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Generalized $\omega\alpha$-Closed Sets in Topological Spaces

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Abstract - The aim of this paper is to introduce a new class of closed sets called g$\omega\alpha$-closed sets using $\omega\alpha$-closed sets in topological spaces. This class is independent of $\omega\alpha$-closed sets. This new class of set lies between the class of α-closed sets and the class of αg-closed sets. Some of their properties are investigated. We also define and study the g$\omega\alpha$-closure and g$\omega\alpha$-interior in topological spaces.

Keywords - *Topological spaces, generalized closed sets, $\omega\alpha$-closed sets, $g\omega\alpha$-closed sets and $g\omega\alpha$-open sets.*

1 Introduction

In 1969 Levine [9] gives the concept and properties of generalized closed (briefly g-closed) sets and the complement of g-closed set is said to be g-open set. In 1982 Mashhour et.al [13] introduced and studied the concept of pre-open set. Later Maki et.al [12], Dontechev [6], Gyanambal [7], Arya and Nour [3] and Bhattacharya and Lahiri [4] introduced and studied the concepts of gp-closed, gsp-closed, gpr-closed, gs-closed, sg-closed and αg-closed and their compliments are respective open sets.

N Jasted [16] introduced and studied the concept of α-sets. Later these sets are called as α-open sets in 1983. Mashhours et.al [14] introduced and studied the concept of α-closed sets, α-closure of set, α-continuous functions, α-open functions and α-closed functions in topological spaces. Maki et.al [10] [11] introduced and studied generalized α-closed sets and α-generalized closed sets. Sundarm and Sheik John [20] defined and studied ω-closed sets in topological spaces and recently S.S.Benchalli et.al [5] studied $\omega\alpha$-closed sets in topological spaces.

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2 Preliminaries

Throughout this paper space (X, τ) and (Y, σ) (or simply X and Y) always denote topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space (X, τ) Cl(A), Int(A) and A^c denote the Closure of A, Interior of A and Compliment of A respectively.

Definition 2.1. *A subset A of a topological space (X, τ) is called,*
(i) Semi-open set [8] if A$\subseteq$Cl(Int(A)) and Semi-closed set if Int(Cl(A))$\subseteq$A.
(ii) Pre-open set [13] if A$\subseteq$Int(Cl(A)) and Pre-closed set if Cl(Int(A))$\subseteq$A.
(iii) α-open set [16] if A$\subseteq$Int(Cl(Int(A))) and α-closed set if Cl(Int(Cl(A)))$\subseteq$A.
(iv) Semi-pre-open set [2] (=β-open set [1]) if A$\subseteq$Cl(Int(Cl(A))) and semi-pre-closed (=β-closed set [1]]) if (Cl(Int(Cl(A)))$\subseteq$A.
(v) Regular-open [7] if A=Int(Cl(A)) and Regular-closed if A=Cl(Int(A)).

The α-closure of A is the smallest α-closed set containing A, and this is denoted by αCl(A). Similarly the semi-closure (resp pre-closure and semi-pre-closure) of a set A in a topological space (X, τ) is the intersection of all semi-closed (resp pre-closed and semi-pre-closed) sets containing A and is denoted by scl(A) (resp pcl(A) and spcl(A)).

Definition 2.2. *A subset of a topological space (X, τ) is called a,*
(i) Generalized closed (briefly g-closed) set [9] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.
(ii) Semi-generalized closed (briefly sg-closed) set [4] if scl(A)$\subseteq U$ whenever A$\subseteq U$ and U is Semi-open in X.
(iii) Generalized semi-closed (briefly gs-closed) set [3] if scl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.
(iv) Generalized α-closed (briefly gα-closed) set [10] if αcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is α-open in X.
(v) α-generalized closed (briefly αg-closed) set [11] if αcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.
(vi) Generalized pre-closed (briefly gp-closed) set [12] if pcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.
(vii) Generalized semi-pre-closed (briefly gsp-closed) set [6] if spcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is open in X.
($viii$) Generalized pre-regular-closed (briefly gpr-closed) set [7] if pcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is regular-open in X.
(ix) Weakly closed (briefly ω-closed) set [21] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is semi-open in X.
(x) Weakly generalized closed (briefly ωg-closed) set [20] if cl(int(A))$\subseteq U$ whenever A$\subseteq U$ and U is open in X.
(xi) Strongly generalized closed (briefly g^*-closed) set [18] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is g-open in X.
(xii) Regular generalized closed (briefly rg-closed) set [17] if cl(A)$\subseteq U$ whenever A$\subseteq U$ and U is regular-open in X.
($xiii$) α-generalized regular closed (briefly αgr-closed) set [23] if αcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is regular-open in X.
(xv) g^*-preclosed (briefly g^*p-closed) [22] if pcl(A)$\subseteq U$ whenever A$\subseteq U$ and U is g-open

in X.
(xiv) $\omega\alpha$ *closed set [5] if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* ω*-open in X.*

The compliment of the above mentioned closed sets are their open sets respectively.

3 g$\omega\alpha$-closed sets in Topological spaces.

In this section we introduce g$\omega\alpha$-closed sets in topological space and study some of their properties.

Definition 3.1. *A subset A of a topological space* (X, τ) *is called a generalized* $\omega\alpha$*-closed (g*$\omega\alpha$*-closed) set if* $\alpha cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\omega\alpha$*-open in X.*

Theorem 3.2. *Every closed set in X is g*$\omega\alpha$*-closed set.*

Proof: *Let A be a closed set in a topological space X, let G be any* $\omega\alpha$*-open sets in X such that* $A \subseteq G$*, Since A is closed, we have* $cl(A) = A$*, but* $\alpha cl(A) \subseteq cl(A)$ *is always true. So* $\alpha cl(A) \subseteq cl(A) \subseteq G$*. Therefore* $\alpha cl(A) \subseteq G$*. Hence A is g*$\omega\alpha$*-closed set.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ *then the set* $A = \{a, c\}$ *is g*$\omega\alpha$*-closed but not closed.*

Theorem 3.4. *Every* α*-closed set in X is g*$\omega\alpha$*-closed set.*

Proof: *Let A be* α*-closed set in a topological space X. Let U be* $\omega\alpha$*-open set in X such that* $A \subseteq U$*. Since A is* α*-closed we have* $\alpha cl(A) = A \subseteq U$*. Therefore* $\alpha cl(A) \subseteq U$*. Hence A is g*$\omega\alpha$*-closed set.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.5. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ *then the set* $A = \{b\}$ *is g*$\omega\alpha$*-closed but not* α*-closed in X.*

Theorem 3.6. *Every g*$\omega\alpha$*-closed set in X is* αg*-closed set in X.*

Proof: *Let A be g*$\omega\alpha$*-closed set in X. Let U be any open set in X, such that* $A \subseteq U$*. Since every open set is* $\omega\alpha$*-open set and A is g*$\omega\alpha$*-closed, we have* $\alpha cl(A) \subseteq U$ *and hence A is* αg*-closed set in X.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.7. *Let* $X = \{a, b, c\}$ *and* $\tau = \{X, \phi, \{a\}\}$ *then the set* $A = \{a, b\}$ *is* αg*-closed but not g*$\omega\alpha$*-closed in X.*

Remark 3.8. *From the theorem 3.4 and 3.6 it follows that $g\omega\alpha$-closed set properly lies between α-closed set and αg-closed set.*

Theorem 3.9. *Every regular-closed (resp ω-closed, $g\alpha$-closed) set is $g\omega\alpha$-closed set.*

Proof: *The proof is obvious from theorem 3.2.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.10. *In Example 3.3 the set $A = \{a, c\}$ is $g\omega\alpha$-closed but not regular-closed (ω-closed, $g\alpha$-closed) set in X.*

Theorem 3.11. *Every $g\omega\alpha$-closed set in X is gs-closed (resp gp-closed, gsp-closed, gpr-closed, rg-closed, ωg-closed, αgr-closed, g^*p-closed) set in X.*

Proof: *Since every open set is $\omega\alpha$-open [5], the proof follows.*

The converse of the above theorem need not be true as seen from the following example.

Example 3.12. *In Example 3.7, the set $A = \{a, b\}$ is gs-closed (gp-closed, gsp-closed, gpr-closed, rg-closed, ωg-closed, αgr-closed) but not $g\omega\alpha$-closed in X.*

Example 3.13. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ then the set $A = \{a, b\}$ is g^*p-closed but not $g\omega\alpha$-closed set in X.*

Remark 3.14. *The concept of $g\omega\alpha$-closed set is independent of the concept of sets namely p-closed, sp-closed, semi-closed, g-closed, sg-closed, g^*-closed, g^*s-closed, $\omega\alpha$-closed sets as seen from the following example.*

Example 3.15. *In Example 3.10, the set $A = \{a, c\}$ is $g\omega\alpha$-closed but not p-closed, sp-closed, semi-closed, sg-closed, g^*s-closed, and the set $B=\{b\}$ is $g\omega\alpha$-closed but not g-closed and g^*-closed in X.*

Example 3.16. *In Example 3.5, the set $A = \{b\}$ is $g\omega\alpha$-closed but not $\omega\alpha$-closed set in X.*

Example 3.17. *Let $X = \{a, b, c, d\}$ and $\tau = \{ X, \phi, \{b, c\}, \{b, c, d\}, \{a, b, c\}\}$ then the set $A = \{b\}$ is p-closed and sp-closed but not $g\omega\alpha$-closed set in X.*

Example 3.18. *Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ then the set $A = \{a\}$ is semi-closed, sg-closed and g^*s-closed but not $g\omega\alpha$-closed set in X .*

Example 3.19. *In Example 3.13, the set $A = \{a, b\}$ is g-closed, g^*-closed, and $\omega\alpha$-closed but not $g\omega\alpha$-closed set in X.*

Theorem 3.20. *Union of two $g\omega\alpha$-closed sets are a $g\omega\alpha$-closed set.*

Proof: *Let A and B be two $g\omega\alpha$-closed sets in (X, τ), let G be any $\omega\alpha$-open set in (X, τ), such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are $g\omega\alpha$-closed sets, $\alpha cl(A) \subseteq G$ and $\alpha cl(B) \subseteq G$. Therefore $\alpha cl(A) \cup \alpha cl(B) = \alpha cl(A \cup B) \subseteq G$. Hence $A \cup B$ is $g\omega\alpha$-closed set.*

Theorem 3.21. *If a subset A of X is $g\omega\alpha$-closed in (X, τ) then $\alpha cl(A)$-A does not contain any non empty $\omega\alpha$-closed set in (X, τ).*

Proof: *Suppose A is $g\omega\alpha$-closed and F be a non empty $\omega\alpha$-closed subset of $\alpha cl(A)$-A. Then $F \subseteq \alpha cl(A) \cap$ (X-A). Since (X-A) is $\omega\alpha$-open and A is $g\omega\alpha$-closed. $\alpha cl(A) \subseteq$ (X-A), therefore $F \subseteq$ (X-$\alpha cl(A)$). Thus $F \subseteq \alpha cl(A) \cap$ (X-$\alpha cl(A)$) $= \phi$. That is F = ϕ. Thus $\alpha cl(A)$-A does not contain any non-empty $\omega\alpha$-closed set in (X, τ).*

However the converse of the above theorem need not be true as seen from the following example.

Example 3.22. *In Example 3.17, the set A = $\{a, b\}$ then $\alpha cl(A)$-A = $\{c, d\}$ does not contain non empty $\omega\alpha$-closed set. But A is not $g\omega\alpha$-closed set in (X, τ).*

Theorem 3.23. *If A is $g\omega\alpha$-closed set in X and $A \subseteq B \subseteq \alpha cl(A)$ then B is also $g\omega\alpha$-closed set in X.*

Proof: *It is given that A is $g\omega\alpha$-closed set in X. To prove B is also $g\omega\alpha$-closed set of X. Let U be an $\omega\alpha$-open set of X, such that $B \subseteq U$. Since $A \subseteq B$, we have $A \subseteq U$. Since A is $g\omega\alpha$-closed, and $\alpha cl(A) \subseteq U$. Now $\alpha cl(B) \subseteq \alpha cl(\alpha cl(A)) = \alpha cl(A) \subseteq U$. So $\alpha cl(B) \subseteq U$. Hence B is $g\omega\alpha$-closed set in X.*

However the converse of the above theorem need not be true as seen from the following example.

Example 3.24. *In Example 3.5, the set A = $\{a\}$ and B = $\{a, b\}$ such that A and B are $g\omega\alpha$-closed sets but $A \subseteq B \nsubseteq \alpha cl(A)$.*

Theorem 3.25. *For each $x \in X$ either x is $\omega\alpha$-closed or x^c is $g\omega\alpha$-closed in X.*

Proof: *Suppose $\{x\}$ is not $\omega\alpha$-closed in X, then $\{x\}^c$ is not $\omega\alpha$-open and the only $\omega\alpha$-open set containing $\{x\}^c$ is the space X itself. Therefore $\alpha cl(\{x\}^c) \subseteq X$. and hence $\{x\}^c$ is $g\omega\alpha$-closed set in (X, τ).*

Theorem 3.26. *Let A be $g\omega\alpha$-closed in (X, τ). Then A is α-closed if and only if $\alpha cl(A)$-A is $\omega\alpha$-closed.*

Proof: *Necessity: Suppose A be α-closed. Then $\alpha cl(A)$ = A and so $\alpha cl(A)$-A = ϕ, which is $\omega\alpha$-closed.*
Sufficiency: Suppose $\alpha cl(A)$-A is $\omega\alpha$-closed. Then $\alpha cl(A)$-A = ϕ, since A is $g\omega\alpha$-closed. That is $\alpha cl(A)$-A or A is α-closed.

Theorem 3.27. *Let $A \subseteq Y \subseteq X$, and suppose that A is $g\omega\alpha$-closed set in X. Then A is $g\omega\alpha$-closed relative to Y.*

Proof: *Let $A \subseteq Y \cap G$ where G is $\omega\alpha$-open. Then $A \subseteq G$ and hence $\alpha cl(A) \subseteq G$. This implies that $Y \cap \alpha cl(A) \subseteq Y \cap G$. Thus A is $g\omega\alpha$-closed relative to Y.*

Now we introduce the following.

Definition 3.28. *A subset A of a topological space (X, τ) is called g$\omega\alpha$-open set if its compliment A^c is g$\omega\alpha$-closed.*

Theorem 3.29. *A subset A of (X, τ) is g$\omega\alpha$-open set if and only if U $\subseteq$ α int(A) whenever U is $\omega\alpha$-closed and U $\subseteq$ A.*

Proof: *Assume that A is g$\omega\alpha$-open in X and U is $\omega\alpha$-closed set of (X, τ) such that U $\subseteq$ A. Then X-A is a g$\omega\alpha$-closed set in (X, τ). Also X-A $\subseteq$ X-U and X-U is $\omega\alpha$-open set of (X, τ). This implies that αcl(X-A) $\subseteq$ X-U. But αcl(X-A) = X-αint(A). Thus X-αint(A) $\subseteq$ X-U. So U $\subseteq$ αint(A).*

Conversely: Suppose U $\subseteq$ αint(A) whenever U is $\omega\alpha$-closed and U $\subseteq$ A, To prove that A is g$\omega\alpha$-open. Let G be $\omega\alpha$-open set of (X, τ) such that X-A $\subseteq$ G. Then X-G $\subseteq$ A. Now X-G is $\omega\alpha$-closed set containing A. So X-G $\subseteq$ αint(A), X-αint(A) $\subseteq$ G, But αcl(X-A) = X-αint(A). Thus αcl(X-A) $\subseteq$ G. That is X-A is g$\omega\alpha$-closed set and hence A is g$\omega\alpha$-open.

Theorem 3.30. *If A is $\omega\alpha$-open and g$\omega\alpha$-closed set then A is α-closed.*

Proof: *Since A $\subseteq$ A and A is $\omega\alpha$-open and g$\omega\alpha$-closed, we have αcl(A) $\subseteq$ A. Thus αcl(A) = A. Hence A is α-closed set of (X, τ).*

Theorem 3.31. *A regular open g$\omega\alpha$-closed set is preclosed and hence clopen.*

Proof: *Let A be regular open g$\omega\alpha$-closed. Since regular open set is $\omega\alpha$-open, αcl(A) $\subseteq$ A. This implies A is α-closed. Since every α-closed (regular) open set is (regular) closed, A is clopen.*

Theorem 3.32. *A set A is g$\omega\alpha$-open in (X, τ) if and only if F $\subseteq$ αint(A) whenever F is $\omega\alpha$-closed in (X, τ) and F $\subseteq$ A.*

Proof: *Suppose F $\subseteq$ αint(A) where F is $\omega\alpha$-closed and F $\subseteq$ A. Let X-A $\subseteq$ G where G is $\omega\alpha$-open in (X, τ). Then G $\subseteq$ X-G and X-G $\subseteq$ αint(A). Thus X-A is g$\omega\alpha$-closed in (X, τ). Hence A is g$\omega\alpha$-open in (X, τ).*

Conversely: Suppose that A is g$\omega\alpha$-open. F $\subseteq$ A and F is $\omega\alpha$-closed in (X, τ). Then X-F is $\omega\alpha$-open and X-A $\subseteq$ X-F. Therefore αcl(X-A) $\subseteq$ X-F. But αcl(X-A) = X-αint(A). Hence F $\subseteq$ αint(A).

Theorem 3.33. *A subset A is g$\omega\alpha$-open in (X, τ) if and only if G = X whenever G is $\omega\alpha$-open and αint(A) $\cup$ (X-G) $\subseteq$ G.*

Proof: *Let A be g$\omega\alpha$-open. G be $\omega\alpha$-open and αint(A) $\cup$ (X-A) $\subseteq$ G. This gives X-G $\subseteq$ (X-αint(A)) $\cap$ (X-(X-A)) = X-αint(A)-(X-A) = αcl(X-A)-(X-A). Since X-A is g$\omega\alpha$-closed and X-G is $\omega\alpha$-closed. Then by theorem 3.32 it follows that X-G = ϕ. Therefore X = G.*

Conversely: Suppose F is $\omega\alpha$-closed and F $\subseteq$ A. Then αint(A) $\cup$ (X-A) $\subseteq$ αint(A) $\cup$ (X-F). It follows that αint(A) $\cup$ (X-F) = X and hence F $\subseteq$ αint(A). Therefore A is g$\omega\alpha$-open in (X, τ).

4 g$\omega\alpha$-Closure and g$\omega\alpha$-Interior

In this section the notion of g$\omega\alpha$-closure and g$\omega\alpha$-interior is defined and some of its basic properties are studied.

Definition 4.1. *For a subset A of (X, τ) g$\omega\alpha$-closure of A is denoted by g$\omega\alpha$cl(A) and is defined as* $g\omega\alpha cl(A) = \cap\{G;\ A \subseteq G,\ G$ *is g$\omega\alpha$-closed in* $(X, \tau)\}$.

Theorem 4.2. *For an $x \in X$, $x \in g\omega\alpha cl(A)$ if and only if $A \cap V \neq \phi$ for every g$\omega\alpha$-open set V containing x.*

Proof: *Let $x \in g\omega\alpha cl(A)$. Suppose there exists a g$\omega\alpha$-open set V containing x such that $V \cap A = \phi$. Then $A \subseteq X\text{-}V$, $g\omega\alpha cl(A) \subseteq X\text{-}V$. This implies $x \notin g\omega\alpha cl(A)$ which is a contradiction. Hence $A \cap V \neq \phi$.*

Conversely, Suppose $x \notin g\omega\alpha cl(A)$ then there exists g$\omega\alpha$-closed set G containing A such that $x \notin G$. Then $x \in X\text{-}G$ and X-G is g$\omega\alpha$-open. Also $(X\text{-}G) \cap A = \phi$ which is a contradiction to the hypothesis, Hence $x \in g\omega\alpha cl(A)$.

Theorem 4.3. *If $A \subseteq X$, then $A \subseteq g\omega\alpha cl(A) \subseteq cl(A)$.*

Proof: *Since every closed set is g$\omega\alpha$-closed, the proof follows.*

Remark 4.4. *Both containment relations in the theorem 4.3 may be proper as seen from the following example.*

Example 4.5. *In Example 3.10, the set $A = \{a\}$ then $g\omega\alpha cl(A) = \{a, c\}$ and $cl(A) = X$, and so $A \subseteq g\omega\alpha cl(A) \subseteq cl(A)$.*

Theorem 4.6. *If A is g$\omega\alpha$-closed, then $g\omega\alpha cl(A) = A$.*

Proof: *Let A be g$\omega\alpha$-closed set in (X, τ). Since $A \subseteq A$ and A is g$\omega\alpha$-closed set, $A \in \{G;\ A \subseteq G,\ G$ is g$\omega\alpha$-closed set $\}$ which implies that $A = \cap\{G;\ A \subseteq G,\ G$ is g$\omega\alpha$-closed set $\} \subseteq A$, that is $g\omega\alpha cl(A) \subseteq A$. But $A \subseteq g\omega\alpha cl(A)$ is always true. Hence $A = g\omega\alpha cl(A)$.*

Theorem 4.7. *If $A \subseteq X$ and A is g$\omega\alpha$-closed, then g$\omega\alpha$cl(A) is the smallest g$\omega\alpha$-closed subset of X containing A.*

Proof: *Let A be g$\omega\alpha$-closed set in (X, τ). Then $g\omega\alpha cl(A) = \cap\{\ G;\ A \subseteq G,\ G$ is g$\omega\alpha$-closed in $(X, \tau)\}$ Since $A \subseteq A$ and A is g$\omega\alpha$-closed set, $g\omega\alpha cl(A) = A$ is the smallest g$\omega\alpha$-closed subset of X containing A.*

However the converse of the above theorem need not be true as seen from the following example.

Example 4.8. *In Example 3.13, the set $A = \{a, c\}$ then $g\omega\alpha cl(A) = X$, which is the smallest g$\omega\alpha$-closed set in X containing A but A is not g$\omega\alpha$-closed in (X, τ).*

Remark 4.9. *The following example shows that for any two subsets A and B of X, $A \subseteq B$ implies $g\omega\alpha cl(A) \neq g\omega\alpha cl(B)$.*

Example 4.10. *In example 3.13, the set* $A = \{c\}$ *and* $B = \{a, c\}$ *then* $A \subseteq B$. *Now* $g\omega\alpha cl(A) = \{c\}$ *and* $g\omega\alpha cl(B) = X$. *Hence* $g\omega\alpha cl(A) \neq g\omega\alpha cl(B)$.

Remark 4.11. *For a subset A of* (X, τ) $g\omega\alpha cl(A) \neq cl(A)$ *as seen from the following example.*

Example 4.12. *In Example 3.13, the set* $A = \{c\} \subseteq X$, $g\omega\alpha cl(A) = \{c\}$ *and* $cl(A) = \{b, c\}$ *Therefore* $g\omega\alpha cl(A) \neq cl(A)$.

Remark 4.13. *For any two subsets A and B of* (X, τ), $g\omega\alpha cl(A) = g\omega\alpha cl(B)$ *does not imply that* $A = B$. *This is shown by the following example.*

Example 4.14. *In Example 3.7, the set* $A = \{a\}$ *and* $B = \{a, c\}$ *then* $g\omega\alpha cl(A) = g\omega\alpha cl(B)$. *But* $A \neq B$.

Theorem 4.15. *Let A and B be the subsets of* (X, τ), *Then,*
1. $g\omega\alpha cl(\phi) = \phi$.
2. $g\omega\alpha cl(X) = X$.
3. $g\omega\alpha cl(A)$ *is* $g\omega\alpha$*-closed set in* (X, τ).
4. *If* $A \subseteq B$ *then* $g\omega\alpha cl(A) \subseteq g\omega\alpha cl(B)$.
5. $g\omega\alpha cl(A \cup B) = g\omega\alpha cl(A) \cup g\omega\alpha cl(B)$.
6. $g\omega\alpha cl(g\omega\alpha cl(A)) = g\omega\alpha cl(A)$.

Proof: *Proof of (1), (2), (3) and (4) are obvious from definition 4.1.*
(5). We know that $g\omega\alpha cl(A) \subseteq g\omega\alpha cl(A \cup B)$ *and* $g\omega\alpha cl(B) \subseteq g\omega\alpha cl(A \cup B) \Rightarrow g\omega\alpha cl(A) \cup g\omega\alpha cl(B) \subseteq g\omega\alpha cl(A \cup B)$*–(i). Now we prove* $g\omega\alpha cl(A \cup B) \subseteq g\omega\alpha cl(A) \cup g\omega\alpha cl(B)$. *let x be any point such that* $x \notin g\omega\alpha(A) \cup g\omega\alpha cl(B)$, *then there exists* $g\omega\alpha$*-closed sets P and Q such that* $A \subseteq P$ *and* $B \subseteq Q$, $x \notin P$ *and Q, then* $x \notin P \cup Q$, $A \cup B \subseteq P \cup Q$ *and* $P \cup Q$ *is* $g\omega\alpha$*-closed set by Theorem 3.20, thus* $x \notin g\omega\alpha cl(A \cup B) \Rightarrow g\omega\alpha cl(A \cup B) \subseteq g\omega\alpha cl(A) \cup g\omega\alpha cl(B)$*–(ii). From (i) and (ii)* $g\omega\alpha cl(A \cup B) = g\omega\alpha cl(A) \cup g\omega\alpha(B)$.
(6). Let P be $g\omega\alpha$*-closed set containing A. Then by definition 4.1* $g\omega\alpha cl(A) \subseteq P$. *Since P is* $g\omega\alpha$*-closed set and contains* $g\omega\alpha cl(A)$ *and is contained in every* $g\omega\alpha$*-closed set containing A, it follows* $g\omega\alpha cl(g\omega\alpha cl(A)) \subseteq g\omega\alpha cl(A)$. *Therefore* $g\omega\alpha cl(g\omega\alpha cl(A)) = g\omega\alpha cl(A)$.

Theorem 4.16. *Let A and B be subset of* (X, τ) *then* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(A) \cap g\omega\alpha cl(B)$.

Proof: *Since* $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$, *by theorem 4.15 (4),* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(A)$ *and* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(B)$. *Thus* $g\omega\alpha cl(A \cap B) \subseteq g\omega\alpha cl(A) \cap g\omega\alpha cl(B)$.

In general $\mathrm{g}\omega\alpha\mathrm{cl(A)} \cap \mathrm{g}\omega\alpha\mathrm{cl(B)} \subseteq \mathrm{g}\omega\alpha\mathrm{cl(A \cap B)}$ as seen from the following example.

Example 4.17. *In Example 3.18, the set* $A = \{a\}$ *and* $B = \{b\}$ *then* $g\omega\alpha cl(A) = \{a, c\}$ *and* $g\omega\alpha cl(B) = \{b, c\}$ *and* $g\omega\alpha cl(A \cap B) = \phi$. *Hence* $g\omega\alpha cl(A) \cap g\omega\alpha cl(B) \subseteq g\omega\alpha cl(A \cap B)$.

Now we introduce the following.

Definition 4.18. *For a subset A of (X, τ) $g\omega\alpha$-interior of A is denoted by $g\omega\alpha int(A)$ and is defined as $g\omega\alpha int(A) = \cup\{$ G; $G \subseteq A$ and G is $g\omega\alpha$-open in $(X, \tau)\}$. that is $g\omega\alpha int(A)$ is the union of all $g\omega\alpha$-open sets contained in A.*

Theorem 4.19. *Let A be subset of (X, τ) then $g\omega\alpha int(A)$ is the largest $g\omega\alpha$-open subset of X contained in A if A is $g\omega\alpha$-open.*

Proof: *Let $A \subseteq X$ be $g\omega\alpha$-open, then $g\omega\alpha int(A) = \cup\{$ G; $G \subseteq A$ and G is $g\omega\alpha$-open in (X, τ) $\}$ Since $A \subseteq A$ and A is $g\omega\alpha$-open, A = $g\omega\alpha int(A)$ is the largest $g\omega\alpha$-open subset of X contained in A.*

The converse of the above theorem need not be true as seen from the following example.

Example 4.20. *In Example 3.18, the set A = $\{b, c\}$, then $g\omega\alpha int(A) = \{b\}$ is $g\omega\alpha$-open in (X, τ), but A is not $g\omega\alpha$-open in (X, τ).*

Remark 4.21. *For any subset A of X, $int(A) \subseteq g\omega\alpha int(A) \subseteq A$.*

Remark 4.22. *For a subset A of X, $g\omega\alpha int(A) \neq int(A)$ as seen from the following example.*

Example 4.23. *In Example 3.5, the set A = $\{b\}$, then $g\omega\alpha int(A) = \{b\}$ and int(A) = ϕ hence $g\omega\alpha int(A) \neq int(A)$.*

Remark 4.24. *For any two subsets A and B of X $g\omega\alpha int(A) = g\omega\alpha int(B)$ does not imply that A = B. That is shown by the following example.*

Example 4.25. *In Example 3.7, the set A = $\{b\}$ and B = $\{c\}$ then $g\omega\alpha int(A) = \phi = g\omega\alpha int(B)$. But $A \neq B$.*

Remark 4.26. *For any two subsets A and B of X, $g\omega\alpha int(A) \cup g\omega\alpha int(B) \neq g\omega\alpha int(A \cup B)$.*

Example 4.27. *In Example 3.18 the set A = $\{b, c\}$ and B = $\{a, c\}$ now $g\omega\alpha int(A) = \{b\}$ and $g\omega\alpha int(B) = \{a\}$ and $g\omega\alpha int(A \cup B) = g\omega\alpha intX = X$. Hence $g\omega\alpha int(A) \cup g\omega\alpha int(B) \neq g\omega\alpha int(A \cup B)$.*

Theorem 4.28. *For any subset A of X [X-$g\omega\alpha int(A)$] = [$g\omega\alpha cl$(X-A)].*

Proof: *Let $X \in$ X-$g\omega\alpha int(A)$, then X is not in $g\omega\alpha int(A)$, that is every $g\omega\alpha$-open set G containing x is such that $G \subseteq A$. This implies every $g\omega\alpha$-open set G containing x intersects X-A. That is $G \cap$ (X-A) $\neq \phi$. Then by theorem 4.2 $x \in g\omega\alpha cl$(X-A) and therefore [X-$g\omega\alpha int(A)$] $\subseteq$ [$g\omega\alpha cl$(X-A)].*
Conversely; Let $x \in g\omega\alpha cl$(X-A), then every $g\omega\alpha$-open set G containing x intersects X-A, That is, $G \cap$ (X-A) $\neq \phi$. That is every $g\omega\alpha$-open set G containing x is such that $G \subseteq A$. Then by definition 4.18, x not in $g\omega\alpha int(A)$, that is $x \in$ [X-$g\omega\alpha int(A)$]; and so [$g\omega\alpha cl$(X-A)] $\subseteq$ [X-$g\omega\alpha int(A)$]. Thus [X-$g\omega\alpha int(A)$] = [$g\omega\alpha cl$(X-A)].

5 g$\omega\alpha$-Neighborhoods and g$\omega\alpha$-Limit points

In this section we define the notion of g$\omega\alpha$-neighborhood, g$\omega\alpha$-limit point and g$\omega\alpha$-derived set of a set and show some of their basic properties and analogous to those for open sets.

Definition 5.1. *Let (X, τ) be a topological space and let $x \in X$. A subset N of X is said to be g$\omega\alpha$-neighborhood of a point $x \in X$ if there exists an g$\omega\alpha$-open set G such that $x \in G \subseteq N$.*

Definition 5.2. *Let (X, τ) be a topological space and A be a subset of X, A subset N of X is said to be g$\omega\alpha$-neighborhood of A if there exists an g$\omega\alpha$-open set G such that A $\in G \subseteq N$.*

The collection of all g$\omega\alpha$-neighborhood of x $\in$ X is called the g$\omega\alpha$-neighborhood system at x and shall be denoted by g$\omega\alpha$N(x).

Theorem 5.3. *A subset A of a topological space is g$\omega\alpha$-open if it is a g$\omega\alpha$-neighborhood of each of its points.*

Proof: *Let a subset G of a topological space be g$\omega\alpha$-open. Then for every $x \in X$, x $\in G \subseteq G$, and therefore G is a g$\omega\alpha$-neighborhood of each of its points.*

The converse of the above theorem need not be true as seen from the following example.

Example 5.4. *In Example 3.7 the set $A = \{b, c\}$ is g$\omega\alpha$-neighborhood of each of its points b and c but A is not g$\omega\alpha$-open.*

Theorem 5.5. *Let (X, τ) be a topological space. If A is g$\omega\alpha$-closed subset of X and x $\in$ g$\omega\alpha$cl(A) if and only if for any g$\omega\alpha$-neighborhood N of x in (X, τ), $N \cap A \neq \phi$.*

Proof: *Let us assume that there is a g$\omega\alpha$-neighborhood N of the point x in (X, τ) such that $N \cap A = \phi$. There exist an g$\omega\alpha$-open set G of X such that $X \in G \subseteq N$. Therefore we have $G \cap A = \phi$ and so $x \in$ X-G. Then g$\omega\alpha$cl(A) $\in$ X-G and therefore x $\notin$ g$\omega\alpha$cl(A), which is the contradiction to the hypothesis $x \in$ g$\omega\alpha$cl(A). Therefore $N \cap A \neq \phi$.*

Conversely: Suppose that $x \notin$ g$\omega\alpha$cl(A). Then there exists a g$\omega\alpha$-closed set G of (X, τ) such that $A \subseteq G$ and $x \notin G$. Thus $x \in$ X-G and X-G is g$\omega\alpha$-open in (X, τ) and hence X-G is a g$\omega\alpha$-neighborhood of x in (X, τ). But $A \cap$ (X-G) $= \phi$ which is a contradiction. Hence $x \in$ g$\omega\alpha$cl(A).

Theorem 5.6. *Let (X, τ) be a topological space and $x \in X$. Let g$\omega\alpha$N(x) be the collection of all g$\omega\alpha$-neighborhood of x. Then,*
1. g$\omega\alpha$N(x) $\neq \phi$ and $x \in$ each member of g$\omega\alpha$N(x).
2. The intersection of the any two members of g$\omega\alpha$N(x) is again a member of g$\omega\alpha$N(x).
3. If $N \in$ g$\omega\alpha$N(x) and $M \subseteq N$, then $M \in$ g$\omega\alpha$N(x).
4. Each member $N \in$ g$\omega\alpha$N(x) is a superset of a member G$\in$g$\omega\alpha$N(x) where G is a g$\omega\alpha$-open set.

Proof: *(1). Since X is $g\omega\alpha$-open set containing p, it is a $g\omega\alpha$-neighborhood of every $p \in X$. Hence there exists atleast one $g\omega\alpha$-neighborhood namely X for each $p \in X$ there is $g\omega\alpha N(p) \neq \phi$. Let $N \in g\omega\alpha N(p)$, N is a $g\omega\alpha$-neighborhood of p, then there exists a $g\omega\alpha$-open set G such that $p \in G \subseteq N$ so $p \in N$. Therefore $p \in$ every member N of $g\omega\alpha N(p)$.*
(2). Let $N \in g\omega\alpha N(p)$and $M \in g\omega\alpha N(p)$. Then by definition 5.1, there exists $g\omega\alpha$-open set G and F such that $p \in G \subseteq N$ and $p \in F \subseteq M$. Hence $p \in G \cap F \subseteq M \cap N$. Note that $G \cap F$ is a $g\omega\alpha$-open set. Therefore it follows that $N \cap M$ is a $g\omega\alpha$-neighborhood of p. Hence $N \cap M \in g\omega\alpha N(p)$.
(3). If $N \in g\omega\alpha N(p)$ then there is an $g\omega\alpha$-open set G such that $p \in G \subseteq N$. Since M $\subseteq$ N, M is $g\omega\alpha$-neighborhood of p. Hence $M \in g\omega\alpha N(p)$.
(4). Let $N \in g\omega\alpha N(p)$ then there exists a $g\omega\alpha$-open set G, such that $p \in G \subseteq N$. Since G is $g\omega\alpha$-open and $p \in G$, G is $g\omega\alpha$-neighborhood of P.therefore $G \in g\omega\alpha N(p)$ and also $G \subseteq N$.

Definition 5.7. *Let (X, τ) be a topological space and A be a subset of X. Then a point x $\in$ X is called a $g\omega\alpha$-limit point of A if and only if every $g\omega\alpha$-neighborhood of x contains a point of A distinct from x. That is $[N\text{-}\{x\}] \cap A \neq \phi$ for each $g\omega\alpha$-neighborhood N of x. Also equivalently if and only if every $g\omega\alpha$-open set G containing x contains a point of A other then x.*

In a topological space (X, τ) the set of all $g\omega\alpha$-limit points of a given subset A of X is called a $g\omega\alpha$-derived set of A and is denoted by $g\omega\alpha d(A)$.

Theorem 5.8. *Let A and B be subset of a topological space (X, τ). Then,*
1. $g\omega\alpha d(\phi) = \phi$.
2. If $A \subseteq B$, then $g\omega\alpha d(A) \subseteq g\omega\alpha d(B)$.
3. If $x \in g\omega\alpha d(A)$, then $x \in g\omega\alpha d[A\text{-}\{x\}]$.
4. $g\omega\alpha d(A \cup B) = g\omega\alpha d(A) \cup g\omega\alpha d(B)$.
5. $g\omega\alpha d(A \cap B) \subseteq g\omega\alpha d(A) \cap g\omega\alpha d(B)$.
Proof: *(1). Let x be any point of X and $x \in g\omega\alpha d(\phi)$. That is x is a $g\omega\alpha$-limit point of ϕ. Then for every $g\omega\alpha$-open set G containing x, we should have $[G\text{-}\{x\}] \cap\phi \neq \phi$ which is impossible. Hence $g\omega\alpha d(\phi) = \phi$.*
(2). If $x \in g\omega\alpha d(A)$, that is if x is $g\omega\alpha$-limit point of A, then by Definition 5.7 $[G\text{-}\{x\}] \cap A \neq \phi$ for every $g\omega\alpha$-open set G containing x. Since $A \subseteq B$ implies $[G\text{-}\{x\}] \cap A \subseteq [G\text{-}\{x\}] \cap B$. Thus if x is a $g\omega\alpha$-limit point of A it is also a $g\omega\alpha$-limit point of B, that is $x \in g\omega\alpha d(B)$. Hence $g\omega\alpha d(A) \subseteq g\omega\alpha d(B)$.
(3). If $x \in g\omega\alpha d(A)$, by definition 5.7 every $g\omega\alpha$-open set G containing x contains at least one point other than x of $A\text{-}\{x\}$. Hence x is $g\omega\alpha$-limit point of $A\text{-}\{x\}$ and it belongs to $g\omega\alpha d[A\text{-}\{x\}]$. Therefore $x \in g\omega\alpha d(A) \Rightarrow x \in g\omega\alpha d[A\text{-}\{x\}]$.
(4). Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, from (1) $g\omega\alpha d(A) \cup g\omega\alpha d(B) \subseteq g\omega\alpha d(A \cup B)$. To prove other way If $x \notin g\omega\alpha d(A) \cup g\omega\alpha d(B)$, then $x \notin g\omega\alpha d(A)$ and $x \notin g\omega\alpha d(B)$. Hence there exists $g\omega\alpha$-neighborhoods G_1 and G_2 of x such that $G_1 \cap(A\text{-}\{x\}) = \phi$ and $G_2 \cap (B\text{-}\{x\}) = \phi$ Since $G_1 \cap G_2$ is $g\omega\alpha$-neighborhood of x, we have $(G_1 \cap G_2) \cap [(A \cup B)\text{-}\{x\}] = \phi$. Therefore $x \notin g\omega\alpha d(A \cup B)$. Hence $g\omega\alpha d(A \cup B) = g\omega\alpha d(A) \cup g\omega\alpha d(B)$.
(5). Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (2) $g\omega\alpha d(A \cap B) \subseteq g\omega\alpha d(A)$ and $g\omega\alpha d(A \cap B) \subseteq g\omega\alpha d(B)$. Consequently $g\omega\alpha d(A \cap B) \subseteq g\omega\alpha d(A) \cap g\omega\alpha d(B)$.

Theorem 5.9. *Let (X, τ) be a topological space and A be a subset of X. If A is $g\omega\alpha$-closed,then $g\omega\alpha d(A) \subseteq A$.*

Proof: *Let A be $g\omega\alpha$-closed, Now we will show that $g\omega\alpha d(A) \subseteq A$. Since A is $g\omega\alpha$-closed, X-A is $g\omega\alpha$-open. To each $x \in$ X-A there exists $g\omega\alpha$-neighborhood G of x such that $G \subseteq$ X-A. Since A $\cap$ (X-A) = ϕ, the $g\omega\alpha$-neighborhood G contains no point of A and so X is not a $g\omega\alpha$-limit point of A. Thus no point of X-A can be $g\omega\alpha$-limit point of A that is, A contains all its $g\omega\alpha$-limit points. that is $g\omega\alpha d(A) \subseteq A$.*

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