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A CHARACTERIZATION OF GENERAL LINEAR TRANSFORMATIONS BY USE OF SYMMEDIAN POINTS OF TRIANGLES

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Abstract

In this study, based on a geometric approach, we obtained a new characteristic of general linear transformations, also known as similarities, by use of symmedian points of triangles in complex z −plane.

Keywords: General linear transformations, Möbius transformations, symmedian point of triangle

1. Introduction

Many important transformations have been characterized in both Euclidean and non-Euclidean geometry with the help of functions that preserve triangles or some special points of triangles. For example, in [1], Li and Wang proved that if $f: \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is a bijection that preserves lines, then f is an affine transformation. Moreover, the authors showed that a bijection $f: \mathbb{R}^n \to \mathbb{R}^n$ $(n > 1)$ is an affine transformation if and only if f is triangle preserving. In [2], J. Lester proved that if $f: X \to X$ preserves triangles with area 1, where X is a finite dimensional real inner product space, then f must be in the form $f(x) = w(x) + t$, where $w(x)$ is an orthogonal transformation and t is a fixed element of X. Later, this result of Lester was examined by W.Benz [4] in non-finite dimensional real inner product spaces. In [2] and [3], J. Lester considered the transformations that preserve triangles of perimeter 1 and proved that these transformations are Euclidean motions. In [5], O. Demirel proved that if a mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ preserves the Fermat-Torricelli points of the triangles in Δ, where Δ be the set of all triple points $\{A, B, C\}$ in $\mathbb{R}^n$ such that the largest angle of the triangle ABC is less than $2\pi/3$, then f is an affine transformation. O. Demirel et al., in [6], gave a new characteristic of general linear transformations (similarities or affine transformations) in complex z −plane by use of the Steinhaus' Problem on partition of a triangle. In complex analysis and geometry, a Möbius transformation (fractional linear transformation) of the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is a rational function of the form

$$f(z) = \frac{az+b}{cz+d}$$

of one complex variable z; here the coefficients a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. Möbius transformations have a very important role in both complex analysis and hyperbolic geometry. The set of all Möbius transformations forms a group under composition and cross-ratios are invariant under Möbius transformations. That is, if a Möbius transformation maps four distinct points z_1, z_2, z_3, z_4 to four distinct points w_1, w_2, w_3, w_4 respectively, then

$$\frac{z_1 - z_3}{z_2 - z_3}\frac{z_2 - z_4}{z_1 - z_4} = \frac{w_1 - w_3}{w_2 - w_3}\frac{w_2 - w_4}{w_1 - w_4}$$

Möbius transformations preserve angles, map every straight line to a line or circle, and map every circle to a line or circle. Translations ($z \longmapsto z + d$), rotations about origin ($z \longmapsto e^{i\theta}z$), stretch transformations ($z \longmapsto az,\ a \in \mathbb{R}^*, a \neq 1$) , inversions ($z \longmapsto \frac{1}{z}$) and general linear transformation ($z \longmapsto az + b\ , a \neq 0$) are the most basic examples of Möbius transformations.

In this study, we discuss the transformations that preserve the symmedian points of triangles in the complex z −plane. Symmedians are three particular lines associated with every triangle and they are constructed by taking a median of the triangle and reflecting the line over the corresponding angle bisector. More precisely, for a triangle, say ABC, if AA_1 is the median of BC, then the symmetry of AA_1 with respect to the bisector of A (the isogonal conjugate of AA_1) is a symmedian. Clearly the angle formed by the symmedian and the angle bisector has the same measure as the angle between the median and the angle bisector, but it is on the other side of the angle bisector. The isogonal conjugates of the medians in a triangle are called symmedian. Symmedians meet at a single point and this point is called the symmedian point of the triangle (Lemoine point or Grebe point). Many important results can be found in the literature with the help of this point. Some of these features are as follows, but they are not limited to this.

If K is the symmedian point of ABC, then K is the isogonal conjugate of the centroid of ABC. The isogonal conjugate of a point P with respect to ABC is constructed by reflecting the lines PA, PB, PC about the angle bisectors of A,B, C respectively. For the isogonal conjugate AS_a of the median AM_a in a triangle ABC (where point S_a is on side BC), S_a divides side BC in the ratio

$$\frac{BS_a}{S_aC} = \frac{c^2}{b^2}$$

It follows from *Steinter's Ratio Theorem* that the only point X on the side BC of ABC that has the property

$$\frac{BX}{CX} = \frac{c^2}{b^2}$$

is the foot S_a of the symmedian through vertex A. The distance from the symmedian point to the side lines of a triangle is proportional to the corresponding side lengths. This property uniquely determines the symmedian point. If $ABDE$ and $BCFG$ are constructed outside the triangle ABC as in the *Figure 1* below and O is the center of the circle (BDG), then the line BO passes through the symmedian point K of the triangle ABC. This feature will be very useful in the proof of our main theorem.

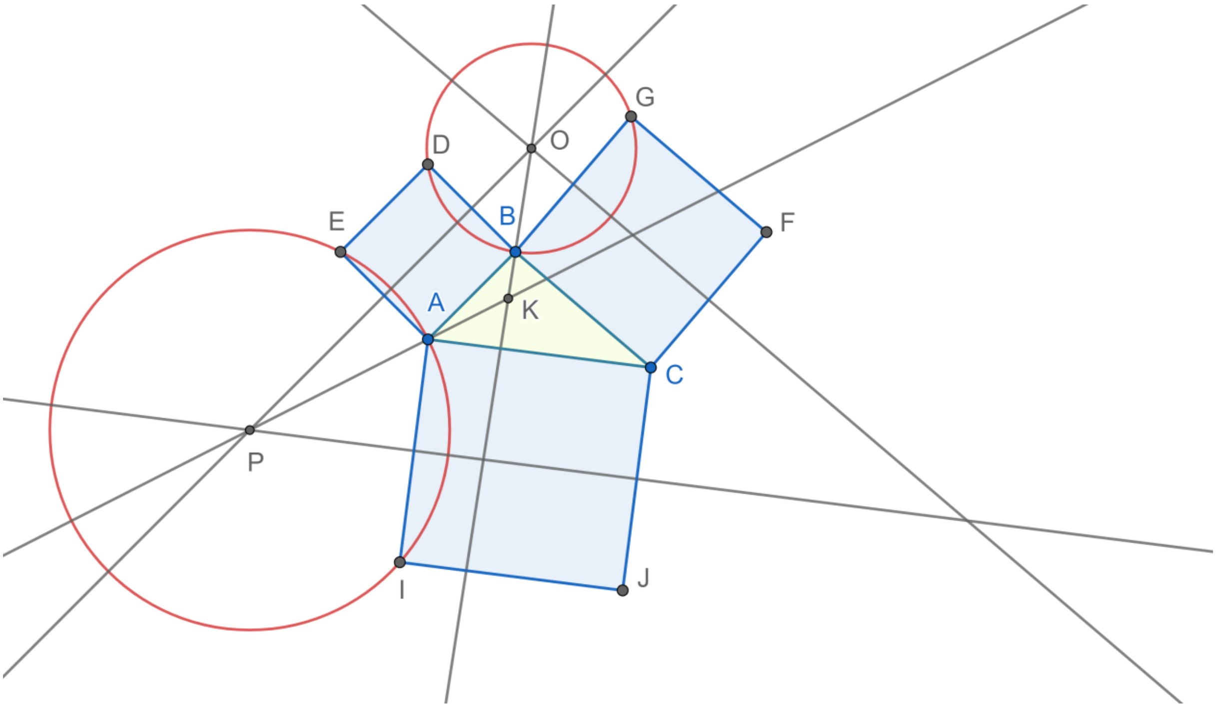

Figure 1. A geometric way to obtain the symmedian point of ABC by using squares whose sides are sides of *ABC*.

For other properties related to symmedian points, we recommend [7] to the readers.

2. Main results

This work was inspired by the excellent work [8] where H. Haruki and T.M. Rassias obtained a new characterization of Möbius transformations by using Apollonius points of triangles in complex z −plane.

Throughout the paper we denote by A' the image of A under f, by AB the geodesic segment between points A and B, by $|AB|$ the distance between points A and B, by ABC the triangle with three ordered vertices A, B and C, and by $\angle BAC$ the angle between AB and AC. Unless otherwise stated, we consider $w = f(z)$ as a nonconstant meromorphic function of a complex variable z in the plane $|z| < \infty$.

Property Sym: Let $w = f(z)$ be an analytic and univalent function in a non-empty region R in the complex plane, and let ABC be an arbitrary triangle in R. If L is the symmedian point of ABC, then L' is the symmedian point of $A'B'C'$.

Lemma 2.1: ([9]) If $w = f(z)$ is an analytic and univalent function in a non-empty region R in the complex plane , then $f'(z) \neq 0$.

Theorem 2.2: $w = f(z)$ has *Property Sym* if and only if $w = f(z)$ is a general linear transformations.

Proof: Let $w = f(z)$ be a similarity defined by $f(z) = az + b$ satisfying $a, b \in \mathbb{C}$ with $a \neq 0$ and L be the symmedian point of an arbitrary triangle ABC. Clearly,

$$f\left(\frac{x+y}{2}\right) = a\left(\frac{x+y}{2}\right) + b = \frac{ax+b}{2} + \frac{ay+b}{2} = \frac{f(x)+f(y)}{2}$$

holds for all $x, y \in \mathbb{C}$ with $x \neq y$. Hence f preserves the midpoints of the line segments. Since f preserves the angles one can clearly get that L' is the symmedian point of triangle $A'B'C'$. Here, f preserves angles, which means that for every point z_0 of the complex $z-$plane, $\text{angle}(r_1, r_2) = \text{angle}(f(r_1), f(r_2))$ holds for every smooth curves r_1 and r_2 meeting at z_0.

Now assume that $w = f(z)$ has *Property Sym*. Since $w = f(z)$ is analytic and univalent in the domain R, by $Lemma$ 2.1

$$f'(z) \neq 0$$

holds for all z in R. Let L be an arbitrary point in R and denote it by x. Hence we get $f'(x) \neq 0$ by $Lemma$ 2.1. Because of $L \in R$, there exists a positive real number δ such that $V(L, \delta)$ is contained in R, where $V(L, \delta)$ is δ-closed circular neighborhood of L. Throughout the proof let ABC denote an arbitrary equilateral triangle which is contained in $V(L, \delta)$ and whose center is at L. Since ABC is an equilateral triangle contained in $V(L, \delta)$, the points A, B, C can be represented by

$$A = x + y, \qquad B = x + wy, \qquad C = x + w^2 y$$

where $w = \frac{-1+\sqrt{3}i}{2}$ and $|y| \leq \delta$. Since $w = f(z)$ is univalent in R, the points A', B', C' are different from each other. Because of A, B, C are not collinear points on the $z-$plane and by the property of analytic functions [10], there exists some sufficiently small positive real number s satisfying $s \leq \delta$ such that A', B', C' are not collinear on the $w-$plane for all y satisfying $0 < |y| \leq s$. It is clear that L is the symmedian point of ABC and by the hypothesis, L' is the symmedian point of $A'B'C$'. Now construct the squares $A'B'SD, A'C'GE, B'C'JT$ with the help of the sides of $A'B'C'$, as shown in the *Figure 2*. Notice that $A'D$ is obtained by rotating $A'B'$ by $\pi/2$ radians in the negative direction (moved in a clockwise motion) around point A'. Similarly, $A'E$ is obtained by rotating $A'C'$ in a positive direction (moved in a counterclockwise motion) around point A' and $B'T$ is obtained by rotating $B'C'$ in a negative direction around point B' by $\pi/2$ radians.

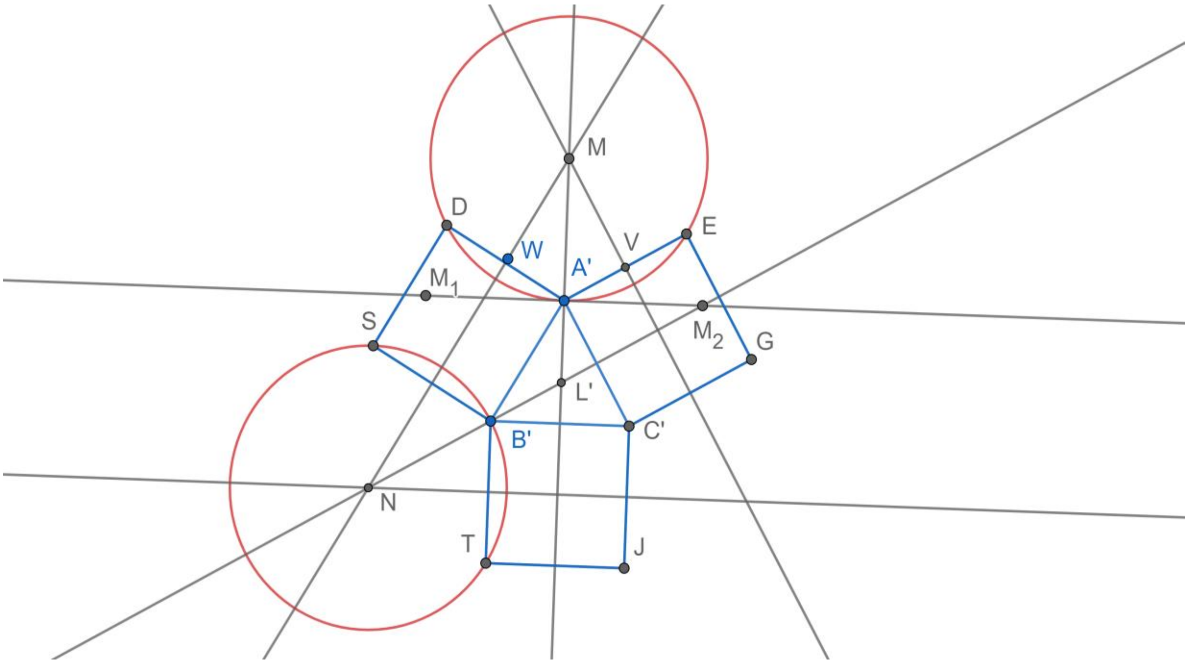

Figure 2. A geometric way to obtain the symmedian point of $A'B'C'$ by using squares whose sides are sides of $A'B'C'$.

Assume

$$W = \frac{A' + D}{2}, \qquad V = \frac{A' + E}{2}$$

and M be the center of the circle passing through A', D, E. By the property of symmedian point, L' should lie on MA'. The equations of the lines MW and MV are

$$|z - D| = |z - A'| \tag{1}$$

and

$$|z - E| = |z - A'| \tag{2}$$

respectively. Notice that in the last two equations above, z represents the complex variable. Since M is the common point of the lines MW and MV

$$|M - D = |M - E| \tag{3}$$

holds. Clearly

$$A' = f(x + y), \qquad B' = f(x + wy), \qquad C' = f(x + w^2 y)$$

and

$$D = (B' - A')(-i) + A' = \big(f(x + wy) - f(x + y)\big)(-i) + f(x + y),$$

$$E = (C' - A')i + A' = \big(f(x + w^2y) - f(x + y)\big)i + f(x + y)$$

holds. Now, we want to get the equation of the line passing through the points M and A'. Let M_1 be the point obtained by rotating M around A' by $\pi/2$ radians in the positive direction, and let M_2 be the point obtained by rotating M around A' by $\pi/2$ radians in the negative direction. In this case, one can easily get

$$M_1 = (M - A')e^{\frac{i\pi}{2}} + A' = (M - A')i + A'$$

and

$$M_2 = (M - A')e^{\frac{-i\pi}{2}} + A' = (M - A')(-i) + A'.$$

The equation of the line MA' is

$$|z - M_1| = |z - M_2|$$

i.e.

$$|z - Mi + A'i - A'| = |z + Mi - A'i - A'|. \tag{4}$$

Let N be the center of the circle passing through the points S, B', T. Obviously, L' should lie on NB' and the equation of NB' is

$$|z - Ni + B'i - B'| = |z + Ni - B'i - B'| \tag{5}$$

similar to (4). From (3), we get

$$\left|\frac{M - D}{M - E}\right| = 1$$

and this implies there exist $\theta \in \mathbb{R}$ with $\theta \neq 2k\pi,\ (k \in \mathbb{Z})$ such that

$$M - D = (M - E)e^{i\theta}$$

holds. Thus we have

$$M = \frac{D - e^{i\theta}E}{1 - e^{i\theta}}.$$

Following the same way, one can easily see that there exist $\alpha \in \mathbb{R}$ with $\alpha \neq 2k\pi, (k \in \mathbb{Z})$ such that

$$N = \frac{T - e^{i\alpha}S}{1 - e^{i\alpha}}.$$

By (4) and (5), we get

$$\left|\frac{z - Mi + A'i - A'}{z + Mi - A'i - A'} \cdot \frac{z + Ni - B'i - B'}{z - Ni + B'i - B'}\right| = 1. \tag{6}$$

$L' = f(x)$ satisfies equation (6). Now define

$$g(y) = \frac{z - Mi + A'i - A'}{z + Mi - A'i - A'} \cdot \frac{z + Ni - B'i - B'}{z - Ni + B'i - B'} \tag{7}$$

and it is clear from (6), $|\boldsymbol{g(y)}| = \mathbf{1 \text{ in } (0 < |y| \leq s)}$.

Since the numerator and the denominator of $g(y)$ in (7) are analytic functions for all y satisfying $0 < |y| \leq s$ and since, by the fact that $w = f(z)$ is univalent in R, the denominator of $g(y)$ in (7) never vanishes in $0 < |y| \leq s$, $g(y)$ is analytic in $0 < |y| \leq s$. Now we prove that $g(y)$ is analytic at $y = 0$. As $y \to 0$, by L'Hôpital's rule, we see that

$$\frac{f(x) - \dfrac{(D - e^{i\theta}E)i}{1 - e^{i\theta}} + f(x+y)i - f(x+y)}{f(x) + \dfrac{(D - e^{i\theta}E)i}{1 - e^{i\theta}} - f(x+y)i - f(x+y)} \longrightarrow -\frac{e^{i\theta}w^2 + w - 2e^{i\theta}}{e^{i\theta}w^2 + w - 2} \tag{8}$$

and

$$\frac{f(x) + \dfrac{(T - e^{i\alpha}S)i}{1 - e^{i\alpha}} - f(x+y)i - f(x+wy)}{f(x) - \dfrac{(T - e^{i\alpha}S)i}{1 - e^{i\alpha}} + f(x+y)i - f(x+wy)} \longrightarrow -\frac{w^2 - 2w + e^{i\alpha}}{w^2 - 2we^{i\alpha} + e^{i\alpha}} \tag{9}$$

hold true since

$D = (B' - A')(-i) + A' = \big(f(x+wy) - f(x+y)\big)(-i) + f(x+y)$,
$E = (C' - A')i + A' = \big(f(x+w^2y) - f(x+y)\big)i + f(x+y)$
$T = (C' - B')(-i) + B' = \big(f(x+w^2y) - f(x+wy)\big)(-i) + f(x+wy)$,
$S = (A' - B')i + B' = \big(f(x+y) - f(x+wy)\big)i + f(x+wy)$.

From (7), (8) and (9), as $y \to 0$, we get

$$g(y) \longrightarrow -\frac{e^{i\theta}w^2 + w - 2e^{i\theta}}{e^{i\theta}w^2 + w - 2} \cdot \left(-\frac{w^2 - 2w + e^{i\alpha}}{w^2 - 2we^{i\alpha} + e^{i\alpha}}\right) = q. \tag{10}$$

If we define $g(0) = q$, by (10) and by Riemann's theorem on removable singularities, the function $g(y)$ is analytic at $y = 0$. Moreover, from $g(0) = q$, the equailty $|g(y)| = 1$ is still satisfied at $y = 0$. By the maximum module principle for analytic functions [10], we obtain $g(y) = K$ in $|y| \leq s$, where K is a complex constant with modulus 1. If we set $y = 0$ in $g(y) = K$, we get $K = q$. By the equations (7), $g(y) = K$, and $K = q$, we obtain

$$\left(f(x)-\frac{\left(D-e^{i\theta}E\right)i}{1-e^{i\theta}}+f(x+y)i-f(x+y)\right)\left(f(x)+\frac{(T-e^{i\alpha}S)i}{1-e^{i\alpha}}-f(x+wy)i-f(x+wy)\right)$$
$$-q\left(f(x)+\frac{\left(D-e^{i\theta}E\right)i}{1-e^{i\theta}}-f(x+y)i+f(x+y)\right)\left(f(x)-\frac{(T-e^{i\alpha}S)i}{1-e^{i\alpha}}+f(x+wy)i\right.$$
$$\left.-f(x+wy)\right)=0 \qquad (11)$$

for all y satisfying $|y| \leq s$.

Using Leibnitz's rule for differentiation with computer aided calculations, differentiating both sides of (11) three times with respect to y, setting $y = 0$, and simplifying the resulting equality yields

$$\frac{f'(x)\cdot f''(x)}{(e^{i\theta}-1)(e^{i\alpha}-1)}(p_1-p_2p_3)=0$$

where

$$p_1 = e^{i\alpha} - 2e^{i\theta} + 5e^{i\theta}e^{i\alpha} + 5,$$

$$p_2 = -\frac{\frac{7}{2}\sqrt{3}e^{i\theta}-\frac{1}{2}e^{i\alpha}-\frac{7}{2}e^{i\theta}+\frac{1}{2}i\sqrt{3}e^{i\alpha}-\frac{5}{2}e^{i\theta}e^{i\alpha}+i\sqrt{3}-\frac{\sqrt{3}}{2}ie^{i\theta}e^{i\alpha}+2}{\frac{1}{2}i\sqrt{3}e^{i\theta}-\frac{7}{2}e^{i\alpha}-\frac{1}{2}e^{i\theta}+\frac{7}{2}i\sqrt{3}e^{i\alpha}-\frac{5}{2}e^{i\theta}e^{i\alpha}+i\sqrt{3}-\frac{\sqrt{3}}{2}ie^{i\theta}e^{i\alpha}+2}$$

and

$$p_3 = 2e^{i\theta} - 13e^{i\alpha} + e^{i\theta}e^{i\alpha} + 1.$$

By the identity theorem (see [11], p. 106) the above equality is valid in$|z| < \infty$. Therefore

$$\frac{f'(z)\cdot f''(z)}{(e^{i\theta}-1)(e^{i\alpha}-1)}(p_1-p_2p_3)=0$$

holds true for all z with $f'(z) \neq 0$. Since $p_1 - p_2p_3 \neq 0$, we can easily see that $f''(z) = 0$ which implies f must be a general linear transformation and is written in the form

$$f(z) = az + b.$$

3. Conclusion

As can be seen from the results above we consider an analytic and univalent function $w = f(z)$ in a non-empty region R in the complex plane and prove that the necessary and sufficient condition for the symmedian point of an arbitrary triangle to be transformed into the symmedian point of the image of the triangle is the transformation $w = f(z)$ to be a general linear transformation.

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