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Hesitant Intuitionistic Fuzzy Ideals of Γ-Semirings

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ABSTRACT: In this paper, we have defined hesitant intuitionistic fuzzy ideals, hesitant intuitionistic fuzzy bi-ideals and hesitant intuitionistic fuzzy quasi-ideals of a Γ-semiring and obtain some of their related properties. We have discussed some inter-relations between these ideals. We also obtain some characterizations of regular Γ-semiring.

Keywords – *Hesitant, Intuitionistic fuzzy ideal, Cartesian product, Regular, Γ-semiring*

1. Introduction

The notion of a semiring as first introduced by Vandiver (1934) in 1934 with two associative binary operations where one distributes over the other. But semirings had appeared in earlier studies on the theory of ideals of rings. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups may hold in semirings, since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure of semiring is not an independent of additive structure of semiring. The additive and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Also, semirings has some applications to the theory of automata, formal languages, optimization theory and other branches of applied mathematics. Ideals of semiring play a central role in the structure theory and useful for many purposes. The notion of Γ-semiring was introduced by Rao (1995) as a generalization of Γ-ring as well as of semiring. Γ-semirings also includes ternary semirings and provide algebraic home to nonpositive cones of totally ordered rings.

The theory of fuzzy sets, proposed by Zadeh (1965), has provided a useful mathematical tool for describing the behavior of the systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. As an important generalization of this notion, Atanassov (1993), introduced the concept of an intuitionistic fuzzy set where besides the degree of membership of each element there was considered a degree of non-membership with (membership value + non-membership value)≤1. In 2010, Torra (2010) introduced the hesitant fuzzy set which permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1. The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy set theory has been applied to several practical problems, see (Jun and Song, 2014; Rodriguez, 2012; Torra and Narukawa, 2009; Wei 2012; Xia and Xu, 2011; Xia et al., 2013; Xu and Xia, 2011; Zhu et al., 2012). Jun and Khan (2015) applied notion of hesitant fuzzy sets to semigroups and investigated several properties. Since then many researchers developed this idea.

The main aim of this paper is to combine the notion of intuitionistic fuzzy set and hesitant fuzzy set and apply it in studying several properties of Γ-semirings. We also obtain some characterizations.

2. Preliminaries

Definition 1. (Rao, 1995) Let S and Γ be two additive commutative semigroups with zero. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ ($((a, \alpha, b) \mapsto a\alpha b)$ satisfying the following conditions:
(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

A subset *A* of a Γ-semiring *S* is called a left (resp. right) ideal of *S* if *A* is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset *A* of a Γ-semiring *S* is called an ideal if it is both left and right ideal of *S*.

A subset A of a Γ-semiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$.
A subset *A* of a Γ-semiring *S* is called a bi-ideal if *A* is closed under addition and $A\Gamma S\Gamma A \subseteq A$.

Definition 2. (Zadeh, 1965) A fuzzy subset of a non-empty set X is defined as a function μ : X → [0,1].

Definition 3. (Torra and Narukawa, 2009) Hesitant fuzzy set on S in terms of a function *H* that when applied to X returns a subset of [0, 1].

Definition 4. (Atanassov, 1983) An intuitionistic fuzzy set defined on a non-empty set X, denoted by IFS(X). An IFS(X) is an object having the form $A = <\mu_A, \lambda_A> = \{x, \mu_A(x), \lambda_A(x) | x \in X\}$ where the fuzzy sets μ_A and λ_A denote the degree of membership(namely $\mu_A(x)$) and the degree of non-membership(namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Throughout this paper unless otherwise mentioned S denotes the Γ-semiring and for any two set P and Q, we use the following notation:

$$\cap \{P, Q\} = P \cap Q and \cup \{P, Q\} = P \cup Q.$$

3. Basic Definitions and Results of Hesitant Intuitionistic Fuzzy Ideals

In this section, the notion of hesitant intuitionistic fuzzy ideals in semiring are introduced and some of their basic properties are investigated.

Definition 5. Let *X* be a non-empty set. A hesitant intuitionistic fuzzy set A in X is a structure $A = \{< x, \mu_A, \lambda_A >: x \in X\}$ which briefly denoted as $A = < \mu_A, \lambda_A >$.

Definition 6. Let $A =< \mu_A, \lambda_A >$ be a non empty hesitant intuitionistic fuzzy subset of a Γ-semiring S. Then $< \mu_A, \lambda_A >$ is called a hesitant intuitionistic fuzzy left ideal [hesitant intuitionistic fuzzy right ideal] of S if

(i) $\mu_A(x + y) \supseteq \cap \{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x + y) \subseteq \cup \{\lambda_A(x), \lambda_A(y)\}$ and

(ii) $\mu_A(x\gamma y) \supseteq \mu_A(y)$, $\lambda_A(x\gamma y) \subseteq \lambda_A(y)$

[respectively $\mu_A(x\gamma y) \supseteq \mu_A(x), \lambda_A(x\gamma y) \subseteq \lambda_A(x)$].

for all $x, y \in S$ and $\gamma \in \Gamma$.

A hesitant intuitionistic fuzzy ideal of a Γ-semiring S is a non empty hesitant intuitionistic fuzzy subset of S which is a hesitant intuitionistic fuzzy left ideal as well as a hesitant intuitionistic fuzzy right ideal of S.

Note that if $< \mu_A, \lambda_A >$ is a hesitant intuitionistic fuzzy left or right ideal of a Γ-semiring S, then $\mu_A(0) \supseteq \mu_A(x)$ and $\lambda_A(0) \subseteq \lambda_A(x)$ for all $x \in S$.

A hesitant intuitionistic fuzzy right ideal is defined similarly. By a hesitant intuitionistic fuzzy ideal $A =< \mu_A, \lambda_A >$, we mean that $A =< \mu_A, \lambda_A >$ is both hesitant intuitionistic fuzzy left and hesitant intuitionistic fuzzy right ideal.

Example 1. Let $S = \Gamma =$,the set of all non positive integers. Then S forms a Γ-semiring with usual addition and multiplication of integers. Define $A =< \mu_A, \lambda_A >$ be a hesitant intuitionistic fuzzy subset of S as follows

$$\begin{aligned} \mu_A(x) &= [0,1] \; if \; x = 0 \\ &= 0.2 \cup (0.3,0.8] \; if \; x \; is \; even \\ &= [0.4,0.7) \; if \; x \; is \; odd \end{aligned}$$

and

$$\begin{aligned} \lambda_A(x) &= \phi \; if \; x = 0 \\ &= [0.5,0.7) if \; x \; is \; even \\ &= (0.1,0.8) \; if \; x \; is \; odd \end{aligned}$$

The hesitant intuitionistic fuzzy subset $< \mu_A, \lambda_A >$ of S defined above is a hesitant intuitionistic fuzzy ideal S.

Throughout this section, we prove results only for hesitant intuitionistic fuzzy left ideals. Similar results can be obtained for hesitant intuitionistic fuzzy right ideals and hesitant intuitionistic fuzzy ideals.

Definition 7. The characteristic hesitant intuitionistic fuzzy set of A is defined to be the structure $\chi_A(x) =< x, \mu_{\chi A}(x), \lambda_{\chi_A}(x): x \in X >$ where

$$\begin{aligned} \mu_{\chi A}(x) &= [0,1] if x \in A \\ &= \phi if x \not\subseteq A. \end{aligned}$$

and

$$\begin{aligned} \lambda_{\chi_A}(x) &= \phi \; if \; x \in A \\ &= [0,1] \; if \; x \not\subseteq A. \end{aligned}$$

Theorem 1. Let I be a non-empty subset of a Γ-semiring S. Then I is a left ideal of S if and only if the characteristic function $\chi_I =< \mu_{\chi I}, \lambda_{\chi_I} >$ is a hesitant intuitionistic fuzzy left ideal of S.

PROOF. Assume that I is a left ideal of S and $x, y \in I$ and $\gamma \in \Gamma$. Suppose χ_I is not a hesitant intuitionistic fuzzy left ideal of S. So, $\mu_{\chi I}(x + y) \subset \cap \{\mu_{\chi I}(x), \mu_{\chi I}(y)\}$ and $\lambda_{\chi_I}(x + y) \supset \cap$ $\{\lambda_{\chi_I}(x), \lambda_{\chi_I}(y)\}$. Since $x, y \in I$, $\mu_{\chi I}(x) = [0,1]$, $\mu_{\chi I}(y) = [0,1]$, $\lambda_{\chi_I}(x) = \phi$ and $\lambda_{\chi_I}(y) =$

ϕ; as a result, $\mu_{\chi I}(x+y)=\phi$ and $\lambda_{\chi_I}=[0,1]$ i.e. $x+y\not\subseteq I$ - contradiction to the fact that I is a left ideal of S. Hence $\mu_{\chi I}(x+y)\supseteq\cap\{\mu_{\chi I}(x),\mu_{\chi I}(y)\}$ and $\lambda_{\chi_I}(x+y)\subseteq\cap\{\lambda_{\chi_I}(x),\lambda_{\chi_I}(y)\}$. Similarly we can show that $\mu_{\chi I}(x\gamma y)\supseteq\mu_{\chi I}(y)$, $\lambda_{\chi I}(x\gamma y)\leq\lambda_{\chi I}(y)$.
Therefore $\chi_I=<\mu_{\chi I},\lambda_{\chi_I}>$ is a hesitant intuitionistic fuzzy left ideal of S.
Conversely, assume that $\chi_I=<\mu_{\chi I},\lambda_{\chi_I}>$ is a hesitant intuitionistic fuzzy left ideal of S for any subset I of S. Let $x,y\in I$ and $\gamma\in\Gamma$. Then $\mu_{\chi I}(x)=[0,1]$, $\mu_{\chi I}(y)=[0,1]$, $\lambda_{\chi_I}(x)=\phi$ and $\lambda_{\chi_I}(y)=\phi$. Now $\mu_{\chi I}(x+y)\supseteq\cap\{\mu_{\chi I}(x),\mu_{\chi I}(y)\}=[0,1]$, $\lambda_{\chi_I}(x+y)\subseteq\cup\{\lambda_{\chi_I}(x),\lambda_{\chi_I}(y)\}=\phi$ and $\mu_{\chi_I}(x\gamma y)\supseteq\mu_{\chi_I}(y)=[0,1]$, $\lambda_{\chi_I}(x\gamma y)\subseteq\lambda_{\chi_I}(y)=\phi$. This implies $x+y,x\gamma y\in I$.
Hence I is a left ideal of S.

Definition 8. Let $A=<\mu_A,\lambda_A>$ and $B=<\mu_B,\lambda_B>$ be two hesitant intuitionistic fuzzy sets of a Γ-semiring S. Define intersection of A and B by

$$A\cap B=<\mu_A,\lambda_A>\cap<\mu_B,\lambda_B>=<\mu_A\cap\mu_B,\lambda_A\cup\lambda_B>.$$

Proposition 1. Intersection of a non-empty collection of hesitant intuitionistic fuzzy left ideals is a hesitant intuitionistic fuzzy left ideal of S.

PROOF. Assume that $A_i=\{<\mu_{A_i},\lambda_{A_i}>:i\in I\}$ be a non-empty family of ideals of S. Let $x,y\in S$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}(\bigcap_{i\in I}\mu_{A_i})(x+y)&=\bigcap_{i\in I}\{\mu_{A_i}(x+y)\}\supseteq\bigcap_{i\in I}\{\cap\{\mu_{A_i}(x),\mu_{A_i}(y)\}\}\\&=\cap\{\bigcap_{i\in I}\mu_{A_i}(x),\bigcap_{i\in I}\mu_{A_i}(y)\}=\cap\{(\bigcap_{i\in I}\mu_{A_i})(x),(\bigcap_{i\in I}\mu_{A_i})(y)\}.\end{aligned}$$

$$\begin{aligned}(\bigcup_{i\in I}\lambda_{A_i})(x+y)&=\bigcup_{i\in I}\{\lambda_{A_i}(x+y)\}\subseteq\bigcup_{i\in I}\{\cup\{\lambda_{A_i}(x),\lambda_{A_i}(y)\}\}\\&=\cup\{\bigcup_{i\in I}\lambda_{A_i}(x),\bigcup_{i\in I}\lambda_{A_i}(y)\}=\cup\{(\bigcup_{i\in I}\lambda_{A_i})(x),(\bigcup_{i\in I}\lambda_{A_i})(y)\}.\end{aligned}$$

Again

$$(\bigcap_{i\in I}\mu_{A_i})(x\gamma y)=\bigcap_{i\in I}\{\mu_{A_i}(x\gamma y)\}\supseteq\bigcap_{i\in I}\{\mu_{A_i}(y)\}=(\bigcap_{i\in I}\mu_{A_i})(y).$$

$$(\bigcup_{i\in I}\lambda_{A_i})(x\gamma y)=\bigcup_{i\in I}\{\lambda_{A_i}(x\gamma y)\}\subseteq\bigcup_{i\in I}\{\lambda_{A_i}(y)\}=(\bigcup_{i\in I}\lambda_{A_i})(y).$$

Hence $\bigcap_{i\in I}A_i=\{<\bigcap_{i\in I}\mu_{A_i},\bigcup_{i\in I}\lambda_{A_i}>:i\in I\}$ is a hesitant intuitionistic fuzzy left ideal of S.

Proposition 2. Let $f:R\to S$ be a morphism of Γ-semirings [2] and $A=<\mu_A,\lambda_A>$ be is a hesitant intuitionistic fuzzy left ideal of S, then $f^{-1}(A)$ is a hesitant intuitionistic fuzzy left ideal of R where $f^{-1}(A)(x)=<f^{-1}(\mu_A)(x),f^{-1}(\lambda_A)(x)>=<\mu_A(f(x)),\lambda_A(f(x))>$.

PROOF. Suppose $f:R\to S$ be a morphism of Γ-semirings. Let $A=<\mu_A,\lambda_A>$ and $r,s\in R$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}f^{-1}(\mu_A)(r+s)&=\mu_A(f(r+s))=\mu_A(f(r)+f(s))\\&\supseteq\cap\{\mu_A(f(r)),\mu_A(f(s))\}=\cap\{f^{-1}(\mu_A)(r),f^{-1}(\mu_A)(s)\}\end{aligned}$$

$$\begin{aligned}f^{-1}(\lambda_A)(r+s)&=\lambda_A(f(r+s))=\lambda_A(f(r)+f(s))\\&\subseteq\cup\{\lambda_A(f(r)),\lambda_A(f(s))\}=\cup\{(f^{-1}(\lambda_A))(r),(f^{-1}(\lambda_A))(s)\}\end{aligned}$$

Again $(f^{-1}(\mu_A))(r\gamma s)=\mu_A(f(r\gamma s))=\mu_A(f(r)\gamma f(s))\supseteq\mu_A(f(s))=(f^{-1}(\mu_A))(s)$.

$$(f^{-1}(\lambda_A))(r\gamma s)=\lambda_A(f(r\gamma s))=\lambda_A(f(r)\gamma f(s))\subseteq\lambda_A(f(s))=(f^{-1}(\lambda_A))(s).$$

Thus $<f^{-1}(\mu_A)(x),f^{-1}(\lambda_A)(x)>$ is a hesitant intuitionistic fuzzy left ideal of R.

Definition 9. Let $A=<\mu_A,\lambda_A>$ and $B=<\mu_B,\lambda_B>$ be hesitant intuitionistic fuzzy subsets of X. The cartesian product of A and B is defined by

$$\begin{aligned}(A\times B)(x,y)&=(<\mu_A,\lambda_A>\times<\mu_B,\lambda_B>)(x,y)=(<\mu_A\times\mu_B,\lambda_A\times\lambda_B>)(x,y)\\&=[\cap\{\mu_A(x),\mu_B(y)\},\cup\{\lambda_A(x),\lambda_B(y)\}]\quad\text{for all } x,y\in X.\end{aligned}$$

Theorem 2. Let $A =< \mu_A, \lambda_A >$ and $B =< \mu_B, \lambda_B >$ be hesitant intuitionistic fuzzy left ideals of a Γ-semiring S. Then $A \times B$ is a hesitant intuitionistic fuzzy left ideal of the Γ-semiring $S \times S$.

PROOF. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}(\mu_A \times \mu_B)((x_1, x_2) + (y_1, y_2)) &= (\mu_A \times \mu_B)(x_1 + y_1, x_2 + y_2)\\ &= \cap \{\mu_A(x_1 + y_1), \mu_B(x_2 + y_2)\}\\ &\supseteq \cap \{\cap \{\mu_A(x_1), \mu_A(y_1)\}, \cap \{\mu_B(x_2), \mu_B(y_2)\}\}\\ &= \cap \{\cap \{\mu_A(x_1), \mu_B(x_2)\}, \cap \{\mu_A(y_1), \mu_B(y_2)\}\}\\ &= \cap \{(\mu_A \times \mu_B)(x_1, x_2), (\mu_A \times \mu_B)(y_1, y_2)\}\end{aligned}$$

$$\begin{aligned}(\lambda_A \times \lambda_B)((x_1, x_2) + (y_1, y_2)) &= (\lambda_A \times \lambda_B)(x_1 + y_1, x_2 + y_2)\\ &= \cup \{\lambda_A(x_1 + y_1), \lambda_B(x_2 + y_2)\}\\ &\subseteq \cup \{\cup \{\lambda_A(x_1), \lambda_A(y_1)\}, \cup \{\lambda_B(x_2), \lambda_B(y_2)\}\}\\ &= \cup \{\cup \{\lambda_A(x_1), \lambda_B(x_2)\}, \cup \{\lambda_A(y_1), \lambda_B(y_2)\}\}\\ &= \cup \{(\lambda_A \times \lambda_B)(x_1, x_2), (\lambda_A \times \lambda_B)(y_1, y_2)\}\end{aligned}$$

and

$$\begin{aligned}(\mu_A \times \mu_B)((x_1, x_2)\gamma(y_1, y_2)) &= (\mu_A \times \mu_B)(x_1\gamma y_1, x_2\gamma y_2)\\ &= \cap \{\mu_A(x_1\gamma y_1), \mu_B(x_2\gamma y_2)\}\\ &\supseteq \cap \{\mu_A(y_1), \mu_B(y_2)\}\\ &= (\mu_A \times \mu_B)(y_1, y_2).\end{aligned}$$

$$\begin{aligned}(\lambda_A \times \lambda_B)((x_1, x_2)\gamma(y_1, y_2)) &= (\lambda_A \times \lambda_B)(x_1\gamma y_1, x_2\gamma y_2)\\ &= \cup \{\lambda_A(x_1\gamma y_1), \lambda_B(x_2\gamma y_2)\}\\ &\subseteq \cup \{\lambda_A(y_1), \lambda_B(y_2)\}\\ &= (\lambda_A \times \lambda_B)(y_1, y_2).\end{aligned}$$

Hence $A \times B$ is a hesitant intuitionistic fuzzy left ideal of $S \times S$.

3. Hesitant Intuitionistic Fuzzy Bi-Ideals and Quasi-Ideals

Definition 10. Let $A =< \mu_A, \lambda_A >$ and $B =< \mu_B, \lambda_B >$ be two hesitant intuitionistic fuzzy sets of a Γ-semiring S. Define composition of A and B by

$$AoB =< \mu_A, \lambda_A > o < \mu_B, \lambda_B >=< \mu_A o \mu_B, \lambda_A o \lambda_B >$$

where

$$\begin{aligned}\mu_A o \mu_B(x) &= \cup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} [\cap_i \{\cap \{\mu_A(a_i), \mu_B(b_i)\}\}]\\ &= \phi, \text{if x cannot be expressed as above}\end{aligned}$$

and

$$\begin{aligned}(\lambda_A o \lambda_B)(x) &= \cap_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} [\cup_i \{\cup \{\lambda_A(a_i), \lambda_B(b_i)\}\}]\\ &= [0,1], \text{if x cannot be expressed as above}\end{aligned}$$

where $x, a_i, b_i \in S$, $\gamma_i \in \Gamma$ and $i = 1, \dots, n$.

Lemma 1. Let $A =< \mu_A, \lambda_A >$, $B =< \mu_B, \lambda_B >$ be two hesitant intuitionistic fuzzy ideal of a Γ-semiring S. Then $AoB \subseteq A \cap B \subseteq A, B$.

PROOF. Suppose $A =< \mu_A, \lambda_A >$, $B =< \mu_B, \lambda_B >$ be two hesitant intuitionistic fuzzy ideal of a Γ-semiring S. Then

$$(\mu_A o \mu_B)(x) = \bigcup_{x=\sum_{i=1}^n a_i\gamma_i b_i} \{\bigcap_i \{\cap \{\mu_A(a_i), \mu_B(b_i)\}\}\}$$

$where\ x, a_i, b_i \in S, \gamma_i \in \Gamma\ and\ i = 1, \ldots, n.$

$$\subseteq \cup \{\bigcap_i \{\mu_A(a_i)\}\}$$

$$\subseteq \bigcup_{x=\sum_{i=1}^n a_i\gamma_i b_i} \{\cap \{\mu_A(\sum_{i=1}^n a_i\gamma_i b_i)\}\} = \mu_A(x)$$

$$(\lambda_A o \lambda_B)(x) = \bigcap_{x=\sum_{i=1}^n a_i\gamma_i b_i} \{\bigcup_i \{\cup \{\lambda_A(a_i), \lambda_B(b_i)\}\}\}$$

$where\ x, a_i, b_i \in S, \gamma_i \in \Gamma\ and\ i = 1, \ldots, n.$

$$= \cap \{\bigcup_i \lambda_A(a_i)\}$$

$$\supseteq \bigcap_{x=\sum_{i=1}^n a_i b_i} \{\cup \{\lambda_A(\sum_{i=1}^n a_i\gamma_i b_i)\}\} = \lambda_A(x)$$

Since this is true for every representation of x, $AoB \subseteq A$.
Similarly, we can prove that $AoB \subseteq B$.
Therefore $AoB \subseteq A \cap B$.
Hence the lemma.

Definition 11. A hesitant intuitionistic fuzzy subset $< \mu_A, \lambda_A >$ of a Γ-semiring S is called hesitant intuitionistic fuzzy bi-ideal if for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$ we have
(i) $\mu_A(x + y) \supseteq \cap \{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x + y) \subseteq \cup \{\lambda_A(x), \lambda_A(y)\}$
(ii) $\mu_A(x\alpha y) \supseteq \cap \{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x\alpha y) \subseteq \cup \{\lambda_A(x), \lambda_A(y)\}$
(iii) $\mu_A(x\alpha y\beta z) \supseteq \cap \{\mu_A(x), \mu_A(z)\}$, $\lambda_A(x\alpha y\beta z) \subseteq \cup \{\lambda_A(x), \lambda_A(z)\}$

Definition 12. A hesitant intuitionistic fuzzy subset $< \mu_A, \lambda_A >$ of a Γ-semiring S is called hesitant intuitionistic fuzzy quasi-ideal if for all $x, y \in S$ we have
(i) $\mu_A(x + y) \supseteq \cap \{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x + y) \subseteq \cup \{\lambda_A(x), \lambda_A(y)\}$
(ii) $(\mu_A o \mu_{\chi_S}) \cap (\mu_{\chi_S} o \mu_A) \subseteq \mu_A$, $(\lambda_A o \lambda_{\chi_S}) \cup (\lambda_{\chi_S} o \lambda_A) \supseteq \lambda_A$.

Theorem 3. A hesitant intuitionistic fuzzy subset $< \mu_A, \lambda_A >$ of a Γ-semiring S is a hesitant intuitionistic fuzzy left ideal of S if and only if for all x,y∈S, we have
(i) $\mu_A(x + y) \supseteq \cap \{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x + y) \subseteq \cup \{\lambda_A(x), \lambda_A(y)\}$
(ii) $\mu_{\chi_S} o \mu_A \subseteq \mu_A$, $\lambda_{\chi_S} o \lambda_A \supseteq \lambda_A$.

PROOF. Assume that $< \mu_A, \lambda_A >$ is a hesitant intuitionistic fuzzy left ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in$S. If x can be expressed as $x = \sum_{i=1}^n a_i\gamma_i b_i$, for $a_i, b_i \in S$, $\gamma_i \in \Gamma$ and i=1,...,n, then we have

$$(\mu_{\chi_S} o \mu_A)(x) = \bigcup_{x=\sum_{i=1}^n a_i\gamma_i b_i} [\bigcap_i \{\cap \{\mu_{\chi_S}(a_i), \mu_A(b_i)\}\}] \subseteq \bigcup_{x=\sum_{i=1}^n a_i\gamma_i b_i} [\bigcap_i \{\cap \{\mu_A(a_i\gamma_i b_i)\}\}]$$

$$\subseteq \bigcup_{x=\sum_{i=1}^n a_i\gamma_i b_i} [\cap \{\mu_A(\sum_{i=1}^n a_i\gamma_i b_i)\}] = \mu_A(x).$$

$$(\lambda_{\chi_S} o \lambda_A)(x) = \bigcap_{x=\sum_{i=1}^n a_i\gamma_i b_i} [\bigcup_i \{\cup \{\lambda_{\chi_S}(a_i), \lambda_A(b_i)\}\}] = \bigcap_{x=\sum_{i=1}^n a_i\gamma_i b_i} [\bigcup_i \{\cup \{\lambda_A(b_i\}\}]$$

$$\geq \bigcap_{x=\sum_{i=1}^n a_i\gamma_i b_i} [\bigcup_i \{\cup \{\lambda_A(a_i\gamma_i b_i)\}\}] \geq \cap [\cup \{\lambda_A(\sum_{i=1}^n a_i\gamma_i b_i)\}] = \lambda_A(x).$$

This implies that $\mu_{\chi_S} o \mu_A \subseteq \mu_A$, $\lambda_{\chi_S} o \lambda_A \supseteq \lambda_A$.

Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of hesitant intuitionistic fuzzy ideal. Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\mu_A(x\gamma y) \supseteq (\mu_{\chi_S} o \mu_A)(x\gamma y) = \bigcup_{x\gamma y=\sum_{i=1}^{n} a_i\gamma_i b_i} [\bigcap_i \{\cap \{\mu_{\chi_S}(a_i), \mu_A(b_i)\}\}]$$
$$\supseteq \mu_A(y)(since\ x\gamma y = x\gamma y).$$

$$\lambda_A(x\gamma y) \subseteq (\lambda_{\chi_S} o \lambda_A)(x\gamma y) = \bigcap_{x\gamma y=\sum_{i=1}^{n} a_i\gamma_i b_i} [\bigcup_i \{\cup \{\lambda_{\chi_S}(a_i), \lambda_A(b_i)\}\}]$$
$$\subseteq \lambda_A(y)(since\ x\gamma y = x\gamma y).$$

Hence $< \mu_A, \lambda_A >$ is a hesitant intuitionistic fuzzy left ideal of S.

Theorem 4. Let $A =< \mu_A, \lambda_A >$ and $B =< \mu_B, \lambda_B >$ be a hesitant intuitionistic fuzzy right ideal and a hesitant intuitionistic fuzzy left ideal of a Γ-semiring S, respectively. Then $A \cap B$ is a hesitant intuitionistic fuzzy quasi-ideal of S.

PROOF. Let x, y be any element of S. Then

$$\begin{aligned}(\mu_A \cap \mu_B)(x + y) &= \cap \{\mu_A(x + y), \mu_B(x + y)\} \\ &\supseteq \cap \{\cap \{\mu_A(x), \mu_A(y)\}, \cap \{\mu_B(x), \mu_B(y)\}\} \\ &= \cap \{\cap \{\mu_A(x), \mu_B(x)\}, \cap \{\mu_A(y), \mu_B(y)\}\} \\ &= \cap \{(\mu_A \cap \mu_B)(x), (\mu_A \cap \mu_B)(y)\}.\end{aligned}$$

$$\begin{aligned}(\lambda_A \cup \lambda_B)(x + y) &= \cup \{\lambda_A(x + y), \lambda_B(x + y)\} \\ &\subseteq \cup \{\cup \{\lambda_A(x), \lambda_A(y)\}, \cup \{\lambda_B(x), \lambda_B(y))\} \\ &= \cup \{\cup \{\lambda_A(x), \lambda_B(x)\}, \cap \{\lambda_A(y), \lambda_B(y)\}\} \\ &= \cup \{(\lambda_A \cup \lambda_B)(x), (\lambda_A \cup \lambda_B)(y)\}.\end{aligned}$$

On the other hand, we have

$$((A \cap B) o \chi_S) \cap (\chi_S o (A \cap B)) \subseteq (A o \chi_S) \cap (\chi_S o B) \subseteq (A \cap B).$$

This completes the proof.

Lemma 2. Any hesitant intuitionistic fuzzy quasi-ideal of S is a hesitant intuitionistic fuzzy bi-ideal of S.

PROOF. Let $A =< \mu_A, \lambda_A >$ be any hesitant intuitionistic fuzzy quasi-ideal of S. It is sufficient to show that $\mu_A(x\alpha y\beta z) \supseteq \cap \{\mu_A(x), \mu_A(z)\}$, $\lambda_A(x\alpha y\beta z) \subseteq \cup \{\lambda_A(x), \lambda_B(z)\}$ and $\mu_A(x\alpha y) \supseteq \cap \{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x\alpha y) \subseteq \cup \{\lambda_A(x), \lambda_B(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. In fact, by the assumption, we have

$$\begin{aligned}\mu_A(x\alpha y\beta z) &\supseteq ((\mu_A o \mu_{\chi_S}) \cap (\mu_{\chi_S} o \mu_A))(x\alpha y\beta z) \\ &= \cap \{(\mu_A o \mu_{\chi_S})(x\alpha y\beta z), (\mu_{\chi_S} o \mu_A)(x\alpha y\beta z)\} \\ &= \cap \{ \bigcup_{x\alpha y\beta z=\sum_{i=1}^{n} a_i\gamma_i b_i} (\cap (\mu_A(a_i), \ \mu_{\chi_S}(b_i))), \bigcup_{x\alpha y\beta z=\sum_{i=1}^{n} a_i\gamma_i b_i} (\cap (\mu_{\chi_S}(a_i)\ , \mu_A(b_i))\} \\ &\supseteq \cap \{\cap (\mu_A(x), \mu_{\chi_S}(z)), \cap (\mu_{\chi_S}(x), \mu_A(z))\}(since\ x\alpha y\beta z = x\alpha y\beta z) \\ &= \cap \{\mu_A(x), \mu_A(z)\}\end{aligned}$$

$$\begin{aligned}\lambda_A(x\alpha y\beta z) &\subseteq (\lambda_A o\lambda_{\chi_S}) \cup (\lambda_{\chi_S} o\lambda_A))(x\alpha y\beta z)\\ &= \cup\{(\lambda_A o\lambda_{\chi_S})(x\alpha y\beta z), (\lambda_{\chi_S} o\lambda_A)(x\alpha y\beta z)\}\\ &= \cup\{\bigcap_{x\alpha y\beta z=\sum_{i=1}^n a_i\gamma_i b_i}(\cup(\lambda_A(a_i),\ \lambda_{\chi_S}(b_i))), \bigcap_{x\alpha y\beta z=\sum_{i=1}^n a_i\gamma_i b_i}(\cup(\lambda_{\chi_S}(a_i)\ , \lambda_A(b_i)))\}\\ &\subseteq \cup\{\cup(\lambda_{\chi_S}(x), \lambda_A(z)), \cup(\lambda_A(x), \lambda_{\chi_S}(z))\}(since\ x\alpha y\beta z = x\alpha y\beta z)\\ &= \cup\{\lambda_A(x), \lambda_A(z)\}\end{aligned}$$

Similarly, we can show that $\mu_A(x\alpha y) \supseteq \cap\{\mu_A(x), \mu_A(y)\}$, $\lambda_A(x\alpha y) \subseteq \cup\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in S$ and $\alpha \in \Gamma$.

Definition 13. A Γ-semiring S is said to be regular if for each $x \in S$, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$.

Theorem 5. Let S be a regular Γ-semiring. Then for any hesitant intuitionistic fuzzy right ideal $A =< \mu_A, \lambda_A >$ and any hesitant intuitionistic fuzzy left ideal $B =< \mu_B, \lambda_B >$ of S we have $AoB = A \cap B$.

PROOF. Let S be a regular Γ-semiring. By Lemma 1, we have $AoB \subseteq A \cap B$.
For any $a \in S$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Then

$$\begin{aligned}(\mu_A o\mu_B)(a) &= \bigcup_{a=\sum_{i=1}^n a_i\gamma_i b_i}\{\cap\{\mu_A(a_i), \mu_B(b_i)\}\} \supseteq \cap\{\mu_A(a\alpha x), \mu_B(a)\}\\ &\supseteq \cap\{\mu_A(a), \mu_B(a)\} = (\mu_A \cap \mu_B)(a).\end{aligned}$$

$$\begin{aligned}(\lambda_A o\lambda_B)(a) &= \bigcap_{a=\sum_{i=1}^n a_i\gamma_i b_i}\{\cup\{\lambda_A(a_i), \lambda_B(b_i)\}\} \subseteq \cup\{\lambda_A(a\alpha x), \lambda_B(a)\}\\ &\subseteq \cup\{\lambda_A(a), \lambda_B(a)\} = (\lambda_A \cup \lambda_B)(a).\end{aligned}$$

Therefore $(A \cap B) \subseteq (AoB)$.
Hence $AoB = A \cap B$.

Theorem 6. Let S be a regular Γ-semiring. Then
(i) $A \subseteq Ao\chi_S oA$ for every hesitant intuitionistic fuzzy bi-ideal $A =< \mu_A, \lambda_A >$ of S.
(ii) $A \subseteq Ao\chi_S oA$ for every hesitant intuitionistic fuzzy quasi-ideal $A =< \mu_A, \lambda_A >$ of S.

PROOF. Suppose that $A =< \mu_A, \lambda_A >$ be any hesitant intuitionistic fuzzy bi-ideal of S and x be any element of S. Since S is regular there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$. Now

$$\begin{aligned}(\mu_A o\mu_{\chi_S} o\mu_A)(x) &= \bigcup_{x=\sum_{i=1}^n a_i\gamma_i b_i}(\cap\{(\mu_A o\mu_{\chi_S})(a_i), \mu_A(b_i)\})\\ &\supseteq \cap\{(\mu_A o\mu_{\chi_S})(x\alpha a), \mu_A(x)\}\\ &= \cap\{\bigcup_{x\alpha a=\sum_{i=1}^n a_i\gamma_i b_i}(\cap\{\mu_A(a_i), (\mu_{\chi_S})(b_i)\}), \mu_A(x)\}\\ &\supseteq \cap\{\mu_A(x), \mu_A(x)\}(since\ x\alpha a = x\alpha a\beta x\alpha a).\\ &= \mu_A(x)\end{aligned}$$

$$\begin{aligned}(\lambda_A o\lambda_{\chi_S} o\lambda_A)(x) &= \bigcap_{x=\sum_{i=1}^n a_i\gamma_i b_i}(\cup\{(\lambda_A o\lambda_{\chi_S})(a_i), \lambda_A(b_i)\})\\ &\subseteq \cup\{(\lambda_A o\lambda_{\chi_S})(x\alpha a), \lambda_A(x)\}\\ &= \cup\{\bigcap_{x\alpha a=\sum_{i=1}^n a_i\gamma_i b_i}(\cup\{(\lambda_A(a_i), \lambda_{\chi_S}(b_i))\}), \lambda_A(x)\}\\ &\subseteq \cup\{\lambda_A(x), \lambda_A(x)\}(since\ x\alpha a = x\alpha a\beta x\alpha a)\\ &= \lambda_A(x)\end{aligned}$$

This implies that $A \subseteq Ao\chi_S oA$.
(i)⇒(ii) This is straight forward from Lemma 2.

Theorem 7. Let S be a regular Γ-semiring. Then

(i). $A \cap B \subseteq AoBoA$ for every hesitant intuitionistic fuzzy bi-ideal $A =< \mu_A, \lambda_A >$ and every hesitant intuitionistic fuzzy ideal $B =< \mu_B, \lambda_B >$ of S.
(ii). $A \cap B \subseteq AoBoA$ for every hesitant intuitionistic fuzzy quasi-ideal $A =< \mu_A, \lambda_A >$ and every hesitant intuitionistic fuzzy ideal $B =< \mu_B, \lambda_B >$ of S.

PROOF. Suppose S is a regular Γ-semiring and $A =< \mu_A, \lambda_A >, B =< \mu_B, \lambda_B >$ be any hesitant intuitionistic fuzzy bi-ideal and hesitant intuitionistic fuzzy ideal of S, respectively and x be any element of S. Since S is regular, there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = x\alpha a\beta x$.

$$(\mu_A o\mu_B o\mu_A)(x) = \bigcup_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} (\cap \{(\mu_A o\mu_B)(a_i), \mu_A(b_i)\})$$
$$\supseteq \cap \{(\mu_A o\mu_B)(x\alpha a), \mu_A(x)\}$$
$$= \cap \{ \bigcup_{x\alpha a=\sum_{i=1}^{n} a_i\gamma_i b_i} (\cap \{(\mu_A(a_i), \mu_B(b_i))\}, \mu_A(x)\}$$
$$\supseteq \cap \{\cap \{\mu_A(x), \mu_B(a\beta x\alpha a), \mu_A(x)\} (since\ x\alpha a = x\alpha a\beta x\alpha a)$$
$$\supseteq \cap \{\mu_A(x), \mu_B(x)\} = (\mu_A \cap \mu_B)(x).$$

$$(\lambda_A o\lambda_B o\lambda_A)(x) = \bigcap_{x=\sum_{i=1}^{n} a_i\gamma_i b_i} (\cup \{(\lambda_A o\lambda_B)(a_i), \lambda_A(b_i)\})$$
$$\subseteq \cup \{(\lambda_A o\lambda_B)(x\alpha a), \lambda_A(x)\}$$
$$= \cup \{\cap (\cup_{x\alpha a=\sum_{i=1}^{n} a_i\gamma_i b_i} \{(\lambda_A(a_i), \lambda_B(b_i))\}), \lambda_A(x)\}$$
$$\subseteq \cup \{\cup \{\lambda_A(x), \lambda_B(a\beta x\alpha a), \lambda_A(x)\} (since\ x\alpha a = x\alpha a\beta x\alpha a)$$
$$\supseteq \cup \{\lambda_A(x), \lambda_B(x)\} = (\lambda_A \cup \lambda_B)(x).$$

(i)⇒(ii) follows from Lemma 2.

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