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$\omega\alpha$-Separation Axioms in Topological Spaces

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Abstract - The aim of this paper is to introduce and study two new classes of spaces,namely $\omega\alpha$-normal and $\omega\alpha$-regular spaces and obtained their properties by utilizing $\omega\alpha$ -closed sets. Recall that a subset A of a topological space (X, τ) is called $\omega\alpha$ -closed if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ω - open in (X, τ). We will present some characterizations of $\omega\alpha$-normal and $\omega\alpha$-regular spaces.

Keywords - *$\omega\alpha$-closed set, $\omega\alpha$-continuous function.*

1 Introduction

Maheshwari and Prasad[8] introduced the new class of spaces called s-normal spaces using semi-open sets. It was further studied by Noiri and Popa[10],Dorsett[6] and Arya[1]. Munshi[9], introduced g-regular and g- normal spaces using g-closed sets of Levine[7]. Later, Benchalli et al [3] and Shik John[12] studied the concept of g^* - pre regular, g^* - pre normal and ω- normal, ω-regular spaces in topological spaces. Recently, Benchalli et al [2,4 ,11] introduced and studied the properties of $\omega\alpha$- closed sets and $\omega\alpha$- continuous functions.

2 Preliminaries

Throughout this paper (X, τ), (Y, σ)(or simply X,Y)denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X the closure, interior and α-closure of A with respect to τ are denoted by $cl(A)$, $int(A)$

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and $\alpha cl(A)$ respectively.

Definition 2.1. *A subset A of a topological space X is called a*
(1) semi-open set [3] if $A \subset cl(int(A))$.
(2) ω-closed set[12] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.
(3) g-closed set[7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.2. *A topological space X is said to be a*

(1) g- regular[10], if for each g-closed set F of X and each point $x \notin F$, there exists disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

(2) α - regular [4], if for each closed set F of X and each point $x \notin F$, there exists disjoint α - open sets U and V such that $F \subseteq V$ and $x \in U$.

(3) ω-regular[12], if for each ω-closed set F of X and each point $x \notin F$, there exists disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

Definition 2.3. *A topological space X is said to be a*

(1) g- normal [10], if for any pair of disjoint g-closed sets A and B, there exists disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(2) α - normal [4], if for any pair of disjoint closed sets A and B, there exists disjoint α-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(3) ω-normal [12], if for any pair of disjoint ω-closed sets A and B, there exists disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition 2.4. [2] *A topological space X is called $T_{\omega\alpha}$ - space if every $\omega\alpha$-closed set in it is closed set.*

Definition 2.5. *A function $f : X \to Y$ is called:*

(1) $\omega\alpha$ - continuous [4] *(resp. ω- continuous* [12]*) if $f^{-1}(F)$ is $\omega\alpha$-closed(resp.ω-closed) set in X for every closed set F of Y.*

(2) $\omega\alpha$ - irresolute [4] *(resp. ω- irrosulte* [12]*) if $f^{-1}(F)$ is $\omega\alpha$-closed (resp.ω-closed) set in X for every $\omega\alpha$ - closed(resp. ω- closed) set F of Y.*

(3) pre-$\omega\alpha$ -closed[4](resp. $\omega\alpha$ -closed[4]) if for each α -closed(resp.closed) set F of X, f(F) is an $\omega\alpha$-closed(resp.$\omega\alpha$- closed) set in Y.

3 $\omega\alpha$-Regular Spaces

In this section, we introduce a new class of spaces called $\omega\alpha$-regular spaces using $\omega\alpha$-closed sets and obtain some of their characterizations.

Definition 3.1. A topological space X is said to be $\omega\alpha$-regular if for each $\omega\alpha$- closed set F and a point $x \notin F$, there exist disjoint open sets G and H such that $F \subseteq G$ and $x \in H$.

We have the following interrelationship between $\omega\alpha$-regularity and regularity.

Theorem 3.2. Every $\omega\alpha$- regular space is regular.

Proof: Let X be a $\omega\alpha$-regular space. Let F be any closed set in X and a point $x \in X$ such that $x \notin$ F. By [2], F is $\omega\alpha$-closed and $x \notin$ F. Since X is a $\omega\alpha$-regular space, there exists a pair of disjoint open sets G and H such that $F \subseteq G$ and $x \in H$. Hence X is a regular space.

Remark 3.3. If X is a regular space and $T_{\omega\alpha}$ - space, then X is $\omega\alpha$- regular.

We have the following characterization.

Theorem 3.4. The following statements are equivalent for a topological space X
(i) X is a $\omega\alpha$- regular space
(ii) For each $x \in X$ and each $\omega\alpha$- open neighbourhood U of x there exists an open neighbourhood N of x such that $cl(N) \subseteq U$.

Proof: (i) $\Rightarrow$ (ii): Suppose X is a $\omega\alpha$- regular space. Let U be any $\omega\alpha$- neighbourhood of x. Then there exists $\omega\alpha$- open set G such that $x \in G \subseteq U$. Now $X - G$ is $\omega\alpha$- closed set and $x \notin X - G$. Since X is $\omega\alpha$- regular, there exist open sets M and N such that $X - G \subseteq M$, $x \in N$ and $M \cap N = \phi$ and so $N \subseteq X - M$. Now $cl(N) \subseteq cl(X - M) = X - M$ and $X - M \subseteq M$. This implies $X - M \subseteq U$. Therefore $cl(N) \subseteq U$.

(ii) $\Rightarrow$ (i): Let F be any $\omega\alpha$- closed set in X and $x \in X - F$ and $X - F$ is a $\omega\alpha$ - open and so $X - F$ is a $\omega\alpha$ - neighbourhood of x. By hypothesis, there exists an open neighbourhood N of x such that $x \in N$ and $cl(N) \subseteq X - F$. This implies $F \subseteq X - cl(N)$ is an open set containing F and $N \cap \{(X - cl(N)\} = \phi$. Hence X is $\omega\alpha$ - regular space.

We have another characterization of $\omega\alpha$ - regularity in the following.

Theorem 3.5. A topological space X is $\omega\alpha$ - regular if and only if for each $\omega\alpha$ - closed set F of X and each $x \in X - F$ there exist open sets G and H of X such that $x \in G$, $F \subseteq H$ and $cl(G) \cap cl(H) = \phi$.

Proof: Suppose X is $\omega\alpha$ - regular space. Let F be a $\omega\alpha$ - closed set in X with $x \notin F$. Then there exists open sets M and H of X such that $x \in M$, $F \subseteq H$ and $M \cap H = \phi$. This implies $M \cap cl(H) = \phi$. As X is $\omega\alpha$ - regular, there exist open sets U and V such that $x \in U$, $cl(H) \subseteq V$ and $U \cap V = \phi$, so $cl(U) \cap V = \phi$. Let $G = M \cap U$, then G and H are open sets of X such that $x \in G$, $F \subseteq H$ and $cl(H) \cap cl(H) = \phi$.

Conversely, if for each $\omega\alpha$ - closed set F of X and each $x \in X - F$ there exists open sets G and H such that $x \in G$, $F \subseteq H$ and $cl(H) \cap cl(H) = \phi$. This implies $x \in G$, $F \subseteq H$ and $G \cap H = \phi$. Hence X is $\omega\alpha$ - regular.

Now we prove that $\omega\alpha$ - regularity is a heriditary property.

Theorem 3.6. Every subspace of a $\omega\alpha$ -regular space is $\omega\alpha$ - regular.

Proof: Let X be a $\omega\alpha$ - regular space. Let Y be a subspace of X. Let $x \in Y$ and F be a $\omega\alpha$ - closed set in Y such that $x \notin F$. Then there is a closed set and so $\omega\alpha$ - closed set A of X with $F = Y \cap A$ and $x \notin A$. Therefore we have $x \in X$, A is $\omega\alpha$ - closed in X such that $x \notin A$. Since X is $\omega\alpha$ - regular, there exist open sets G and H such that $x \in G$, $A \subseteq H$ and $G \cap H = \phi$. Note that $Y \cap G$ and $Y \cap H$ are open sets in Y. Also $x \in G$ and $x \in Y$, which implies $x \in Y \cap G$ and $A \subseteq H$ implies $Y \cap G \subseteq Y \cap H$, $F \subseteq Y \cap H$. Also $(Y \cap G) \cap (Y \cap H) = \phi$. Hence Y is $\omega\alpha$ -regular space.

We have yet another characterization of $\omega\alpha$ -regularity in the following.

Theorem 3.7. The following statements about a topological space X are equivalent:
(i) X is $\omega\alpha$ -regular
(ii) For each $x \in X$ and each $\omega\alpha$ - open set U in X such that $x \in U$ there exists an open set V in X such that $x \in V \subseteq cl(V) \subseteq U$
(iii) For each point $x \in X$ and for each $\omega\alpha$ - closed set A with $x \notin A$, there exists an open set V containing x such that $cl(V) \cap A = \phi$.

Proof: (i)$\Rightarrow$ (ii): Follows from Theorem 3.5.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. Let $x \in X$ and A be an $\omega\alpha$ - closed set of X such that $x \notin A$. Then $X - A$ is a $\omega\alpha$ - open set with $x \in X - A$. By hypothesis, there exists an open set V such that $x \in V \subseteq cl(V) \subseteq X - A$. That is $x \in V$, $V \subseteq cl(A)$ and $cl(A) \subseteq X - A$. So $x \in V$ and $cl(V) \cap A = \phi$.

(iii) $\Rightarrow$ (i): Let $x \in X$ and U be an $\omega\alpha$ - open set in X such that $x \in U$. Then $X - U$ is an $\omega\alpha$ closed set and $x \notin X - U$. Then by hypothesis, there exists an open set V containing x such that $cl(A) \cap (X - U) = \phi$. Therefore $x \in V$, $cl(V) \subseteq U$ so $x \in V \subseteq cl(V) \subseteq U$.

The invariance of $\omega\alpha$ - regularity is given in the following.

Theorem 3.8. Let $f : X \rightarrow Y$ be a bijective, $\omega\alpha$ - irresolute and open map from a $\omega\alpha$ - regular space X into a topological space Y, then Y is $\omega\alpha$ - regular.

Proof: Let $y \in Y$ and F be a $\omega\alpha$ - closed set in Y with $y \notin F$. Since F is $\omega\alpha$ - irresolute, $f^{-1}(F)$ is $\omega\alpha$ - closed set in X. Let $f(x) = y$ so that $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again X is $\omega\alpha$ - regular space, there exist open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq G$, $U \cap V = \phi$. Since f is open and bijective, we have $y \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Hence Y is $\omega\alpha$ - regular space.

Theorem 3.9. Let $f : X \rightarrow Y$ be a bijective, $\omega\alpha$ - closed and open map from a topological space X into a $\omega\alpha$ - regular space Y. If X is $T_{\omega\alpha}$-space, then X is $\omega\alpha$ - regular.

Proof: Let $x \in X$ and F be an $\omega\alpha$ - closed set in X with $x \notin F$. Since X is $T_{\omega\alpha}$-space, F is closed in X. Then $f(F)$ is $\omega\alpha$ - closed set with $f(x) \notin f(F)$ in Y, since f is $\omega\alpha$ - closed. As Y is $\omega\alpha$ - regular, there exist open sets U and V such that $x \in U$ and $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subseteq f^{-1}(V)$. Hence X is $\omega\alpha$ - regular space.

Theorem 3.10. If $f : X \to Y$ is ω - irresolute, pre $\omega\alpha$ - closed, continuous injection and Y is $\omega\alpha$ - regular space, then X is $\omega\alpha$ - regular.

Proof: Let F be any closed set in X with $x \notin F$. Since f is ω - irresolute, pre $\omega\alpha$ - closed by [3], f is $\omega\alpha$ - closed set in Y and $f(x) \notin f(F)$. Since Y is $\omega\alpha$ - regular, there exists open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Thus $x \in f^{-1}(U)$, $F \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence X is $\omega\alpha$ - regular space.

4 $\omega\alpha$-Normal Spaces

In this section, we introduce the concept of $\omega\alpha$ - normal spaces and study some of their characterizations.

Definition 4.1. A topological space X is said to be $\omega\alpha$-normal if for each pair of disjoint $\omega\alpha$- closed sets A and B in X, there exists a pair of disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

We have the following interrelationship.

Theorem 4.2. Every $\omega\alpha$ - normal space is normal.

Proof: Let X be a $\omega\alpha$- normal space. Let A and B be a pair of disjoint closed sets in X. From [2], A and B are $\omega\alpha$- closed sets in X. Since X is $\omega\alpha$- normal, there exists a pair of disjoint open sets G and H in X such that $A \subseteq G$ and $B \subseteq H$. Hence X is normal.

Remark 4.3. The converse need not be true in general as seen from the following example.

Example 4.4. Let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c, d\}\}$ Then the space X is normal but not $\omega\alpha$ - normal, since the pair of disjoint $\omega\alpha$ - closed sets namely, $A = \{a, d\}$ and $B = \{b, c\}$for which there do not exists disjoint open sets G and H such that $A \subseteq G$ and $B \subseteq H$.

Remark 4.5. If X is normal and $T_{\omega\alpha}$-space, then X is $\omega\alpha$ - normal.

Hereditary property of $\omega\alpha$ - normality is given in the following.

Theorem 4.6. A $\omega\alpha$ - closed subspace of a $\omega\alpha$ - normal space is $\omega\alpha$ - normal.

We have the following characterization.

Theorem 4.7. The following statements for a topological space X are equivalent:
(i) X is $\omega\alpha$ - normal
(ii) For each $\omega\alpha$ - closed set A and each $\omega\alpha$ - open set U such that $A \subseteq U$, there exists an open set V such that $A \subseteq V \subseteq cl(V) \subseteq U$
(iii) For any $\omega\alpha$ - closed sets A, B, there exists an open set V such that $A \subseteq V$ and $cl(V) \cap B = \phi$
(iv) For each pair A, B of disjoint $\omega\alpha$ - closed sets there exist open sets U and V such that $A \subseteq U$,$B \subseteq V$ and $cl(U) \cap cl(V) = \phi$.

Proof: (i) $\Rightarrow$ (ii): Let A be a $\omega\alpha$ -closed set and U be a $\omega\alpha$ - open set such that $A \subseteq U$. Then A and $X - U$ are disjoint $\omega\alpha$ - closed sets in X. Since X is $\omega\alpha$ - normal , there exists a pair of disjoint open sets V and W in X such that $A \subseteq V$ and $X - U \subseteq W$. Now $X - W \subseteq X - (X - U)$, so $X - W \subseteq U$ also $V \cap W = \phi$ implies $V \subseteq X - W$, so $cl(V) \subseteq cl(X - W)$ which implies $cl(V) \subseteq X - W$. Therefore $cl(V) \subseteq X - W \subseteq U$. So $cl(V) \subseteq U$. Hence $A \subseteq V \subseteq cl(V) \subseteq U$.

(ii) $\Rightarrow$(iii): Let A and B be a pair of disjoint $\omega\alpha$ - closed sets in X. Now $A \cap B = \phi$, so $A \subseteq X - B$, where A is $\omega\alpha$ - closed and $X - B$ is $\omega\alpha$ - open . Then by (ii) there exists an open set V such that $A \subseteq V \subseteq cl(V) \subseteq X - B$. Now $cl(V) \subseteq X - B$ implies $cl(V) \cap B = \phi$. Thus $A \subseteq V$ and $cl(V) \cap B = \phi$

(iii) $\Rightarrow$(iv): Let A and B be a pair of disjoint $\omega\alpha$ - closed sets in X.Then from (iii) there exists an open set U such that $A \subseteq U$ and $cl(U) \cap B = \phi$. Since $cl(V)$ is closed, so $\omega\alpha$ - closed set.Therefore $cl(V)$ and B are disjoint $\omega\alpha$ - closed sets in X. By hypothesis, ther exists an open set V, such that $B \subseteq V$ and $cl(U) \cap cl(V) = \phi$.

(iv) $\Rightarrow$(i): Let A and B be a pair of disjoint $\omega\alpha$ - closed sets in X.Then from (iv) there exist an open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $cl(U) \cap cl(V) = \phi$. So $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$. Hence X $\omega\alpha$ - normal.

Theorem 4.8. Let X be a topological space. Then X is $\omega\alpha$ - normal if and only if for any pair A, B of disjoint $\omega\alpha$ - closed sets there exist open sets U and V of X such that $A \subseteq U$,$B \subseteq V$ and $cl(U) \cap cl(V) = \phi$.

Theorem 4.9. Let X be a topological space. Then the following are equivalent:
(i) X is normal
(ii) For any disjoint closed sets A and B, there exist disjoint $\omega\alpha$ - open sets U and V such that $A \subseteq U$,$B \subseteq V$.
(iii) For any closed set A and any open set V such that $A \subseteq V$, there exists an $\omega\alpha$ - open set U of X such that $A \subseteq U \subseteq \alpha cl(U) \subseteq V$.

Proof: (i) $\Rightarrow$(ii): Suppose X is normal. Since every open set is $\omega\alpha$ - open [2], (ii) follows.

(ii) $\Rightarrow$(iii): Suppose (ii) holds. Let A be a closed set and V be an open set containing A. Then A and $X - V$ are disjoint closed sets. By (ii), there exist disjoint $\omega\alpha$ - open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$, since $X - V$ is closed, so $\omega\alpha$ - closed. From [2], we have $X - V \subseteq \alpha int(W)$ and $U \cap \alpha int(W) = \phi$ and so we have $\alpha cl(U) \cap \alpha int(W) = \phi$. Hence $A \subseteq U \subseteq \alpha cl(U) \subseteq X - \alpha int(W) \subseteq V$. Thus $A \subseteq U \subseteq \alpha cl(U) \subseteq V$.

(iii) $\Rightarrow$(i): Let A and B be a pair of disjoint closed sets of X.Then $A \subseteq X - B$ and $X - B$ is open. There exists a $\omega\alpha$ - open set G of X such that $A \subseteq G \subseteq \alpha cl(G) \subseteq X - B$. Since A is closed, it is ω - closed, we have $A \subseteq \alpha int(G)$. Take $U = int(cl(int(\alpha int(G))))$ and $V = int(cl(int(X - \alpha cl(G))))$. Then U and V are disjoint open sets of X such that $A \subseteq U$ and $B \subseteq V$. Hence X is normal.

We have the following characterization of $\omega\alpha$ - normality and α- normality.

Theorem 4.10. Let X be a topological space. Then the following are equivalent:
(i) X is α - normal
(ii) For any disjoint closed sets A and B, there exist disjoint $\omega\alpha$ - open sets U and V such that $A \subseteq U$,$B \subseteq V$ and $U \cap V = \phi$.

Proof: (i) $\Rightarrow$(ii): Suppose X is α- normal. Let A and B be a pair of disjoint closed sets of X. Since X is α - normal,there exist disjoint α - open sets U and V such that $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \phi$.

(ii) $\Rightarrow$(i):Let A and B be a pair of disjoint closed sets of X.The by hypothesis there exist disjoint $\omega\alpha$ - open sets U and V such that $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \phi$.Since from [2], $A \subseteq \alpha intU$ and $B \subseteq \alpha intV$ and $\alpha intU \cap \alpha intV = \phi$. Hence X is α- normal.

Theorem 4.11. Let X bea α - normal, then the following hold good:
(i)For each closed set A and every $\omega\alpha$ - open set B such that $A \subseteq B$ ther exists a α-open set U such that $A \subseteq U \subseteq \alpha cl(U) \subseteq B$.
(ii) For every $\omega\alpha$ - closed set A and every open set B containing A, there exist a α - open set U such that $A \subseteq U \subseteq \alpha cl(U) \subseteq B$.

Theorem 4.12. If $f : X \to Y$ is weakly continuous, $\omega\alpha$ - closed injection and Y is $\omega\alpha$ - normal, then X is normal.

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