)\xi \\ & + g(\varphi V, Z)\nabla_U \xi - g(\varphi V, Z)\nabla_U \xi - g(\varphi U, Z)\nabla_V \xi,\end{aligned} \tag{2.28}$$

in which R is the Riemannian curvature tensor of Levi-Civita connection ∇ [7].

Theorem 2.4 A cosymplectic manifold is locally the Riemannian product of an almost Kaehler manifold with the real line [21].

3. α-cosymplectic manifolds with a (NSNM)-connection

In this section, the curvatures of α-cosymplectic manifolds satisfying a (NSNM) connection are going to be examined.

Theorem 3.1 α-cosymplectic manifolds admitting a (NSNM) connection $\overset{*}{\nabla}$ satisfy $\overset{*}{\mathcal{N}}(\mathrm{U},\mathrm{V}) = \overset{*}{\mathrm{d}}\eta(\mathrm{U},\mathrm{V})\xi$.

Proof. If we define

$$\overset{*}{d}\eta(U,V) = (\overset{*}{\nabla}_U\eta)V - (\overset{*}{\nabla}_V\eta)U. \tag{3.1}$$

Using Eqs. (2.6) and (2.23) from (3.1), we get

$$\overset{*}{d}\eta(U,V) = 2g(\varphi U,V). \tag{3.2}$$

Let's define the Nijenhuis tensor $\overset{*}{\mathcal{N}}(U,V)$ admitting a (NSNM) connection as

$$\begin{aligned}\overset{*}{\mathcal{N}}(U,V) = {} & g((\overset{*}{\nabla}_{\varphi U}\varphi)(V),Z) - g((\overset{*}{\nabla}_{\varphi V}\varphi)(U),Z) - g(\varphi((\overset{*}{\nabla}_U\varphi)(V)),Z) \\ & + g(\varphi((\overset{*}{\nabla}_V\varphi)(U)),Z).\end{aligned} \tag{3.3}$$

Taking into account of (2.5) and (2.22) in Eq. (3.3), we obtain

$$\overset{*}{\mathcal{N}}(U,V) = 2g(\varphi U,V)\xi. \tag{3.4}$$

Using Eqs. (3.2) and (3.4), one has

$$\overset{*}{\mathcal{N}}(U,V) - \overset{*}{d}\eta(U,V)\xi = 0. \tag{3.5}$$

The proof comes from (3.5).
Hence, one has the corollary given below:

Corollary 3.2 α-cosymplectic manifolds admitting a (NSNM) connection $\overset{*}{\nabla}$ satisfies the Eq. $\overset{*}{\mathcal{N}}(\mathrm{U},\mathrm{V}) = \overset{*}{\mathrm{T}}(\mathrm{U},\mathrm{V})$.

Lemma 3.3 On an α-cosymplectic manifold with a (NSNM) connection $\overset{*}{\nabla}$, scalar curvature is invariant for a (NSNM) connection $\overset{*}{\nabla}$ and Levi-Civita connection ∇.

Proof. Using Eqs. (2.1), (2.3), (2.4) and (2.5) in (2.28), we get

$$\overset{*}{R}(U,V)Z = R(U,V)Z + 2\alpha g(\varphi U,V)\eta(Z)\xi + \alpha g(\varphi V,Z)U - \alpha g(\varphi U,Z)V. \quad (3.6)$$

When Eq. (3.6) is contracted with respect to U, one has

$$\overset{*}{S}(V,Z) = S(V,Z) + 2n\alpha g(\varphi V,Z). \quad (3.7)$$

With the help of Eq. (2.13), Eq. (3.7) becomes

$$\overset{*}{Q}(V) = Q(V) + 2n\alpha(\varphi V). \quad (3.8)$$

Again contracting Eq. (3.7), we obtain

$$\overset{*}{r} = r, \quad (3.9)$$

in which $\overset{*}{S}$; S, $\overset{*}{Q}$; Q and $\overset{*}{r}$; r are the Ricci tensors, Ricci operators and scalar curvatures of the (NSNM) connection $\overset{*}{\nabla}$ and Levi-Civita connection ∇.

By replacing $U = \xi$ in the Eq. (3.6) and utilizing Eqs. (2.1), (2.3) together with Eq. (2.9), one obtains

$$\overset{*}{R}(\xi,V)Z = \alpha^2[\eta(Z)V - g(V,Z)\xi] + \alpha g(\varphi V,Z)\xi. \quad (3.10)$$

Again by replacing $Z = \xi$ in the Eq. (3.6) and using Eq. (2.8), we have

$$\overset{*}{R}(U,V)\xi = \alpha^2[\eta(U)V - \eta(V)U] + 2\alpha g(\varphi U,V)\xi. \quad (3.11)$$

In view of Eqs. (2.1), (3.6) and $g(R(U,V,Z),W)$=-g(R(U,V,W),Z), we obtain

$$\eta(\overset{*}{R}(U,V)Z) = \alpha^2[g(U,Z)\eta(V) - g(V,Z)\eta(U)] - 2\alpha g(\varphi U,V)\eta(Z). \quad (3.12)$$

When Eq. (3.11) is contracted with respect to U, one obtains

$$\overset{*}{S}(V,\xi) = -2n\alpha^2\eta(V). \quad (3.13)$$

Theorem 3.4 In an α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$, the necessary and sufficient condition for the Ricci tensor $\overset{*}{S}$ of $\overset{*}{\nabla}$ to be skew-symmetric is that the manifold is Ricci-flat.

Proof. When Eqs. (2.3), (3.7) and S(U,V)=S(V,U) are taken into consideration, one has

$$\overset{*}{S}(U,V) + \overset{*}{S}(V,U) = 2S(U,V). \quad (3.14)$$

One can prove this theorem in a clear way by the help of Eq. (3.14).
When Eqs. (2.3), (3.7) together with the equality $S(U,V) = S(V,U)$, one obtains

$$\overset{*}{S}(U,V) - \overset{*}{S}(V,U) = 4n\alpha g(\varphi U, V). \tag{3.15}$$

In view of Eq. (3.15), we can state the following corollary.

Corollary 3.5 In an α-cosymplectic manifold of dimension ≥ 3 and with $\alpha \neq 0$ admitting the (NSNM) connection $\overset{*}{\nabla}$, the Ricci tensor $\overset{*}{S}$ of $\overset{*}{\nabla}$ is non-symmetric.

Theorem 3.6 Let Riemmanian curvature tensor of $\overset{*}{\nabla}$ in an α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$ vanishes, then the manifold is an Einstein manifold.

Proof. On taking $\overset{*}{R}(U,V)Z = 0$ in Eq. (3.6), we have

$$R(U,V)Z = -2\alpha g(\varphi U, V)\eta(Z)\xi - \alpha g(\varphi V, Z)U + \alpha g(\varphi U, Z)V. \tag{3.16}$$

When Eq. (3.16) is contracted with respect to U, one obtains

$$S(V,Z) = -2n\alpha g(\varphi V, Z). \tag{3.17}$$

The proof comes from Eq. (3.17).
Hence, one has the corollary given below:

Corollary 3.7 If Riemmanian curvature tensor of $\overset{*}{\nabla}$ in an α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$ vanishes and $\alpha = 0$ in Eq. (3.17), then the manifold is Ricci-flat.

Theorem 3.8 In an α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$, the necessary and sufficient conditions for the conformal curvature tensor of $\overset{*}{\nabla}$ coincides with those of ∇ is that the conharmonic curvature tensore of $\overset{*}{\nabla}$ is equal to that of ∇.

Proof. The conformal curvature tensor of $\overset{*}{\nabla}$ is defined as [24]

$$\begin{aligned}\overset{*}{C}(U,V)Z = \overset{*}{R}(U,V)Z - \frac{1}{(2n-1)}[\overset{*}{S}(V,Z)U - \overset{*}{S}(U,Z)V + g(V,Z)\overset{*}{Q}U \\ -g(U,Z)\overset{*}{Q}V] + \frac{\overset{*}{r}}{2n(2n-1)}[g(V,Z)U - g(U,Z)V].\end{aligned} \tag{3.18}$$

Using (3.6), (3.7), (3.8) and (3.9) in the Eq. (3.18), we get

$$\begin{aligned}\overset{*}{C}(U,V)Z - C(U,V)Z = 2\alpha g(\varphi U, V)\eta(Z)\xi + \alpha g(\varphi V, Z) - \alpha g(\varphi U, Z)V \\ -\frac{1}{2n-1}[2n\alpha g(\varphi V, Z)U - 2n\alpha g(\varphi U, Z)V \\ +2n\alpha g(V,Z)\varphi U - 2n\alpha g(U,Z)\varphi V]\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}C(U,V)Z = R(U,V)Z - \frac{1}{(2n-1)}[S(V,Z)U - S(U,Z)V + g(V,Z)QU \\ -g(U,Z)QV] + \frac{r}{2n(2n-1)}[g(V,Z)U - g(U,Z)V].\end{aligned} \tag{3.20}$$

One can describe the conharmonic curvature tensor $\overset{*}{\nabla}$ as follows [24]

$$\overset{*}{L}(U,V)Z = \overset{*}{R}(U,V)Z - \frac{1}{(2n-1)}[\overset{*}{S}(V,Z)U - \overset{*}{S}(U,Z)V + g(V,Z)\overset{*}{Q}U - g(U,Z)\overset{*}{Q}V]. \tag{3.21}$$

Using (3.6), (3.7), (3.8) and (3.9) in the Eq. (3.21), we have

$$\overset{*}{L}(U,V)Z - L(U,V)Z = 2\alpha g(\varphi U,V)\eta(Z)\xi + \alpha g(\varphi V,Z) - \alpha g(\varphi U,Z)V - \frac{1}{2n-1}[2n\alpha g(\varphi V,Z)U - 2n\alpha g(\varphi U,Z)V + 2n\alpha g(V,Z)\varphi U - 2n\alpha g(U,Z)\varphi V] \tag{3.22}$$

where

$$L(U,V)Z = R(U,V)Z - \frac{1}{(2n-1)}[S(V,Z)U - S(U,Z)V + g(V,Z)QU - g(U,Z)QV]. \tag{3.23}$$

One can prove this theorem in a clear way by the help of Eqs. (3.19) and (3.22).

Theorem 3.9 In an α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$, the necessary and sufficient condition for the concircular curvature tensor coincides with curvature tensor is scalar curvature of $\overset{*}{\nabla}$ to be zero.

Proof. The concircular curvature tensor [25] of a Riemannian manifold is described by the following formula

$$\mathcal{Z}(U,V)Z = R(U,V)Z - \frac{r}{2n(2n+1)}[g(V,Z)U - g(U,Z)V]. \tag{3.24}$$

One can prove the theorem easily by using Eqs. (3.9) and (3.24) together with $\overset{*}{r} = r$.

4. Semi-symmetric α-cosymplectic manifolds admitting the (NSNM)-connection $\overset{*}{\nabla}$

In this section, semi-symmetric α-cosymplectic manifolds admitting the (NSNM) connection $\overset{*}{\nabla}$ are going to be investigated.

Definition 4.1 A $(2n+1)$-dimensional contact metric manifold M with a (NSNM) connection is known as semi-symmetric if $(\overset{*}{R}(U,V).\overset{*}{R})(Z,W)F = 0$ [24].

Theorem 4.2 A semi-symmetric α-cosymplectic manifold admitting (NSNM) connection $\overset{*}{\nabla}$ is a cosymplectic manifold.

Proof. Suppose M be a semi-symmetric contact metric manifold admitting the (NSNM) connection $\overset{*}{\nabla}$. Which implies

$$\left(\overset{*}{R}(U,V)\overset{*}{R}\right)(Z,W)F-\overset{*}{R}\left(\overset{*}{R}(U,V)Z,W\right)F-\overset{*}{R}\left(Z,\overset{*}{R}(U,V)W\right)F \qquad (4.1)$$
$$-\overset{*}{R}(Z,W)\overset{*}{R}(U,V)F=0.$$

By changing $U=\xi$ in (4.1) and with the help of (2.1), (2.2), (3.10), (3.12) Eqs. on simplification, we get

$$\begin{gathered}\alpha^2\overset{*}{R}(Z,W,F,V)-\alpha g(\overset{*}{R}(Z,W)F,\varphi V)-3\alpha^3 g(\varphi V,W)\eta(F)\eta(Z)\\ +\alpha^4 g(V,Z)g(W,F)-\alpha^3 g(V,Z)g(\varphi W,F)-2\alpha^3 g(\varphi Z,V)\eta(F)\eta(W)\\ -\alpha^4 g(V,W)g(Z,F)+\alpha^3 g(W,V)g(\varphi Z,F)+\alpha^3 g(\varphi V,W)g(Z,F)\\ -\alpha^2 g(\varphi V,W)g(\varphi Z,F)-2\alpha^3 g(V,F)g(\varphi Z,W)-2\alpha^2 g(\varphi V,F)g(\varphi Z,W)=0.\end{gathered} \qquad (4.2)$$

Putting $V=\xi$ in Eq. (4.2) and making the necessary simplifications, we obtain

$$\alpha^2\overset{*}{R}(Z,W,F,\xi)+\alpha^4\eta(Z)g(W,F)-\alpha^3\eta(Z)g(\varphi W,F) \qquad (4.3)$$
$$-\alpha^4\eta(W)g(Z,F)+\alpha^3\eta(W)g(\varphi Z,F)-2\alpha^3\eta(F)g(\varphi Z,W)=0.$$

On contracting Z and F in (4.3), we have

$$\alpha^2[\overset{*}{S}(W,\xi)-(2n+2)\alpha^2\eta(W)]=0. \qquad (4.4)$$

Putting (2.11), (3.13) in (4.4) and by means of simplification, one obtains

$$\alpha^4(2n+1)\eta(W)=0. \qquad (4.5)$$

The Eq. (4.5) implies that either $\alpha=0$ or $n=\frac{-1}{2}$ (this contradicts $n>1$).
Thus from (4.5), one achieves the proof.

Corollary 4.3 A semi-symmetric α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$ is locally the Riemannian product of an almost Kaehler manifold with the real line.

5. Ricci semi-symmetric α-cosymplectic manifolds admitting (NSNM)-connection $\overset{*}{\nabla}$

In this section, Ricci-symmetric α-cosymplectic manifolds admitting the (NSNM) connection $\overset{*}{\nabla}$ are going to be investigated.

Definition 5.1 A $(2n+1)$-dimensional contact metric manifold M with the (NSNM) connection is said to be Ricci semi-symmetric if $\overset{*}{R}(U,V).\overset{*}{S}$ vanish identically [24].

Theorem 5.2 A Ricci semi-symmetric α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$ is an Einstein manifold.

Proof. In a $(2n+1)$-dimensional Ricci semi-symmetric contact metric manifold M having a (NSNM) connection, we have

$$\overset{*}{S}(\overset{*}{R}(U,V)Z,W)+\overset{*}{S}(Z,\overset{*}{R}(U,V)W)=0. \tag{5.1}$$

By changing $Z=\xi$ and with the help of Eqs. (3.11), (3.13) on simplification, we have

$$\alpha^2\eta(U)\overset{*}{S}(V,W)-\alpha^2\eta(V)\overset{*}{S}(U,W)-2n\alpha^2 g(\varphi U,V)\eta(W)-2n\alpha^4 g(U,W)\eta(V) \\ +2n\alpha^4 g(V,W)\eta(U)+4n\alpha^3 g(\varphi U,V)\eta(W)=0. \tag{5.2}$$

Taking $V=\xi$ in Eq. (5.2) with $\alpha\neq 0$ and using Eq. (3.13), we get

$$\overset{*}{S}(U,W)=-2n\alpha^2 g(U,W). \tag{5.3}$$

As the Ricci curvature tensor of an α-cosymplectic manifold admitting the (NSNM) connection is invariant with respect to the (NSNM) connection $\overset{*}{\nabla}$ and Levi-Civita connection ∇, one may say

$$S(U,W)=-2n\alpha^2 g(U,W). \tag{5.4}$$

Thus the proof is complete.

Ricci soliton of data (g,V,λ) is defined by the Eq. (2.15), where g,V and λ are Riemannian metric, a vector field and real constant. Naturally two situations appear regarding the vector field $V: V\in Span(\xi)$ and $V\perp Span(\xi)$. Here we discuss first case that is $V\in Span(\xi)$. Ricci soliton of data (g,ξ,λ) on an α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$ can be defined as under:

$$\left(\overset{*}{\mathfrak{L}}_\xi g\right)(U,V)+2\overset{*}{S}(U,V)+2\lambda g(U,V)=0, \tag{5.5}$$

for all $U,V\in TM$. Here $\overset{*}{\mathfrak{L}}_\xi g$ is the Lie-derivative of Riemmanian metric g with respect to ξ admitting the (NSNM) connection $\overset{*}{\nabla}$.

Theorem 5.3 On a Ricci semi-symmetric α-cosymplectic manifold admitting the (NSNM) connection $\overset{*}{\nabla}$, the Ricci soliton of data (g,ξ,λ) is steady, expanding as $\alpha=0$, $\alpha\neq 0$, respectively.

Proof. $\overset{*}{\mathfrak{L}}_\xi g$ is defined as

$$\begin{aligned}\overset{*}{\mathfrak{L}}_\xi g(U,V)&=\xi g(U,V)-g(\overset{*}{\mathfrak{L}}_\xi U,V)-g(U,\overset{*}{\mathfrak{L}}_\xi V\\ &=\xi g(U,V)+g\left(\overset{*}{\nabla}_\xi U,V\right)+g\left(\overset{*}{\nabla}_\xi V,U\right)-g\left(\overset{*}{\nabla}_U\xi,V\right)-g\left(\overset{*}{\nabla}_V\xi,U\right)\\ &=\left(\overset{*}{\nabla}_\xi g\right)(U,V)-g\left(\overset{*}{\nabla}_U\xi,V\right)-g\left(\overset{*}{\nabla}_V\xi,U\right).\end{aligned} \tag{5.6}$$

Taking into account of (2.4), (2.21) and (2.25) in Eq. (5.6), we have

$$\overset{*}{\mathfrak{L}}_\xi g(U,V)=-2\alpha g(\varphi U,\varphi V). \tag{5.7}$$

Making use of (5.7) in (5.5), we get

$$-2\alpha g(\varphi U,\varphi V)+2\overset{*}{S}(U,V)+2\lambda g(U,V)=0. \tag{5.8}$$

By utilizing Eq. (5.3) in Eq. (5.8), we obtain

$$-2\alpha g(\varphi U,\varphi V)-4n\alpha^2 g(U,V)+2\lambda g(U,V)=0. \tag{5.9}$$

Using $U=V=\xi$ in (5.9), one has

$$\lambda=2n\alpha^2. \tag{5.10}$$

The proof comes from (5.10).

6. Conclusion

In this article, firstly, basic definitions and propositions of the α-cosymplectic manifolds and (NSNM) connection are given. Then, several results have been obtained on Ricci semi-symmetric and semi-symmetric α-cosymplectic manifolds with respect to (NSNM) connection. In addition to, Ricci solitons are investigated for α-cosymplectic manifolds with non-symmetric non-metric connection. The works on this subject will be useful tools for the applications of contact geometry with different connections.

References

[1] Ageshe, N. S. and Chafle, M. R., "A semi-symmetric non-metric connection on a Riemannian manifold", Indian J. Pure Appl. Math., 23 (1992) : 339-409.

[2] Aktan, N., Yıldırım, M., Murathan, C., "Almost f-cosymplectic manifolds", Mediterr J. Math., 11(2) (2014) : 775-787.

[3] Bagewadi, C. S. and Ingalahalli, G., "Ricci solitons in Lorentzian α-Sasakian manifolds", Acta Math. Acad. P. N., Vol-28 (2012) : 59-68.

[4] Barman, A. and De, U. C., "Semi-symmetric non-metric connections on Kenmotsu manifolds", Romanian J. Math. Comp. Sci., 5 (2014) : 13-24.

[5] Beyendi, S., Ayar, G., Aktan, N, "On a type of α-cosymplectic manifolds, Commun", Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68 (2019) : 852-861.

[6] Beyendi, S., Yıldırım, M., "On generalized weakly symmetric α-cosymplectic manifolds", Hacet. J. Math. Stat, Vol: 50 (6) (2021) : 1745-1755.

[7] Blair, D. E., "Contact manifolds in Riemannian geometry", Lecture Notes in Math. 509, Springer-Verlag, Berlin, (1976).

[8] Chaubey, S. K., "On semi-symmetric non-metric connection", Prog. of Math., 41-42 (2007) : 11-20.

[9] Chaubey, S. K., "Almost contact metric manifolds admitting semisymmetric non-metric connection", Bulletin of Mathematical Analysis and Applications, 3(2) (2011) : 552-560.

[10] Chaubey, S. K. and Ojha, R. H., "On a semi-symmetric non-metric connection", Filomat, Vol-26:2 (2012) : 63-69.

[11] Chaubey, S. K. and Pandey, A. C., "Some properties of a semisymmetric Non-metric connection on a Sasakian manifold", Int. J. Contemp. Math. Science, Vol-8, No. 16 (2013) : 789-799.
[12] De, U. C. and Biswas, S. C., "On a type of semi-symmetric non-metric connection on a Riemannian manifold", Ganita, 48 (1997) : 91-94.
[13] Erken, I. K., "On a classification of almost α-cosymplectic manifolds", Khayyam J. Math., 5 (2019) : 1-10.
[14] Friedmann, A. and Schouten, J. A., "Uber die geometrie der halbsymmetrischen ubertragung", Math. Zeitschr. 21 (1924) : 211-223.
[15] Goldberg, S. I. and Yano, K., "Integrability of almost cosymplectic structure", Pac. J. Math., 31 (1969) : 373-381.
[16] Hayden, H. A., "Subspace of a space with torsion", Proc. London Math. Soc., 34 (1932) : 27– 50.
[17] Haseeb, A., "On concircular curvature tensor with respect to the semi-symmetric non-metric connection in a Kenmotsu manifold", Kyungpook Math. J., 56 (2016) : 951-964.
[18] Kenmotsu, K., "A class of almost contact Riemannian manifolds", Tohoku Math. J., 24(1) (1972) : 93-103.
[19] Kim, T. W., Pak, H. K., "Canonical foliations of certain classes of almost contact metric structures", Acta Math, Sinica, Eng. Ser. Aug., 21(4) (2005) : 841-846.
[20] Lie, S., "Geometrie der Berührungstransformationen", B.G. Teubner, Leipzig, (1896).
[21] Olszak, Z., "On almost cosymplectic manifolds", Kodai Math., 4 (2) (1981) : 239- 250.
[22] Öztürk, H., Murathan, C., Aktan, N., Vanli, A. T., "Almost α-cosymplectic f-manifolds", An. Stiint. Univ. Al. I. Cuza Iasi Mat., (N.S.) 60(1) (2014) : 211–226.
[23] Pankaj, Chaubey, S. K. and Prasad, R., "Trans-Sasakian manifolds with respect to a non-symmetric nonmetric connection", Global Journal of Advanced Research on Classical and Modern Geometries, Vol.7 (2018) : Issue 1, 1-10.
[24] Pankaj, Chaubey, S. K. and Prasad, R., "Sasakian manifolds admitting a non-symmetric non-metric connection", Palestina journal of mathematic, Vol. 9(2) (2020) : 698-710.
[25] Prasad, R. and Pankaj, "Some curvature tensors on a trans-Sasakian manifold with respect to semisymmetric non-metric connection", J.Nat. Acad. Math, Sp. Volume (2009) : 55-64.
[26] Schouten, J. A., "Ricci-calculus", Springer-Verlag. Berlin, 1954.
[27] Singh, A., Mishra, C. K., Kumar, L. And Patel, S., "Characterization of Kenmotsu manifolds admitting a non-symmetric non-metric connection", J. Int. Acad. Phys. Sci., Vol. 26, No. 3 (2022) : 265-274.
[28] Vaisman, K., "Conformal changes of almost contact metric manifolds", In Lecture Notes in Mathematics 792, Springer, Berlin, (1980) : 435-443.
[29] Yano, K., "On semisymmetric metric connections", Rev. Roumaine Math. Press Apple. 15 (1970) : 1579-1586.