Journal of New Results in Science (JNRS)

ISSN: 1304-7981
http://dergipark.gov.tr/en/pub/jnrs
Research Article
Open Access

Volume: 9
Issue: 3
Year: 2020
Pages: 38-50

***Received*:** 07.12.2020 ***Accepted*:** 22.12.2020

Mixed-Integer Programming to Solve Distribution Problems

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ABSTRACT: Mixed-integer linear and quadratic programming problems are considered to solve distribution problems in this paper. The first problem is the distribution of proctors with respect to the student placements to the class- rooms by mixed-integer linear programming whereas the second problem is the fair distribution of the workloads for teaching assistants in a department formulated by mixed-integer quadratic programming. Three approaches to find the solution for mixed-integer quadratic programming problem are proposed and a comparative example is given to measure the effects for the suggested criterion.

Keywords – *Integer programming, Linear programming, Quadratic programming, Distribution problems*

1. Introduction

Workload distribution is considered as one of the major challenges for administrators at a workplace. In academia, teaching assistants (TAs) have a lot of departmental duties such as teaching in problem sessions, proctoring in exams, entering notes to system, arranging classrooms/proctors for midterms etc. While having these duties, they have a limited time while finishing their graduate studies, which may cause stress, anxiety and other mental problems. That is why, workload distribution becomes prominent in terms of equity. Such kind of academic workload distribution issues are analyzed in various studies in social sciences such as in (Bitzer, 2007), (Kenny, 2018), (Parsons and Slabbert, 2001) and the references therein. Even though there are studies for workload imbalance (Ku et al.,2018), as to the authors knowledge the methodology of comparative studies using workload imbalance criterion in literature is limited (Baykasoglu et al., 2009). Therefore, the aim of this study is to propose an idea for distributing workload among TAs in a department.

Proctor distribution problem is also an another challenging problem in this context. The less proctor is used; the time is spared for graduate students having TA duties so that they can spend their time for their graduate studies. This paper deals with two distribution problems. The main objective of the first problem is to distribute the proctors with respect to the student placements to the classrooms. The problem is formulated in mixed-integer linear programming (MILP) form. The goal of the second problem is to distribute the workloads fairly for TAs in a department. This problem is formulated mixed-integer quadratic programming (MIQP) which is known as an NP-hard problem (Bliek et al., 2014) and (Park and Boyd, 2018). That is why, three approaches for approximate solutions are proposed. In the first approach, rounding is made after solving quadratic programming problem. In the second approach, MIQP is converted into MILP by approximating the quadratic term. Last and third approach is an algorithmic approach, based on the assignment of the workload by sorting the TAs by their previous workloads. Numerical examples are given to compare these approaches.

2. Constrained Optimization Preliminaries

A nonlinear optimization problem can be stated as follows.

$$\begin{aligned} &\min_x f(x) \\ subject\ to\ &g_i(x) \leq 0, \quad i = 1,..,m \\ &h_j(x) = 0, \quad j = 1,..,n \end{aligned} \tag{2.1}$$

Here $f(\cdot)$ is called the objective function, whereas $g_i(\cdot)$ and $h_j()$, are inequality and equality constraints in residual form, respectively. The necessary and sufficient conditions for the nonlinear optimization problem are presented in (Karush, 1939) and (Kuhn and Tucker, 1951), known as Karush-Kuhn-Tucker (KKT) conditions and represented below.

2.1 KKT Conditions

To state KKT conditions, a function called Lagrangian is constructed for the problem (2.1).

$$\mathcal{L}(x, v, u, s) = f(x) + \sum_{i=1}^{m} (u_i^T\, g_i(x) + s_i^2) + \sum_{j=1}^{\ell} v_j^T\, h_j(x), \tag{2.2}$$

Here u_i and v_j are called the conjugate variables whereas s_i are the slack variables. Thus, the necessary KKT conditions can be stated as follows.

Theorem 1. Assume that the functions $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable at $x^* \in \mathbb{R}^n$. If x^* is the optimal solution of the problem (2.1) and satisfies the conditions below, then there exists μ_i $(i = 1, \dots, m)$ and λ_j $(j = 1, \dots, \ell)$ satisfying the following conditions:

1. Stationarity Condition:

- To maximize $f(x)$:

$$\nabla f(x^*) - \sum_{i=1}^{m} u_i^T\, \nabla g_i(x^*) - \sum_{j=1}^{\ell} v_j^T\, \nabla h_j(x^*) = 0,$$

- To minimize $f(x)$:

$$-\nabla f(x^*) - \sum_{i=1}^{m} u_i^T\, \nabla g_i(x^*) - \sum_{j=1}^{\ell} v_j^T\, \nabla h_j(x^*) = 0,$$

2. Primal Feasibility Condition:

$g_i(x^*) \leq 0, i = 1,..,m$
$h_j(x^*) \leq 0, j = 1,..,\ell$

3. Dual Feasibility Condition:

$u_i \geq 0, i = 1, \dots m$

4. Complementary Slackness Condition:

$u_i^T g_i(x^*) = 0, i = 1,..,m$

Besides the introduced necessary conditions above, if $f(\cdot)$ is a convex, $g_i(\cdot)$ are convex and $h_j(\cdot)$ are affine functions, then these conditions are also sufficient conditions.

2.2 Mixed-Integer Linear Programming

In the case, where the objective function, equality and inequality constraints are linear in terms of independent variables, this problem is described as constrained linear optimization problem and is defined as

$$\begin{aligned} &\min_x f^T x \\ &subject\ to\ A_{eq}x = b_{eq} \\ &\qquad A_{ineq}x \leq b_{ineq} \end{aligned} \tag{2.3}$$

Here f is a vector, A_{eq}, A_{ineq}, b_{eq} and b_{ineq} are matrices for equality and inequality constraints with appropriate dimensions.

If all or some of the variables of an optimization problem are limited to take integer values, the problem is considered as an MILP problem. This problem is defined as follows.

$$\begin{aligned} &\min_x f^T x \\ &subject\ to\ A_{eq}x = b_{eq} \\ &\qquad A_{ineq}x \leq b_{ineq} \\ &\qquad x_i \in D \subset \mathrm{Z} \end{aligned} \tag{2.4}$$

To apply KKT conditions to (3), we first should define the equality and inequality constraints in residual form.

$$\begin{aligned} &min\ f(x) = f^T x \\ &subject\ to\ h(x) = A_{eq}x - b_{eq} = 0 \\ &\qquad g(x) = A_{ineq}x - b_{ineq} \leq 0 \end{aligned} \tag{2.5}$$

The Lagrangian of the problem (3) will be

$$\mathcal{L}(\mathrm{x}, \mathrm{v}, \mathrm{u}, \mathrm{s}) = f^T x + v^T(A_{eq}x - b_{eq}) + u^T(A_{ineq}x - b_{ineq} + s^2) \tag{2.6}$$

So the KKT conditions of the problem (2.3) are stated as follows.

1. Stationarity Condition: In order to obtain the stationarity condition, the gradients are calculated as

$$\nabla (f^T x) = \nabla (x^T f) = f$$
$$\nabla (v^T A_{eq} x) = \nabla (x^T A_{eq}^T v) = A_{eq}^T v$$
$$\nabla (u^T A_{ineq} x) = \nabla (x^T A_{ineq}^T v) = A_{ineq}^T u$$

According to the gradients, the stationarity conditions are obtained as

$$\nabla L = \mathrm{f} + A_{eq}^T v + A_{ineq}^T u = 0$$
$$\frac{\partial L}{\partial v} = 0 \Rightarrow h(x) = A_{eq}x - b_{eq} = 0$$
$$\frac{\partial L}{\partial u} = 0 \Rightarrow g(x) = A_{ineq}x - b_{ineq} + s^2 = 0$$

2. Primal Feasibility Condition: From primal feasibility condition,

$$s^2 \geq 0,$$

is obtained.

3. Dual Feasibility Condition: From dual feasibility condition,

$$u^2 \geq 0,$$

4. Complementary Slackness Condition: From complementary slackness condition

$$\frac{\partial L}{\partial s} = 2s^T u = 0,$$

is obtained.

Remark 1 The case that primary feasibility condition is active (i.e. $s^2 = 0$), means that the inequality constraint is active. That is why, this condition cannot be active in both proctor distribution problem as well as workload assignment problem. So, the general expression of the KKT conditions under that consideration will be

$$\underbrace{\begin{bmatrix} 0 & A_{eq}^T & 0 \\ A_{eq} & 0 & 0 \\ A_{ineq} & 0 & I \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \\ s^2 \end{bmatrix}^T = \underbrace{\begin{bmatrix} -f \\ b_{eq} \\ b_{ineq} \end{bmatrix}}_{b}. \tag{2.7}$$

This equation can be solved as

$$x = A^{-1}b \tag{2.8}$$

for any invertible A.

Early results to find the integer solutions of linear programming problems was proposed by Ralph E. Gomory in (Gomory, 1960). The proposed method of Gomory is called integer programming or cutting-plane method. In this method, it is aimed to obtain an integer solution by adding valid inequalities consecutively to the relaxed linear programming problem of integer decision variables.

2.3 Mixed-Integer Quadratic Programming

The problem, in which the objective function is second-order and the constraints are linear, is named as Constrained Quadratic Programming problems. If some variables are restricted to take integer values, it is called Mixed-Integer Quadratic Programming (MIQP) problem and this problem is defined as follows.

$$\begin{aligned} &\min_{x} x^T H x + f^T x \\ &subject\ to\ A_{eq}x = b_{eq} \\ &\qquad A_{ineq}x \leq b_{ineq} \\ &\qquad x_i \in D \subset Z\ for\ some\ i \in \{1, \dots, n\} \end{aligned} \tag{2.9}$$

Here H is a symmetric matrix, f is a vector, A_{eq}, A_{ineq}, b_{eq} and b_{ineq} are matrices and vectors for equality and inequality constraints in appropriate dimensions. This problem is classified as an NP-Hard problem (Bliek et al., 2014) and (Park and Boyd, 2018). As to the authors knowledge, numerical computing programs such as MATLAB does not have MIQP library. Therefore, the problem can be solved by quadratic programming and the solution can be rounded or integer programming can be applied by using linear approximations.

$$\begin{aligned} &\min_{x,z} z + f^T x \\ &subject\ to\ A_{eq}x = b_{eq} \\ &\qquad x^T H x - z \leq 0 \\ &\qquad x_i \in D \subset Z\ for\ some\ i \in \{1, \dots, n\} \\ &\qquad z \geq 0\ (-z \leq 0) \end{aligned} \tag{2.10}$$

If the quadratic term in the inequality constraint is approximated around the point $x = x_0$, we have

$$x^T H x - z = -x_0^T H x_0 + 2x_0^T H x - z + \mathcal{O}(\| x - x_0 \|_2). \tag{2.11}$$

Here $|| \cdot ||_2$ is the L_2 norm of a vector. So, the problem is converted into a mixed-integer linear problem.

$$\begin{aligned} &\min_{x,z} \begin{bmatrix} f \\ 1 \end{bmatrix}^T \begin{bmatrix} x \\ z \end{bmatrix} \\ &subject\ to\ \begin{bmatrix} A_{eq}^T \\ 0^T \end{bmatrix}^T \begin{bmatrix} x \\ z \end{bmatrix} = \mathrm{b_{eq}} \\ &\qquad \begin{bmatrix} 2Hx_0 \\ -1 \end{bmatrix}^T \begin{bmatrix} x \\ z \end{bmatrix} \leq x_0^T H x_0 \\ &\qquad x_i \in D \subset \mathrm{Z} \\ &\qquad z \geq 0\ (-z \leq 0) \end{aligned} \tag{2.12}$$

See (Kelley, 1960) for more details.

The distribution problem examined in this study is in the form of:

$$\begin{aligned} &\min_{x} f(x) = x^T H x + f^T x \\ &subject\ to\ h(x) = A_{eq}x - b_{eq} = 0 \\ &\qquad g_1(x) = x - 2 \leq 0 \\ &\qquad g_2(x) = -x \leq 0 \end{aligned} \tag{2.13}$$

The Lagrangian will be

$$\mathcal{L}(x, v, u_1, u_2, s_1, s_2) = f(x) + v^T h(x) + u_1^T(g_1(x) + s_1^2) + u_2^T(g_2(x) + s_2^2) \tag{2.14}$$

To obtain the KKT conditions for this problem, the gradients have to be calculated.

$$\begin{aligned} &\nabla(x^T H x) = (\nabla x^T) H x + (\nabla x^T) H^T x = (H + H^T) x = 2Hx \\ &\nabla(f^T x) = \nabla(x^T f) = f \\ &\nabla\left(v^T A_{eq}^x\right) = \nabla\left(x^T A_{eq}^T v\right) = A_{eq}^T v \\ &\nabla(u_1^T x) = u_1 \\ &\nabla(u_2^T(-x)) = -u_2. \end{aligned} \tag{2.15}$$

According to the calculated gradients the stationary condition will be

$$\begin{aligned} &\nabla L = 2Hx + f + A_{eq}^T v + u_1 - u_2 = 0 \\ &\frac{\partial L}{\partial v} = 0 \Rightarrow h(x) = A_{eq} x - b_{eq} = 0 \\ &\frac{\partial L}{\partial u_1} = 0 \Rightarrow x - 2 + s_1^2 = 0 \\ &\frac{\partial L}{\partial u_2} = 0 \Rightarrow -x + s_2^2 = 0 \end{aligned} \tag{2.16}$$

From primal feasibility condition, we have

$$s_1^2 \geq 0, \qquad s_2^2 \geq 0 \tag{2.17}$$

whereas the dual feasibility condition will be

$$u_1^2 \geq 0, \qquad u_2^2 \geq 0 \tag{2.18}$$

The complementary slackness condition is obtained as

$$\frac{\partial L}{\partial s_1} = 2 s_1^T u_1 = 0, \qquad \frac{\partial L}{\partial s_2} = 2 s_2^T u_2 = 0. \tag{2.19}$$

If we assume that primal feasibility condition is active, i.e. $s_1^2 = 0$ or $s_2^2 = 0$, the results will not be applicable, so these conditions are taken as inactive. So, the general solution under this assumption is as follows.

$$\underbrace{\begin{bmatrix} 2H & A_{eq}^T & 0 & 0 \\ A_{eq} & 0 & 0 & 0 \\ I & 0 & I & 0 \\ -I & 0 & 0 & I \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \\ s_1^2 \\ s_2^2 \end{bmatrix} = \underbrace{\begin{bmatrix} -f \\ b_{eq} \\ 2 \\ 0 \end{bmatrix}}_{b} \tag{2.20}$$

If the matrix A is invertible, then the solution will be

$$x = A^{-1} b, \tag{2.21}$$

Proposition 1: The following statements hold.
1. $f(x) = x^T Hx + f^T x$ is convex if and only if H is positive semi-definite.
2. $h(x) = A_{eq}x - b_{eq}$ is affine.
3. $g_1(x) = x - 2$ *and* $g_2(x) = -x$ are convex.
Proof: 1. Let H be a positive semi-definite matrix. For all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, we have

$$\alpha(1-\alpha)(x_1 - x_2)^T H(x_1 - x_2) \geq 0$$

and

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &= (\alpha x_1 + (1-\alpha)x_2)^T H(\alpha x_1 + (1-\alpha)x_2) + f^T(\alpha x_1 + (1-\alpha)x_2) \\ &\leq (\alpha x_1 + (1-\alpha)x_2)^T H(\alpha x_1 + (1-\alpha)x_2) + f^T(\alpha x_1 + (1-\alpha)x_2) \\ &\quad +\alpha(1-\alpha)(x_1 - x_2)^T H(x_1 - x_2) \\ &= \alpha^2 x_1^T Hx_1 + \alpha(1-\alpha)x_2 Hx_1 + \alpha(1-\alpha)x_1 Hx_2 \\ &\quad +(1 - 2\alpha + \alpha^2)x_2^T Hx_2 \; (\alpha - \alpha^2)x_1^T Hx_1 \\ &\quad -\alpha(1-\alpha)x_2 Hx_1 - \alpha(1-\alpha)x_1 Hx_2 \\ &\quad + (\alpha - \alpha^2)x_2^T Hx_2 + f^T(\alpha x_1 + (1-\alpha)x_2) \\ &= \alpha(x_1^T \mathrm{H} x_1 f^T x_1) + (1-\alpha)(x_2^T \mathrm{H} x_2 f^T x_2) \end{aligned}$$

which implies

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2).$$

Therefore, f is a convex function. Necessity part can also be shown in similar fashion.
2. For all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} h(\alpha x_1 + (1-\alpha)x_2) &= A_{eq}(\alpha x_1 + (1-\alpha)x_2) - b_{eq} \\ &= A_{eq}(\alpha x_1 + (1-\alpha)x_2) - (\alpha + 1 - \alpha)b_{eq} \\ &= \alpha\left(A_{eq}x_1 + b_{eq}\right) + (1-\alpha)\left(A_{eq}x_2 + b_{eq}\right) \\ &= \alpha h(x_1) + (1-\alpha)h(x_2) \end{aligned}$$

so that h is an affine function.
3. The proof is similar to the one above and thus it is omitted.

Note that, the KKT conditions (2.20) are necessity and sufficient conditions for optimal solution.

3. Distribution Problems

In this section, problem statements of proctor distribution problem and workload assignment problem are introduced. Then, we will also refer to the solution methodologies to these problems namely solve-and-round, MIQP approximation and sort-and-distribute approaches.

3.1 Proctor Distribution Problem

The proctor distribution problem can be formulated as a minimization problem which has a linear objective function with linear constraints which is essentially MILP problem.

This problem can be stated as

$$\begin{aligned} &\min_{x,y} f(x,y) = \sum_{i=1}^{N} y_i \\ &subject\ to\ \frac{x_1}{30} \le y_1\ , \frac{x_2}{30} \le y_2 \dots , \frac{x_N}{30} \le y_N \\ &\quad x_1 + x_2 + \dots + x_N = M \\ &\quad x_i \in Z \cap [0, c_i] \\ &\quad y_i \in Z^+ \cup \{0\} \end{aligned} \tag{3.1}$$

Here $x = [x_1 \quad x_2 \quad \cdots \quad x_N]^T$ denotes the students, $c = [c_1 \quad c_2 \quad \cdots \quad c_N]^T$ denotes the classroom capacities and $y = [y_1 \quad y_2 \quad \cdots \quad y_N]^T$ are the proctors to be distributed.

One proctor is assigned for 30 students. So the objective function will be

$$f(x,y) = \sum_{i=1}^{N} y_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{2Nx1}^{T} \begin{bmatrix} x \\ y \end{bmatrix}_{2Nx1} \tag{3.2}$$

where the equality and inequality constraints will be

$$\begin{aligned} &[(1/30)I_N,\ I_{-N}]_{Nx2N} \begin{bmatrix} x \\ y \end{bmatrix}_{2Nx1} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2NX1} \\ &\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2Nx1}^{T} \begin{bmatrix} x \\ y \end{bmatrix}_{2Nx1} = M_{1X1} \end{aligned} \tag{3.3}$$

3.2 Workload Assignment Problem

The workload assignment problem can be defined as a minimization problem which has a quadratic objective function with linear constraints as

$$\begin{aligned} &\min_{x} f(x_1, x_2, x_3) = \sum_{i=1}^{14} (2x_{1,i} + 2x_{2,i} + 2x_{3,i} + e_i)^2 \\ &subject\ to\ x_{1,1} + x_{1,2} + \dots + x_{1,14} = 4 \\ &\quad x_{2,1} + x_{2,2} + \dots + x_{2,14} = 8 \\ &\quad x_{3,1} + x_{3,2} + \dots + x_{3,14} = 8 \\ &\quad 0 \le x_{j,i} \le 2\ , i \in \mathcal{I} = \{1, 2, \dots, 14\}\ and\ j \in \mathcal{J} = \{1, 2, 3\}. \end{aligned} \tag{3.4}$$

where $x = [x_1^T \quad x_2^T \quad x_3^T]^T$, $x_1 = [x_{1,1} \quad x_{1,2} \quad \cdots \quad x_{1,14}]^T$, $x_2 = [x_{2,1} \quad x_{2,2} \quad \cdots \quad x_{2,14}]^T$ and $x_3 = [x_{3,1} \quad x_{3,2} \quad \cdots \quad x_{3,14}]^T$ are the vectors for different tasks whereas $x_{j,i}$ denotes for the i^{th} task of the j^{th} TA where $j = 1, 2, 3$ and $i = 1, \dots, 14$. $e_i = c_i - \mu$ is defined as a residual term where c_i is the missing workload or the overload of the i^{th} TA coming from the past and μ is the average workload to be done for one TA. So the objective function will be

$$f(x_1, x_2, x_3) = \sum_{i=1}^{14} 4(x_{1,i} + x_{2,i} + x_{3,i})^2 + 4(x_{1,i} + x_{2,i} + x_{3,i})e_i + e_1^2, \tag{3.5}$$

which can also be written in a compact form as

$$f(x_1, x_2, x_3)$$
$$= \sum_{i=1}^{14} 4(x_{1,i}^2 + x_{2,i}^2 + x_{3,i}^2 + 2x_{1,i}\, x_{2,i} + 2x_{1,i}x_{3,i} + 2x_{2,i}x_{3,i})$$
$$+4\big(x_{1,i} + x_{2,i} + x_{3,i}\big)e_i + e_i^2$$
$$= \sum_{i=1}^{14} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix}^T \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} + \begin{bmatrix} e_i \\ e_i \\ e_i \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix}^T + e_i^2.$$

$$= \begin{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} \\ \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} x_{1,14} \\ x_{2,14} \\ x_{3,14} \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix} & & & \\ & \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} \\ \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} x_{1,14} \\ x_{2,14} \\ x_{3,14} \end{bmatrix} \end{bmatrix} \quad (3.6)$$
$$+ \begin{bmatrix} \begin{bmatrix} 4e_1 \\ 4e_1 \\ 4e_1 \end{bmatrix} \\ \begin{bmatrix} 4e_2 \\ 4e_2 \\ 4e_2 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 4e_{14} \\ 4e_{14} \\ 4e_{14} \end{bmatrix} \end{bmatrix}^T \begin{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} \\ \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} x_{1,14} \\ x_{2,14} \\ x_{3,14} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{14} \end{bmatrix}^T \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{14} \end{bmatrix},$$

where the equality constraints will be

$$\underbrace{[I_3, I_3, \dots, I_3]}_{14\ times} \begin{bmatrix} \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} \\ \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} x_{1,14} \\ x_{2,14} \\ x_{3,14} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} x_{1,1} + x_{1,2} + \cdots + x_{1,4} \\ x_{2,1} + x_{2,2} + \cdots + x_{2,14} \\ x_{3,1} + x_{3,2} + \cdots + x_{3,14} \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix}. \quad (3.7)$$

We use three approaches to solve this problem.
– **Solve-and-Round Approach:** In order to solve the quadratic programming problem, Optimization Toolbox of MATLAB is used. The function *quadprog* used for this approach uses KKT conditions to solve the problem which is explained in Section 2.

– **MIQP Approximation Approach:** The linearization method to solve the MIQP is given in Section 2. The obtained MILP is solved via *intlinprog* function of MATLAB Optimization Toolbox.
– **Sort-and-Distribute Approach (Adaköy et al., 2017):** In this approach, the people are sorted by swapping and comparing them one by one according to workload. Then, as long as the load to be distributed is positive, the workload is distributed. The distribution is started from the minimum while two units of workload is distributed per cycle. After performing the task distribution in each cycle, the comparison, swap and sort processes are made according to workload. Finally, the workload to be distributed is reduced by 2 units and the next cycle is passed. Obtained results are given in the following section.

4 Results and Discussion

4.1 Example for Proctor Distribution Problem

In the numerical example, we distribute 612 students to the 10 classes having total capacity of 710. The results are given in Table 1.

Table 1. The Distribution of Students and Proctors

	Capacity	Students	Proctors
Class 1	76	60	2
Class 2	32	30	1
Class 3	89	89	3
Class 4	95	90	3
Class 5	78	60	2
Class 6	83	76	3
Class 7	82	60	2
Class 8	57	57	2
Class 9	76	60	2
Class 10	42	30	1

In Table 1, 21 proctors are used in total which is the minimum number of total proctors to be used.

4.2 Example for Workload Assignment Problem

The remaining workloads or overloads from the last term and the error terms are presented in Table 2.

Table 2. The Remaining Workloads or Overloads from the Last Term and the Error Terms

	c_i	e_i
TA 1	28	-11.4
TA 2	33	-6.4
TA 3	38	-1.4
TA 4	40	0.6
TA 5	33	-6.4
TA 6	46	6.6
TA 7	35	-4.4
TA 8	32	-7.4
TA 9	55	15.6
TA 10	38	-1.4
TA 11	34	-5.4
TA 12	42	2.6
TA 13	32	-7.4
TA 14	25	-14.4

The results obtained according to quadratic programming, solve-and-round, MIQP approximation and sort-and-distribute approaches are given in Tables 4, 5, 6 and 7 in

Appendix, respectively. A criterion for workload imbalance is given as the sum of squared errors (SSE):

$$SSE = \sum_{i=1}^{14} {e_i'}^2 = \sum_{i=1}^{14} (2x_{1,i} + 2x_{2,i} + 2x_{3,i} + e_i)^2$$
$$= \sum_{i=1}^{14} (2x_{1,i} + 2x_{2,i} + 2x_{3,i} + c_i - \mu)^2 \quad (4.1)$$

According to solve-and-round, MIQP approximation and sort-and-distribute approaches, SSE values are given in Table 3. From the three approaches considered, the least SSE value is obtained in the quadratic optimization approach is rounded, SSE value become higher than the SSE values of the other approaches. Therefore, the most equitable integer distribution will be made when the sort-and-distribute approach is used.

Table 3. SSE Values According to Quadratic Programming, Solve-and-Round, MIQP Approximation and Sort-and-Distribute Approaches.

Approach	SSE
Quadratic Optimization	365.24
Solve-and-Round	377.24
MIQP Approximation	435.24
Sort-and-Distribute	367.24

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Appendix A. Workloads and Error Terms for Various Approaches

Table 4. Workloads and Error Terms According to Quadratic Optimization Approach

	$x_{1,i}$	$x_{2,i}$	$x_{3,i}$	c_i	$2x_{1,i}+2x_{2,i}+2x_{3,i}+c_i$	e_i'
TA 1	0.87	1.69	1.69	28	36.50	-2.90
TA 2	0.21	0.77	0.77	33	36.50	-2.90
TA 3	0.00	0.00	0.00	38	38.00	-1.40
TA 4	0.00	0.00	0.00	40	40.00	0.60
TA 5	0.21	0.77	0.77	33	36.50	-2.90
TA 6	0.00	0.00	0.00	46	46.00	6.60
TA 7	0.14	0.30	0.30	35	36.50	-2.90
TA 8	0.24	1.00	1.00	32	36.50	-2.90
TA 9	0.00	0.00	0.00	55	55.00	15.60
TA 10	0.00	0.00	0.00	38	38.00	-1.40
TA 11	0.18	0.53	0.53	34	36.50	-2.90
TA 12	0.00	0.00	0.00	42	42.00	-2.60
TA 13	0.24	1.00	1.00	32	36.50	-2.90
TA 14	1.91	1.92	1.92	25	36.50	-2.90
Sum	4	8	8	511	551	-0.6

Table 5. Workloads and Error Terms According to Solve-and-Round Approach

	$x_{1,i}$	$x_{2,i}$	$x_{3,i}$	c_i	$2x_{1,i}+2x_{2,i}+2x_{3,i}+c_i$	e_i'
TA 1	1	2	2	28	36.50	-1.4
TA 2	0	1	1	33	36.50	-2.4
TA 3	0	0	0	38	38.00	-1.4
TA 4	0	0	0	40	40.00	0.6
TA 5	0	1	1	33	36.50	-2.4
TA 6	0	0	0	46	46.00	6.6
TA 7	0	0	0	35	36.50	-4.4
TA 8	0.5	1	1	32	36.50	-2.4
TA 9	0	0	0	55	55.00	15.6
TA 10	0	0	0	38	38.00	-1.4
TA 11	0	0	0	34	36.50	-5.4
TA 12	0	0	0	42	42.00	2.6
TA 13	0.5	1	1	32	36.50	-2.4
TA 14	2	2	2	25	36.50	-2.4
Sum	4	8	8	511	551	-0.6

Table 6. Workloads and Error Terms According to MIQP Approach

	$x_{1,i}$	$x_{2,i}$	$x_{3,i}$	c_i	$2x_{1,i}+2x_{2,i}+2x_{3,i}+c_i$	e_i'
TA 1	2	2	2	28	40	0.6
TA 2	0	0	0	33	33	-6.4
TA 3	0	0	0	38	38	-1.4
TA 4	0	0	0	40	40	0.6
TA 5	0	0	0	33	33	-6.4
TA 6	0	0	0	46	46	6.6
TA 7	0	0	0	35	35	-4.4
TA 8	0	2	2	32	40	0.6
TA 9	0	0	0	55	55	15.6
TA 10	0	0	0	38	38	-1.4
TA 11	0	0	0	34	34	-5.4
TA 12	0	0	0	42	42	2.6
TA 13	0	2	2	32	40	0.6
TA 14	2	2	2	25	37	-2.4
Sum	4	8	8	511	551	-0.6

Table 7. Workloads and Error Terms According to Sort-and-Distribute Approach

	$x_{1,i}$	$x_{2,i}$	$x_{3,i}$	c_i	$2x_{1,i} + 2x_{2,i} + 2x_{3,i} + c_i$	e_i'
TA 1	0	2	2	28	36	-3.4
TA 2	1	0	1	33	37	-2.4
TA 3	0	0	0	38	38	-1.4
TA 4	0	0	0	40	40	0.6
TA 5	1	0	1	33	37	-2.4
TA 6	0	0	0	46	46	6.6
TA 7	0	1	0	35	37	-2.4
TA 8	0	1	1	32	36	-3.4
TA 9	0	0	0	55	55	15.6
TA 10	0	0	0	38	38	-1.4
TA 11	0	1	0	34	36	-3.4
TA 12	0	0	0	42	42	2.6
TA 13	2	1	1	32	36	-3.4
TA 14	2	2	2	25	37	-2.4
Sum	4	8	8	511	551	-0.6