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f-Asymptotically $\mathscr{I}_{\sigma\theta}$-Equivalence of Real Sequences

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Abstract

In this manuscript, we present the ideas of asymptotically $[\mathscr{I}_{\sigma\theta}]$-equivalence, asymptotically $\mathscr{I}_{\sigma\theta}(f)$-equivalence, asymptotically $[\mathscr{I}_{\sigma\theta}(f)]$-equivalence and asymptotically $\mathscr{I}(S_{\sigma\theta})$-equivalence for real sequences. In addition to, investigate some connections among these new ideas and we give some inclusion theorems about them.

1. Introduction

Before starting article, we give the basic concepts and properties of statistical convergence, ideal convergence, invariant mean and invariant convergence, asymptotically equivalence and modulus function. Throughout this study, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers. Statistical convergence and ideal convergence have recently begun to attract interest in science and engineering as well as by mathematicians. The idea of convergence of a real sequence was extended to statistical convergence by Fast [1] and Schoenberg [2] independently, and then statistical convergence has been studied by many authors. Kostyrko et al. [3] firstly, introduced the notion of $\mathscr{I}$-convergence as a generalization of statistical convergence.

Invariant convergence has recently been gaining more and more interest among mathematicians working on summability theory. Several authors including Raimi [4], Schaefer [5], Mursaleen and Edely [6], Mursaleen [7], Savaş [8–10], Nuray and Savaş [11], Pancaroğlu and Nuray [12] studied on σ-convergent sequences and some properties of σ-convergence. The notion of lacunary strong σ-convergence was defined by Savaş [10]. Then, Savaş and Nuray [13] introduced the notion of σ-statistical convergence and also, defined lacunary σ-statistical convergence and examined some inclusion theorems with examples. After that, Nuray et al. [14] defined the notions of σ-uniform density of a subset A of $\mathbb{N}$, $\mathscr{I}_\sigma$-convergence and examined connections between $\mathscr{I}_\sigma$-convergence and σ-convergence and also, $\mathscr{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Also, Pancaroğlu and Nuray [12] studied statistical lacunary σ-summability. Recently, Ulusu and Nuray [15] investigated the concepts of lacunary $\mathscr{I}_\sigma$-convergence and lacunary $\mathscr{I}_\sigma$-Cauchy sequence of real numbers.

The concept of asymptotically equivalence and applications are of interest to scientists working on convergence types. Marouf [16] peresented ideas of asymptotically equivalence. Patterson and Savaş [17, 18] denoted the ideas of asymptotically lacunary statistically equivalence and asymptotically $\sigma\theta$-statistical equivalence of real sequences. Ulusu [19, 20] studied the notion asymptotically ideal invariant equivalence and asymptotically lacunary $\mathscr{I}_\sigma$-equivalence.

Modulus function and its various applications are used in many sub-disciplines in the field of mathematics. Nakano [21] denoted the notion f modulus function. Maddox [22], Pehlivan [23] and several authors using a modulus function f, define some new concepts and give some inclusion theorems with examples. Kumar and Sharma [24] using a modulus function f, investigated lacunary $\mathscr{I}$-equivalence of real sequences.

Now, let's give some basic and important concepts, lemma and properties that are related to our work subject and we will use in our article,

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by citing the authors we give in the references (see [3, 9, 10, 14–16, 21–26]).

Let σ be a mapping such that $\sigma : \mathbb{N}^{+} \to \mathbb{N}^{+}$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞, the space of bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

1. $\psi(a_n) \geq 0$, when the sequence (a_n) has $a_n \geq 0$, for all n,
2. $\psi(e) = 1$, where $e = (1,1,1,\ldots)$ and
3. $\psi(a_{\sigma(n)}) = \psi(a_n)$ for all $(a_n) \in \ell_\infty$.

The mappings σ are supposed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $m,n \in \mathbb{N}^{+}$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(a_n) = \lim a_n$ for all $(a_n) \in c$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Throughout this study, let $\theta = \{k_r\}$ be a lacunary sequence.

The concept of lacunary strong σ-convergence was defined as below:

$$L_\theta = \left\{ a = (a_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |a_{\sigma^k(m)} - K| = 0 \right\},$$

uniformly in $m = 1,2,\ldots$.

If for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : |a_{\sigma^k(n)} - K| \geq \varepsilon \right\} \right| = 0,$$

uniformly in $n = 1,2,\ldots$, then the sequence $a = (a_k)$ is $S_{\sigma\theta}$-convergent to K.

Let $\mathscr{I}$ be a family of subsets of $2^{\mathbb{N}}$. If the following conditions holds, then we named $\mathscr{I} \subseteq 2^{\mathbb{N}}$ an ideal:

(i) $\emptyset \in \mathscr{I}$,

(ii) For any $C, D \in \mathscr{I}$, we get $C \cup D \in \mathscr{I}$,

(iii) For any $C \in \mathscr{I}$ and any $D \subseteq C$, we get $D \in \mathscr{I}$.

An ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named a non-trivial if $\mathbb{N} \notin \mathscr{I}$ and a non-trivial ideal $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is named admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$. Throughout this study, we let $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.

Let $H \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| H \cap \left\{ \sigma(n), \sigma^2(n), \ldots, \sigma^m(n) \right\} \right| \text{ and } S_m = \max_n \left| H \cap \left\{ \sigma(n), \sigma^2(n), \ldots, \sigma^m(n) \right\} \right|.$$

If the limits

$$\underline{V}(H) = \lim_{m \to \infty} \frac{s_m}{m} \text{ and } \overline{V}(H) = \lim_{m \to \infty} \frac{S_m}{m}$$

exist, then they are named a lower σ-uniform density and an upper σ-uniform density of the set H, respectively. If $\underline{V}(H) = \overline{V}(H)$, then $V(H) = \underline{V}(H) = \overline{V}(H)$ is named the σ-uniform density of H.

Denote by $\mathscr{I}_\sigma$ the class of all $H \subseteq \mathbb{N}$ with $V(H) = 0$. Obviously, $\mathscr{I}_\sigma$ is an admissible ideal in $\mathbb{N}$.

A sequence $a = (a_k)$ is told to be $\mathscr{I}_\sigma$-convergent to K if for each $\varepsilon > 0$, the set $H_\varepsilon = \left\{ k : |a_k - K| \geq \varepsilon \right\}$ belongs to $\mathscr{I}_\sigma$, i.e., $V(H_\varepsilon) = 0$. It is denoted by $\mathscr{I}_\sigma - \lim_{k \to \infty} a_k = K$.

Let $\theta = \{k_r\}$ be a lacunary sequence, $H \subseteq \mathbb{N}$ and

$$s_r = \min_n \left\{ \left| H \cap \{ \sigma^m(n) : m \in I_r \} \right| \right\} \text{ and } S_r = \max_n \left\{ \left| H \cap \{ \sigma^m(n) : m \in I_r \} \right| \right\}.$$

If the limits

$$\underline{V_\theta}(H) = \lim_{r \to \infty} \frac{s_r}{h_r} \text{ and } \overline{V_\theta}(H) = \lim_{r \to \infty} \frac{S_r}{h_r}$$

exist, then they are named a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set H, respectively. If $\underline{V_\theta}(H) = \overline{V_\theta}(H)$, then $V_\theta(H) = \underline{V_\theta}(H) = \overline{V_\theta}(H)$ is named the lacunary σ-uniform density of H.

Denoted by $\mathscr{I}_{\sigma\theta}$ the class of all $H \subseteq \mathbb{N}$ with $V_\theta(H) = 0$. Obviously, $\mathscr{I}_{\sigma\theta}$ is an admissible ideal in $\mathbb{N}$.

A sequence (a_k) is told to be lacunary $\mathscr{I}_\sigma$-convergent or $\mathscr{I}_{\sigma\theta}$-convergent to K if for each $\varepsilon > 0$, $H_\varepsilon = \left\{ k : |a_k - K| \geq \varepsilon \right\} \in \mathscr{I}_{\sigma\theta}$, i.e., $V_\theta(H_\varepsilon) = 0$. It is denoted by $\mathscr{I}_{\sigma\theta} - \lim_{k \to \infty} a_k = K$.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically equivalent if $\lim_{k\to\infty}\frac{a_k}{v_k}=1$, (denoted by $a\sim v$).

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically lacunary invariant equivalent of multiple K if

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_{\sigma^k(m)}}{v_{\sigma^k(m)}}-K\right|=0,$$

uniformly in m (denoted by $a \overset{N^K_{\sigma\theta}}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically $N_{\sigma\theta}$-equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically lacunary invariant statistical equivalent of multiple K if for each $\varepsilon>0$,

$$\lim_r \frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{a_{\sigma^k(m)}}{v_{\sigma^k(m)}}-K\right|\geq\varepsilon\right\}\right|=0,$$

uniformly in m (denoted by $a \overset{S^K_{\sigma\theta}}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically lacunary invariant statistical equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically lacunary $\mathscr{I}$-equivalent of multiple K provided that for each $\varepsilon>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}$$

(denoted by $a \overset{\mathscr{I}(N^K_\theta)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly asymptotically lacunary $\mathscr{I}$-equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically lacunary statistical $\mathscr{I}$-equivalent of multiple K if for each $\varepsilon>0$ and $\gamma>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\left|\left\{k\in I_r:\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\right|\geq\gamma\right\}\in\mathscr{I}$$

(denoted by $a \overset{\mathscr{I}(S^K_\theta)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically lacunary $\mathscr{I}$-statistical equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically $\mathscr{I}_{\sigma\theta}$-equivalent of multiple K if for each $\varepsilon>0$,

$$\widetilde{H}_\varepsilon=\left\{k\in I_r:\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta},$$

i.e., $V_\theta(\widetilde{H}_\varepsilon)=0$. It is denoted by $a \overset{\mathscr{I}^K_{\sigma\theta}}{\sim} v$. If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically $\mathscr{I}_{\sigma\theta}$-equivalent.

A function $f:[0,\infty)\to[0,\infty)$ is called a modulus if

1. $f(t)=0$ if and if only if $t=0$,
2. $f(t+v)\leq f(t)+f(v)$,
3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(t)=t^p$, $0<p<1$) or bounded (for example $f(t)=\frac{t}{t+1}$).

Throughout this study, let f be a modulus function.

Two non-negative $a=(a_k)$ and $v=(v_k)$ are told to be strongly f-asymptotically lacunary $\mathscr{I}$-equivalent of multiple K provided that for each $\varepsilon>0$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{k\in I_r}f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}$$

(denoted by $a \overset{\mathscr{I}^K(N^f_\theta)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly f-asymptotically lacunary $\mathscr{I}$-equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{[\mathscr{I}^K_\sigma]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly asymptotically $\mathscr{I}$-invariant equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be f-asymptotically $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$,

$$\left\{k\in\mathbb{N}: f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{\mathscr{I}_\sigma^K(f)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply f-asymptotically $\mathscr{I}$-invariant equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly f-asymptotically $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$,

$$\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{[\mathscr{I}_\sigma^K(f)]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly f-asymptotically $\mathscr{I}$-invariant equivalent.

Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be asymptotically $\mathscr{I}$-invariant statistical equivalent of multiple K if for each $\varepsilon>0$ and each $\gamma>0$,

$$\left\{n\in\mathbb{N}: \frac{1}{n}\left|\left\{k\leq n: \left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\right|\geq\gamma\right\}\in\mathscr{I}_\sigma$$

(denoted by $a \overset{\mathscr{I}(S_\sigma^K)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply asymptotically $\mathscr{I}$-invariant statistical equivalent.

Lemma 1.1. *[23] Let $0<\lambda<1$. Then, we have $f(s)\leq 2f(1)\lambda^{-1}s$, for each $s\geq\lambda$.*

2. Main results

Now, we give the original definitions of our article and explain the theorems that are original, together with their proofs. Our theorems give many features and necessity relations between these new concepts.

Definition 2.1. *Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly asymptotically lacunary $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$*

$$\left\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta}$$

(denoted by $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly asymptotically lacunary $\mathscr{I}$-invariant equivalent.

Definition 2.2. *Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be f-asymptotically lacunary $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$*

$$\left\{k\in I_r: f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta},$$

(denoted by $a \overset{\mathscr{I}_{\sigma\theta}^K(f)}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply f-asymptotically lacunary $\mathscr{I}$-invariant equivalent.

Definition 2.3. *Two non-negative sequences $a=(a_k)$ and $v=(v_k)$ are told to be strongly f-asymptotically lacunary $\mathscr{I}$-invariant equivalent of multiple K if for each $\varepsilon>0$*

$$\left\{r\in\mathbb{N}: \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right)\geq\varepsilon\right\}\in\mathscr{I}_{\sigma\theta}$$

(denoted by $a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$). If we let $K=1$, then $a=(a_k)$ and $v=(v_k)$ are told to be simply strongly f-asymptotically lacunary $\mathscr{I}$-invariant equivalent.

Theorem 2.4. *For two non-negative sequences $a=(a_k)$ and $v=(v_k)$ we have $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v \Rightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$.*

Proof. Let $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v$ and $\varepsilon>0$ be given. For $0\leq s\leq\lambda$, select $0<\lambda<1$ such that $f(s)<\varepsilon$. Then, we have

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right)=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|\leq\lambda}} f\left(\left|\frac{a_k}{v_k}-K\right|\right)+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|>\lambda}} f\left(\left|\frac{a_k}{v_k}-K\right|\right)$$

and so by Lemma 1.1

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right)<\varepsilon+\left(\frac{2f(1)}{\lambda}\right)\frac{1}{h_r}\sum_{k\in I_r}\left|\frac{a_k}{v_k}-K\right|.$$

Thus, for each $\gamma > 0$ we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{a_k}{v_k} - K \right| \geq \frac{(\gamma - \varepsilon)\lambda}{2f(1)} \right\}.$$

Since $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v$, the next set and so the first set in the foregoing statement pertain to $\mathscr{I}_{\sigma\theta}$. This proves that

$$a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v.$$

□

Theorem 2.5. *If* $\lim_{s \to \infty} \frac{f(s)}{s} = \alpha > 0$*, then for two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *we have*

$$a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v \Leftrightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v.$$

Proof. In Theorem 2.4, we showed that $a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v \Rightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$. Now, we must show that

$$a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v \Rightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v.$$

For all $a \geq 0$, if we let $\lim_{a \to \infty} \frac{f(a)}{a} = \alpha > 0$, then we have $f(a) \geq \alpha a$. Assume that $a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$. Since

$$\frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \frac{1}{h_r} \sum_{k \in I_r} \alpha \left(\left| \frac{a_k}{v_k} - K \right| \right) = \alpha \left(\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{a_k}{v_k} - K \right| \right)$$

holds, hence for each $\varepsilon > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \alpha\varepsilon \right\}.$$

Since $a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$, the next set and so the first set in the foregoing statement pertains to $\mathscr{I}_{\sigma\theta}$. This proves that

$$a \overset{[\mathscr{I}_{\sigma\theta}^K]}{\sim} v \Leftrightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v.$$

□

Definition 2.6. *Two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *are told to be asymptotically lacunary* $\mathscr{I}$*-invariant statistical equivalent of multiple* K *if for any* $\varepsilon > 0$ *and any* $\gamma > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \in \mathscr{I}_{\sigma\theta}$$

(denoted by $a \overset{\mathscr{I}(S_{\sigma\theta}^K)}{\sim} v$*). If we let* $K = 1$*, then* $a = (a_k)$ *and* $v = (v_k)$ *are told to be simply asymptotically lacunary* $\mathscr{I}$*-invariant statistical equivalent.*

Theorem 2.7. *For two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *we have*

$$a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v \Rightarrow a \overset{\mathscr{I}(S_{\sigma\theta}^K)}{\sim} v.$$

Proof. Granted that $a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$ and $\varepsilon > 0$ be given. Since

$$\frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon}} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq f(\varepsilon) \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \right|$$

holds, hence for each $\gamma > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{a_k}{v_k} - K \right| \geq \varepsilon \right\} \right| \geq \gamma \right\} \subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{a_k}{v_k} - K \right| \right) \geq \gamma f(\varepsilon) \right\}.$$

Since $a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v$, the next set and so the first set in the foregoing statement pertains to $\mathscr{I}_{\sigma\theta}$ and hence, $a \overset{\mathscr{I}(S_{\sigma\theta}^K)}{\sim} v$. □

Theorem 2.8. *If* f *is bounded, then for two non-negative sequences* $a = (a_k)$ *and* $v = (v_k)$ *we have*

$$a \overset{\mathscr{I}(S_{\sigma\theta}^K)}{\sim} v \Leftrightarrow a \overset{[\mathscr{I}_{\sigma\theta}^K(f)]}{\sim} v.$$

Proof. In Theorem 2.7, we showed that $a \overset{[\mathscr{I}_{\sigma\theta}^{K}(f)]}{\sim} v \Rightarrow a \overset{\mathscr{I}(S_{\sigma\theta}^{K})}{\sim} v$. Now, we must show that

$$a \overset{\mathscr{I}(S_{\sigma\theta}^{K})}{\sim} v \Rightarrow a \overset{[\mathscr{I}_{\sigma\theta}^{K}(f)]}{\sim} v.$$

Granted that f is bounded and $a \overset{\mathscr{I}(S_{\sigma\theta}^{K})}{\sim} v$. Hence, there exists a positive real number L such that $|f(a)| \leq L$, for all $a \geq 0$. Further using this fact, we have

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{a_k}{v_k}-K\right|\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|\geq\varepsilon}} f\left(\left|\frac{a_k}{v_k}-K\right|\right)+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{a_k}{v_k}-K\right|<\varepsilon}} f\left(\left|\frac{a_k}{v_k}-K\right|\right) \\ &\leq \frac{L}{h_r}\left|\left\{k\in I_r : \left|\frac{a_k}{v_k}-K\right|\geq\varepsilon\right\}\right|+f(\varepsilon). \end{aligned}$$

This proves that $a \overset{[\mathscr{I}_{\sigma\theta}^{K}(f)]}{\sim} v$. □

3. Conclusion

In the present study, using modulus function and lacunary sequences, we investigated the types of asymptotically ideal invariant equivalence for real sequences and give theorems about some properties. These new concepts can be examine for set sequences.

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