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New Travelling Wave Solutions of Conformable Cahn-Hilliard Equation

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Abstract

In this article, two methods are proposed to solve the fractional Cahn-Hilliard equation. This model describes the process of phase separation with nonlocal memory effects. Cahn-Hilliard equations have numerous applications in real-world scenarios, e.g., material sciences, cell biology, and image processing. Different types of solutions have been obtained. For this, the fractional complex transformation has been used to convert fractional differential equation to ordinary differential equation of integer order. As a result, these solutions are new solutions that do not exist in the literature.

1. Introduction

Shallow water areas show nonlinear effects during the propagation and transformation of swelling waves. Nonlinear equations such as Cahn-Hilliard equation exhibit significant spectral energy transfer for finite amplitude waves in shallow areas above the flat seafloor [1]. Chaotic oscillations usually occur in nonlinear dynamical systems (especially in watersheds). This systems can be represented by Cahn-Hilliard equation with nonlinear oscillations and external periodic excitation [2].
Fractional differential equations (FDEs) are generalizations of known differential equations (ODEs). Fractional order partial differential equations (fPDEs) are used effectively in many fields of science because they give more realistic results in modeling real life problems. [3–6]. Many useful methods have been presented for exact solutions of fPDEs as the $\left(\frac{G'}{G}\right)$-expansion [7–9], the sub-equation [10, 11], the exp-function [12–15], the first integral [16], the functional variable [17, 18], the modified simplest equation [19, 20]. The Kudryashov method [21]. With the help of these methods, solutions of fPDEs in many different forms are calculated.
In the next section, the extended tanh method and the sine-cosine method are introduced. In the next section, we will find the traveling wave solutions of the fractional Cahn-Hilliard equation (fCHE) via the this methods. We will talk about the data obtained in the last section.

2. Presentment of the Methods

2.1. The conformable derivative

The basic limit definition of this derivative is [22]:

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi & , \quad 0 < \alpha < 1 \\ (f^{(n)}(t))^{(\alpha - n)} & , \quad n \leq \alpha < n+1, \quad n \geq 1. \end{cases} \tag{2.1}$$

Some important properties of the conformable derivative were summarized in [23, 24]. Now, we briefly describe the definition and theorems of conformable derivatives:

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Definition 2.1. *Let $g:(0,\infty)\to R$ be a function. The conformable derivative of g for order α is defined by*

$$T_{\alpha}(g(k))=\lim_{\varepsilon\to 0}\frac{g\left(k+\varepsilon k^{1-\alpha}\right)-g(k)}{\varepsilon} \tag{2.2}$$

for all $k>0,\quad \alpha\in(0,1)$.

Theorem 2.2. *If a function $g:[0,\infty)\to R$ is $\alpha-$differentiable at $t_0>0,\ \alpha\in(0,1],\ g$ is continuous at t_0.*

Theorem 2.3. *Let f and g be $\alpha-$differentiable at a point $t>0,\ \alpha\in(0,1]$:*

$$\begin{aligned}
&T_{\alpha}(\alpha f+bg)=aT_{\alpha}(f)+bT_{\alpha}(g),\quad \text{for all } a,b\in R.\\
&T_{\alpha}(t^{p})=pt^{p-\alpha},\quad \text{for all } p\in R.\\
&T_{\alpha}(\lambda)=0,\quad \text{for all constant functions } f(t)=\lambda.\\
&T_{\alpha}(fg)=fT_{\alpha}(g)+gT_{\alpha}(f).\\
&T_{\alpha}\left(\frac{f}{g}\right)=\frac{gT_{\alpha}(f)-fT_{\alpha}(g)}{g^{2}}.
\end{aligned} \tag{2.3}$$

If g is differentiable:

$$T_{\alpha}(g)(t)=t^{1-\alpha}\frac{dg}{dt}(t). \tag{2.4}$$

A nonlinear conformable partial differential equations (CPDEs) with two independent variables are:

$$P\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}},\frac{\partial^{\alpha}u}{\partial x^{\alpha}},\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}},\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}},\ldots\right)=0,\qquad 0<\alpha\leq 1, \tag{2.5}$$

$$u(x,t)=U(\xi),\qquad \xi=k\frac{x^{\alpha}}{\alpha}-c\frac{t^{\alpha}}{\alpha}, \tag{2.6}$$

$u(x,t)$ is a traveling wave solution and c is a constant and will be calculated later.

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}=-c\frac{\partial}{\partial\xi},\qquad \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}=c^{2}\frac{\partial^{2}}{\partial\xi^{2}},\qquad \frac{\partial^{\alpha}u}{\partial x^{\alpha}}=k\frac{\partial}{\partial\xi},\qquad \frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}=k^{2}\frac{\partial^{2}}{\partial\xi^{2}},\ldots$$

using equation (2.6), nonlinear CPDE equation (2.5) can be reduced to a nonlinear ODE equation (2.7):

$$Q(U,U',U'',U''',\ldots)=0. \tag{2.7}$$

where the prime denotes the derivation with respect to ξ.

2.2. The extended $\tanh-\coth$ method

In [25], this method is summarized as follows:
From the Y independent variable and its derivatives:

$$Y=\tanh(\xi)\quad \text{or}\quad Y=\coth(\xi), \tag{2.8}$$

$$\begin{aligned}
\frac{d}{d\xi} &= \left(1-Y^{2}\right)\frac{d}{dY},\\
\frac{d^{2}}{d\xi^{2}} &= \left(1-Y^{2}\right)\left(-2Y\frac{d}{dY}+\left(1-Y^{2}\right)\frac{d^{2}}{dY^{2}}\right).
\end{aligned} \tag{2.9}$$

The $\tanh-\coth$ method:

$$U(\xi)=S(Y)=\sum_{k=0}^{m}a_{k}Y^{k}, \tag{2.10}$$

where $a_k(k=0,1,2,\ldots,m)$ are constants.
Equation (2.10) can be expanded as follows [26].

$$U(\xi)=S(Y)=\sum_{k=0}^{m}a_{k}Y^{k}+\sum_{k=1}^{m}b_{k}Y^{-k}, \tag{2.11}$$

where $a_k(k=0,1,2,\ldots,m)$, $b_k(k=0,1,2,\ldots,m)$ are constants. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (2.7).

2.3. The sine-cosine method

In [27], this method is summarized as follows:
The solutions of nonlinear equations is

$$u(x,t)=\left\{\begin{array}{ll}\lambda\sin^{\beta}(\mu\xi), & |\xi|\leq\frac{\pi}{\mu},\\ 0, & \text{otherwise}\end{array}\right\}, \tag{2.12}$$

$$u(x,t)=\left\{\begin{array}{ll}\lambda\cos^{\beta}(\mu\xi), & |\xi|\leq\frac{\pi}{2\mu},\\ 0, & \text{otherwise}\end{array}\right\} \tag{2.13}$$

where λ,μ and β are parameters to be calculated. μ and c in wave transform are the wave number and the wave speed.
From equation (2.12):

$$\begin{aligned}
&u(\xi)=\lambda\sin^{\beta}(\mu\xi),\\
&u^{n}(\xi)=\lambda^{n}sin^{n\beta}(\mu\xi),\\
&(u^{n})_{\xi}=n\mu\beta\lambda^{n}cos(\mu\xi)sin^{n\beta-1}(\mu\xi),\\
&(u^{n})_{\xi\xi}=-n^{2}\mu^{2}\beta^{2}\lambda^{n}sin^{n\beta}(\mu\xi)+n\mu^{2}\lambda^{n}\beta(n\beta-1)sin^{n\beta-2}(\mu\xi),
\end{aligned} \tag{2.14}$$

and from equation (2.13):

$$\begin{aligned}
&u(\xi)=\lambda\cos^{\beta}(\mu\xi),\\
&u^{n}(\xi)=\lambda^{n}\cos^{n\beta}(\mu\xi),\\
&(u^{n})_{\xi}=-n\mu\beta\lambda^{n}\sin(\mu\xi)\cos^{n\beta-1}(\mu\xi),\\
&(u^{n})_{\xi\xi}=-n^{2}\mu^{2}\beta^{2}\lambda^{n}\cos^{n\beta}(\mu\xi)+n\mu^{2}\lambda^{n}\beta(n\beta-1)\cos n^{n\beta-2}(\mu\xi).
\end{aligned} \tag{2.15}$$

We substitute equation (2.15) or (2.14) in the reduced equation (2.7). We can sum all terms with the same power in $cos^{k}(\mu\xi)$ or $sin^{k}(\mu\xi)$ and set their coefficients to zero to get a system of algebraic equations. So, we can get all values of parameters μ,β and λ.

3. Applications

Let us investigate traveling wave solutions of space-time fCHE in order to demonstrate the effectiveness of the methods.

3.1. Exact solutions of the fCHE

3.1.1. The extended tanh method

Now we will investigate the solutions of equation (3.1) with the extended tanh method. We will then interpret the results obtained.

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-r\frac{\partial^{\alpha}u}{\partial x^{\alpha}}-6u\left(\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\right)^{2}-\left(3u^{2}-1\right)\frac{\partial^{2\alpha}u}{\partial x^{2\alpha}}+\frac{\partial^{4\alpha}u}{\partial x^{4\alpha}}=0,\quad 0<\alpha\leq 1, \tag{3.1}$$

nonlinear wave equation. We have applied intended tanh method on the Conformable CHE for gaining the exact solutions:

$$u(x,t)=U(\xi),\quad \xi=k\frac{x^{\alpha}}{\alpha}-c\frac{t^{\alpha}}{\alpha}.$$

Equation (3.1) can be reduced to equation (3.2):

$$-cU-krU-3k^{2}U^{2}U'+k^{2}U'+k^{4}U'''=0, \tag{3.2}$$

c,r and k are constants.
Balancing U''' with $U^{2}U'$ yields $m=1$. So, the solution of equation is equation (3.3) format.

$$U(\xi)=a_{0}+a_{1}Y+b_{1}Y^{-1},\quad Y=\tanh(\xi),\quad Y'=1-Y^{2}. \tag{3.3}$$

Here a_0, a_1, b_1, k and c are constants such that $a_1 \neq 0$. Substituting equation (3.3) into equation (3.2), collecting the coefficients of Y^i $(i = -4, \ldots 4)$ and set it to zero we obtain the system

$$\begin{aligned} 3a_1^3k^2 - 6a_1k^4 &= 0, \\ 6a_0a_1^2k^2 &= 0, \\ 3a_0^2a_1k^2 - 3a_1^3k^2 + 3a_1^2b_1k^2 + 8a_1k^4 - a_1k^2 &= 0, \\ 6a_0a_1^2k^2 - a_1kr - a_1c &= 0, \\ (a_1+b_1)(k^2 - 2k^4 - 3a_0^2k^2 - 3k^2a_1b_1) - a_0(kr+c) &= 0, \\ 6a_0b_1^2k^2 - b_1kr - b_1c &= 0, \\ 3a_0^2b_1k^2 + 3a_1b_1^2k^2 - 3b_1^3k^2 + 8b_1k^4 - b_1k^2 &= 0, \\ 6a_0b_1^2k^2 &= 0, \\ 3b_1^3k^2 - 6b_1k^4 &= 0. \end{aligned} \tag{3.4}$$

The exact solutions obtained from this system of equations are:

$$\begin{aligned} u_1(x,t) &= \frac{1}{\tanh\left(\pm\frac{\sqrt{2}}{2}x^\alpha \pm \frac{\sqrt{2}}{2}rt^\alpha\right)}, \\ u_2(x,t) &= -\frac{1}{\tanh\left(\pm\frac{\sqrt{2}}{4}x^\alpha \pm \frac{\sqrt{2}}{4}rt^\alpha\right)}, \\ u_3(x,t) &= \tanh\left(\pm\frac{\sqrt{2}}{2}x^\alpha \pm \frac{\sqrt{2}}{2}rt^\alpha\right), \\ u_4(x,t) &= -\tanh\left(\pm\frac{\sqrt{2}}{4}x^\alpha \pm \frac{\sqrt{2}}{4}rt^\alpha\right), \\ u_5(x,t) &= \pm\frac{1}{2}\tanh\left(\pm\frac{\sqrt{2}}{4}x^\alpha \pm \frac{\sqrt{2}}{4}rt^\alpha\right) \pm \frac{1}{2\tanh\left(\pm\frac{\sqrt{2}}{4}x^\alpha \pm \frac{\sqrt{2}}{4}rt^\alpha\right)}, \\ u_6(x,t) &= \pm\frac{i\sqrt{2}}{2}\tanh\left(\pm\frac{i}{2}x^\alpha \pm \tfrac{i}{2}rt^\alpha\right) \pm \frac{1}{2}\frac{i\sqrt{2}}{\tanh\left(\pm\frac{i}{2}x^\alpha \pm \frac{i}{2}rt^\alpha\right)}, \end{aligned} \tag{3.5}$$

where r is arbitrary constant.

3.1.2. The sine-cosine method

Now we will investigate the solutions of equation (3.1) with this method. We will then interpret the results obtained. We have applied the method on the Conformable CHE for gaining the exact solutions. Using the ξ wave transform, equation (3.1) can be reduced to equation (3.6):

$$-cU - krU - 3k^2U^2U' + k^2U' + k^4U''' + q = 0, \tag{3.6}$$

c, r and k are constants, q is a constant of integration.
Balancing U''' with U^2U' yields $m = 1$. So, the solution of equation is equation (3.7) or equation (3.8) format:

$$u(x,t) = \lambda \cos^\beta(\mu\xi), \tag{3.7}$$

$$u(x,t) = \lambda \sin^\beta(\mu\xi). \tag{3.8}$$

We substitute equation (3.7) into the reduced equation obtained from equation (3.6).

$$
\begin{aligned}
\frac{(-k^2\lambda\mu + 11k^4\lambda^3\mu^3 + qsin(\mu\xi))}{sin(\mu\xi)} &= 0,\\
\frac{(-c\lambda sin(\mu\xi) - kr\lambda sin(\mu\xi))}{sin(\mu\xi)} &= 0,\\
\frac{(-12\lambda\mu^3k^4 + 3\lambda^3\mu k^2 + a_1\mu k^2)}{sin(\mu\xi)} &= 0,\\
\frac{(6\lambda\mu^3k^4 - 3\lambda^3\mu k^2)}{sin(\mu\xi)} &= 0.
\end{aligned} \tag{3.9}
$$

After the necessary arrangements have been made the following exact solutions obtained:

$$
u_{7,8}(x,t) = \frac{\sqrt{3}}{3}\cos^{-1}\left(\pm\frac{r^2}{6c}\left(\frac{-1}{r}x^\alpha - t^\alpha\right)\right). \tag{3.10}
$$

where r and c are arbitrary constants.

4. Conclusion

In this paper we have been successfully obtained traveling wave solutions of the fractional Cahn-Hilliard equation. Many of the results obtained are new solutions that do not exist in the literature. This hyperbolic solutions are significant for the explanation of many physical phenomena, ocean engineering and science. This implies that the methods are more powerful and effective in finding the exact solutions of nonlinear fractional differential equations. Many new solutions can be found with this methods. Using this methods possible to solve other similar nonlinear equations and systems of equations. The obtained solutions in this research have been found by aid of Maple packet program.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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