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ON $I_{\pi g\alpha^\star}$-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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Abstract – In this paper, a new class of sets called $I_{\pi g\alpha^\star}$-closed sets is introduced and its properties are studied in ideal topological space. Moreover $I_{\pi g\alpha^\star}$-continuity and the notion of quasi-$\alpha^\star$-I-normal spaces are introduced.

Keywords – *π-open set, $I_{\pi g\alpha^\star}$-closed set, $I_{\pi g\alpha^\star}$-continuity, quasi-$\alpha^\star$-I-normal space.*

1 Introduction and Preliminaries

An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I). $A^*(I)$ = {x $\in$ X | U $\cap$ A $\notin I$ for each open neighborhood U of x} is called the local function of A with respect to I and τ [9]. When there is no chance for confusion $A^*(I)$ is denoted by A^*. For every ideal topological space (X, τ, I), there exists a topology τ^* finer than τ, generated by the base $\beta(I, \tau)$ = {U\I | U$\in \tau$ and $I \in I$}. In general $\beta(I, \tau)$ is not always a topology [8]. Observe additionally that $cl^*(A) = A^* \cup A$ [14] defines a Kuratowski closure operator for τ^*. $int^\star(A)$ will denote the interior of A in (X, $\tau^\star$).

In this paper, we define and study a new notion $I_{\pi g\alpha^\star}$-closed set by using the notion of $\alpha_I^\star$-open set. Some new notions depending on $I_{\pi g\alpha^\star}$-closed sets such as $I_{\pi g\alpha^\star}$-open sets, $I_{\pi g\alpha^\star}$-continuity and $I_{\pi g\alpha^\star}$-irresoluteness are also introduced and a decomposition of $\alpha^\star$-I-continuity is given. Also by using $I_{\pi g\alpha^\star}$-closed sets characterizations of quasi-$\alpha^\star$-I-normal spaces are obtained. Several preservation theorems for quasi-$\alpha^\star$-I-normal spaces are given.

Throughout this paper, space (X, τ) (or simply X) always means topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by cl(A) and int(A), respectively.

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A subset A of a topological space (X, τ) is said to be regular open [13](resp. regular closed [13]) if A = int(cl(A)) (resp. A = cl(int(A))).

The finite union of regular open sets is said to be π-open [16] in (X, τ). The complement of a π-open set is π-closed [16].

A subset A of a topological space (X, τ) is said to be α-open [10] if A $\subseteq$ int(cl(int(A))) and the complement of an α-open set is called α-closed [10].

The intersection of all α-closed sets containing A is called the α-closure [10] of A and is denoted by αcl(A).

Note that αcl(A) = A $\cup$ cl(int(cl(A))).

A subset A of a space (X, τ) is said to be πg-closed [2] (resp. $\pi g\alpha$-closed [1]) if cl(A) $\subseteq$ U (resp. αcl(A) $\subseteq$ U) whenever A $\subseteq$ U and U is π-open in X.

A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be m-π-closed [4] if f(V) is π-closed in (Y, σ) for every π-closed in (X, τ).

A function f : (X, τ) $\rightarrow$ (Y, σ) is said to be πg-continuous [2] (resp. $\pi g\alpha$-continuous [1]) if f^{-1}(V) is πg-closed (resp. $\pi g\alpha$-closed) in (X, τ) for every closed set V of (Y, σ).

A space (X, τ) is said to be quasi-normal [16] if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V.

An ideal I is said to be codense [3] if $\tau \cap I = \emptyset$.

A subset A of an ideal topological space X is said to be $\star$-dense-in-itself [7](resp. $\alpha^\star$-I-open or $\alpha_I^\star$-open [15], t-I-set [6], α-I-open [6]) if A $\subseteq$ $A^\star$ (resp. A $\subseteq$ $int^\star(cl(int^\star(A)))$, int(A) = $int(cl^\star(A))$, A $\subseteq$ $int(cl^\star(int(A)))$).

The complement of $\alpha_I^\star$-open is $\alpha_I^\star$-closed.

A subset A of an ideal topological space X is said to be $I_{\pi g}$-closed [11] if $A^\star \subseteq$ U whenever A $\subseteq$ U and U is π-open in X.

A function f : (X, τ, I) $\rightarrow$ (Y, σ) is said to be $I_{\pi g}$-continuous [11] if f^{-1}(V) is $I_{\pi g}$-closed in (X, τ, I) for every closed set V of (Y, σ).

Lemma 1.1. *[12] Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $A \subseteq A^\star$, then $A^\star = cl(A^\star) = cl(A) = cl^\star(A)$.*

Theorem 1.2. *[11] Every πg-closed set is $I_{\pi g}$-closed but not conversely.*

Theorem 1.3. *[11] For a function f : (X, τ, I) $\rightarrow$ (Y, σ), the following holds:*
Every πg-continuous function is $I_{\pi g}$-continuous but not conversely.

Theorem 1.4. *[1] Every πg-closed set is $\pi g\alpha$-closed but not conversely.*

Proposition 1.5. *[6] Every α-I-open set is α-open but not conversely.*

2 $I_{\pi g\alpha^\star}$-closed Sets

Theorem 2.1. *For a function f : (X, τ) $\rightarrow$ (Y, σ), the following holds:*
Every πg-continuous function is $\pi g\alpha$-continuous but not conversely.

Example 2.2. *Let X = {a, b, c, d}, τ ={X, $\emptyset$, {a}, {b}, {a, b}, {b, c}, {a, b, c}}, Y = {x, y, z} and σ = {Y, $\emptyset$, {y}, {y, z}}. Define a function f : (X, τ) $\rightarrow$ (Y, σ) as follows f(a) = f(b) = y, f(c)= x and f(d) = z. Then f is $\pi g\alpha$-continuous function but it is not an πg-continuous.*

Definition 2.3. *Let (X, τ, I) be an ideal topological space and let A be a subset of X. The union of all $\alpha_I^\star$-open sets contained in A is called the $\alpha_I^\star$-interior of A and is denoted by $\alpha_I^\star int(A)$.*

Definition 2.4. *Let (X, τ, I) be an ideal topological space and let A be a subset of X. The intersection of all $\alpha_I^\star$-closed sets containing A is called the $\alpha_I^\star$-closure of A and is denoted by $\alpha_I^\star cl(A)$.*

Lemma 2.5. *Let (X, τ, I) be an ideal topological space. For a subset A of X, the followings hold:*

1. *$\alpha_I^\star cl(A) = A \cup cl^\star(int(cl^\star(A)))$,*
2. *$\alpha_I^\star int(A) = A \cap int^\star(cl(int^\star(A)))$.*

Definition 2.6. *A subset A of an ideal topological space (X, τ, I) is called $I_{\pi g\alpha^\star}$-closed if $\alpha_I^\star cl(A) \subseteq$ U whenever $A \subseteq U$ and U is π-open in X.*

The complement of $I_{\pi g\alpha^\star}$-closed set is said to be $I_{\pi g\alpha^\star}$-open.

Proposition 2.7. *Every α-open set is $\alpha_I^\star$-open but not conversely.*

Proof. Let A be α-open set. Then A $\subseteq$ int(cl(int(A))) which implies A $\subseteq$ int*(cl(int*(A))). Hence A is $\alpha_I^\star$-open set.

Example 2.8. *Let X and τ be as in Example 2.2 and $I = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a, c\}$ is $\alpha_I^\star$-open set but not an α-open set.*

Theorem 2.9. *Every $\star$-dense-in-itself and $I_{\pi g\alpha^\star}$-closed set is a $\pi g\alpha$-closed set.*

Proof. Let A $\subseteq$ U, and U is π-open in X. Since A is $I_{\pi g\alpha^\star}$-closed, $\alpha_I^\star$cl(A) $\subseteq$ U. By Lemmas 1.1 and 2.5, $\alpha_I^\star$cl(A) = A $\cup$ cl*(int(cl*(A))) = A $\cup$ cl(int(cl(A))) = αcl(A). Then, αcl(A) $\subseteq$ U. So A is $\pi g\alpha$-closed.

Theorem 2.10. *Every π-open and $I_{\pi g\alpha^\star}$-closed set is t-I-set.*

Proof. $\alpha_I^\star$cl(A) $\subseteq$ A, since A is π-open and $I_{\pi g\alpha^\star}$-closed. We have cl*(int(cl*(A))) $\subseteq$ A and int(cl*(A)) $\subseteq$ cl*(int(cl*(A))) $\subseteq$ A. It implies int(cl*(A)) $\subseteq$ int(A). Always int(A) $\subseteq$ int(cl*(A)). Therefore int(A) = int(cl*(A)), which shows that A is t-I-set.

Theorem 2.11. *Let A be $I_{\pi g\alpha^\star}$-closed in (X, τ, I). Then $\alpha_I^\star cl(A) \setminus A$ does not contain any non-empty π-closed set.*

Proof. Let F be a π-closed set such that F $\subseteq$ $\alpha_I^\star$cl(A) $\setminus$ A. Then F $\subseteq$ X $\setminus$ A implies A $\subseteq$ X $\setminus$ F. Therefore $\alpha_I^\star$cl(A) $\subseteq$ X $\setminus$ F. That is F $\subseteq$ X $\setminus$ $\alpha_I^\star$cl(A). Hence F $\subseteq$ $\alpha_I^\star$cl(A) $\cap$ (X $\setminus$ $\alpha_I^\star$cl(A))= $\emptyset$. This shows F = $\emptyset$.

Theorem 2.12. *If A is $I_{\pi g\alpha^\star}$-closed and $A \subseteq B \subseteq \alpha_I^\star cl(A)$, then B is $I_{\pi g\alpha^\star}$-closed.*

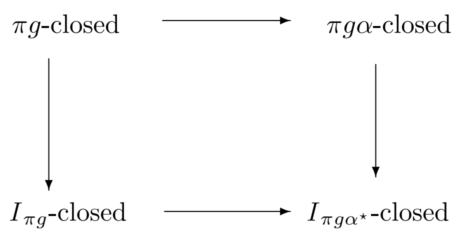
Proof. Let A be $I_{\pi g\alpha^\star}$-closed and B $\subseteq$ U, where U is π-open. Then A $\subseteq$ B implies A $\subseteq$ U. Since A is $I_{\pi g\alpha^\star}$-closed, $\alpha_I^\star$cl(A) $\subseteq$ U. B $\subseteq$ $\alpha_I^\star$cl(A) implies $\alpha_I^\star$cl(B) $\subseteq$ $\alpha_I^\star$cl(A). Therefore $\alpha_I^\star$cl(B) $\subseteq$ U and hence B is $I_{\pi g\alpha^\star}$-closed.

Proposition 2.13. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then the following properties hold:*

1. *If A is $\pi g\alpha$-closed, then A is $I_{\pi g\alpha^\star}$-closed,*
2. *If A is $I_{\pi g}$-closed, then A is $I_{\pi g\alpha^\star}$-closed.*

Proof. The proof is obvious.

Remark 2.14. *From Theorem 1.2, Theorem 1.4 and Proposition 2.13, we have the following diagram.*



where none of these implications is reversible as shown in the following examples.

Example 2.15. *(1) Let X and τ be as in Example 2.2. Then $\{c\}$ is $\pi g\alpha$-closed set but not an πg-closed.*
(2) In Example 2.8, $\{a\}$ is $I_{\pi g\alpha^\star}$-closed set but not $\pi g\alpha$-closed.
(3) In Example 2.8, $\{c\}$ is $I_{\pi g\alpha^\star}$-closed set but not $I_{\pi g}$-closed.

Remark 2.16. *The union of two $I_{\pi g\alpha^\star}$-closed sets need not be $I_{\pi g\alpha^\star}$-closed.*

Example 2.17. *In Example 2.8, $\{b\}$ and $\{c\}$ are $I_{\pi g\alpha^\star}$-closed sets but their union $\{b,\ c\}$ is not $I_{\pi g\alpha^\star}$-closed.*

Remark 2.18. *The intersection of two $I_{\pi g\alpha^\star}$-closed sets need not be $I_{\pi g\alpha^\star}$-closed.*

Example 2.19. *Let $X = \{a,\ b,\ c,\ d\}$, $\tau = \{X,\ \emptyset,\ \{a\},\ \{b\},\ \{a,\ b\},\ \{a,\ b,\ c\},\ \{a,\ b,\ d\}\}$ and $I = \{\emptyset,\ \{a\}\}$. Then $A = \{a,\ b,\ c\}$ and $B = \{a,\ b,\ d\}$ are $I_{\pi g\alpha^\star}$-closed sets but $A \cap B = \{a,\ b\}$ is not $I_{\pi g\alpha^\star}$-closed set.*

Definition 2.20. *[5] An ideal topological space (X, τ, I) is said to be $\star$-extremally disconnected if the $\star$-closure of every open subset of X is open.*

Theorem 2.21. *[5] For an ideal topological space (X, τ, I), the following properties are equivalent:*

1. *X is $\star$-extremally disconnected,*
2. *$cl^\star(int(V)) \subseteq int(cl^\star(V))$ for every subset V of X.*

Theorem 2.22. *Let (X, τ, I) be a $\star$-extremally disconnected ideal topological space. Then every subset of X is $I_{\pi g\alpha^\star}$-closed if and only if every π-open set is t-I-set.*

Proof. Necessity: It is obvious from Theorem 2.10.
Sufficiency: Suppose that every π-open set is t-I-set. Let A be a subset of X and U be π-open such that A $\subseteq$ U. By hypothesis $\mathrm{cl}^\star(\mathrm{int}(\mathrm{cl}^\star(\mathrm{A}))) \subseteq \mathrm{int}(\mathrm{cl}^\star(\mathrm{A})) \subseteq \mathrm{int}(\mathrm{cl}^\star(\mathrm{U})) = \mathrm{int}(\mathrm{U}) \subseteq \mathrm{U}$. Then $\alpha_I^\star\mathrm{cl}(\mathrm{A}) \subseteq \mathrm{U}$. So A is $I_{\pi g\alpha^\star}$-closed.

Theorem 2.23. *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is $I_{\pi g\alpha^\star}$-open if and only if $F \subseteq \alpha_I^\star int(A)$ whenever F is π-closed and $F \subseteq A$.*

Proof. Necessity: Let A be $I_{\pi g\alpha^\star}$-open and F be π-closed such that F $\subseteq$ A. Then X\A $\subseteq$ X\F where X\F is π-open. $I_{\pi g\alpha}$-closedness of X\A implies $\alpha_I^\star\mathrm{cl}(\mathrm{X}\backslash\mathrm{A}) \subseteq \mathrm{X}\backslash\mathrm{F}$. Then $\mathrm{F} \subseteq \alpha_I^\star\mathrm{int}(\mathrm{A})$.
Sufficiency: Suppose F is π-closed and F $\subseteq$ A implies $\mathrm{F} \subseteq \alpha_I^\star\mathrm{int}(\mathrm{A})$. Let X\A $\subseteq$ U where U is π-open. Then X\U $\subseteq$ A where X\U is π-closed. By hypothesis $\mathrm{X}\backslash\mathrm{U} \subseteq \alpha_I^\star\mathrm{int}(\mathrm{A})$. That is $\alpha_I^\star\mathrm{cl}(\mathrm{X}\backslash\mathrm{A}) \subseteq \mathrm{U}$. So, A is $I_{\pi g\alpha^\star}$-open.

Definition 2.24. *A subset A of an ideal topological space (X, τ, I) is called N_I-set if $A = U \cup V$ where U is π-closed and V is $\alpha_I^\star$-open.*

Proposition 2.25. *Every π-closed set is N_I-set but not conversely.*

Example 2.26. *In Example 2.19, $\{a\}$ is N_I-set but not π-closed set.*

Proposition 2.27. *Every $\alpha_I^\star$-open set is N_I-set but not conversely.*

Example 2.28. *In Example 2.19, $\{a,\ c,\ d\}$ is N_I-set but not $\alpha_I^\star$-open set.*

Proposition 2.29. *Every $\alpha_I^\star$-open set is $I_{\pi g\alpha^\star}$-open but not conversely.*

Proof. Let A be $\alpha_I^\star$-open set. Then A ⊆ int*(cl(int*(V))). Assume that F is π-closed and F ⊆ A. Then F ⊆ int*(cl(int*(V))) which implies F ⊆ A ∩ int*(cl(int*(V))) = $\alpha_I^\star$int(A) by Lemma 2.5. Hence, by Theorem 2.23, A is $I_{\pi g\alpha^\star}$-open.

Example 2.30. *In Example 2.19, {a, d} is $I_{\pi g\alpha^\star}$-open set but not $\alpha_I^\star$-open set.*

Theorem 2.31. *For a subset A of (X, τ, I) the following conditions are equivalent:*

1. *A is $\alpha_I^\star$-open,*
2. *A is $I_{\pi g\alpha^\star}$-open and a N_I-set.*

Proof. (1) ⇒ (2) It is obvious.
(2) ⇒ (1) Let A be $I_{\pi g\alpha^\star}$-open and a N_I-set. Then there exist a π-closed set U and $\alpha_I^\star$-open set V such that A = U ∪ V. Since U ⊆ A and A is $I_{\pi g\alpha^\star}$-open, by Theorem 2.23, U ⊆ $\alpha_I^\star$int(A) and U ⊆ int*(cl(int*(A))). Also, V ⊆ int*(cl(int*(V))) ⊆ int*(cl(int*(A))). Then A ⊆ int*(cl(int*(A))). So A is $\alpha_I^\star$-open.

The following examples show that concepts of $I_{\pi g\alpha^\star}$-open set and N_I-set are independent.

Example 2.32. *Let (X, τ, I) be the same ideal topological space as in Example 2.19. Then {c, d} is N_I-set but not $I_{\pi g\alpha^\star}$-open set.*

Example 2.33. *Let (X, τ, I) be the same ideal topological space as in Example 2.19. Then {a, c} is $I_{\pi g\alpha^\star}$-open set but not a N_I-set.*

3 $I_{\pi g\alpha^\star}$-continuity and $I_{\pi g\alpha^\star}$-irresoluteness

Definition 3.1. *A function f : (X, τ, I) → (Y, σ) is said to be $I_{\pi g\alpha^\star}$-continuous (resp. $\alpha^\star$-I-continuous) if $f^{-1}(V)$ is $I_{\pi g\alpha^\star}$-closed (resp. $\alpha_I^\star$-closed) in X for every closed set V of Y.*

Definition 3.2. *A function f : (X, τ, I) → (Y, σ, J) is said to be $I_{\pi g\alpha^\star}$-irresolute if $f^{-1}(V)$ is $I_{\pi g\alpha^\star}$-closed in X for every $J_{\pi g\alpha^\star}$-closed set V of Y.*

Definition 3.3. *A function f : (X, τ, I) → (Y, σ) is said to be N_I-continuous if $f^{-1}(V)$ is N_I-set in (X, τ, I) for every closed set V of (Y, σ.*

Theorem 3.4. *A function f : (X, τ, I) → (Y, σ) is $\alpha^\star$-I-continuous if and only if it is N_I-continuous and $I_{\pi g\alpha^\star}$-continuous.*

Proof. This is an immediate consequence of Theorem 2.31.

The composition of two $I_{\pi g\alpha^\star}$-continuous functions need not be $I_{\pi g\alpha^\star}$-continuous. Consider the following Example:

Example 3.5. *Let X = {a, b, c, d}, τ = {X, ∅, {b}, {d}, {b, d}, {b, c, d}} and I = {∅, {c}, {d}, {c, d}}. Let Y = {x, y, z}, σ = {Y, ∅, {y, z}}, J = {∅, {x}}, Z = {1, 2} and η = {Z, ∅, {1}}. Define f : (X, τ, I) → (Y, σ, J) by f(a)= f(c)= x, f(b)= y and f(d)= z and g : (Y, σ, J) → (Z, η) by g(x)= 1 and g(y)= g(z)= 2. Then f and g are $I_{\pi g\alpha^\star}$-continuous. {2} is closed in (Z, η), $(g \circ f)^{-1}(\{2\}) = f^{-1}(g^{-1}(\{2\})) = f^{-1}(\{y, z\})$ = {b, d} which is not $I_{\pi g\alpha^\star}$-closed in (X, τ, I). Hence g ∘ f is not $I_{\pi g\alpha^\star}$-continuous.*

Theorem 3.6. *Let f : (X, τ, I) → (Y, σ, J) and g : (Y, σ, J) → (Z, η, K) be any two functions. Then*

1. *g ∘ f is $I_{\pi g\alpha^\star}$-continuous, if g is continuous and f is $I_{\pi g\alpha^\star}$-continuous,*
2. *g ∘ f is $I_{\pi g\alpha^\star}$-continuous, if g is $J_{\pi g\alpha^\star}$-continuous and f is $I_{\pi g\alpha^\star}$-irresolute,*
3. *g ∘ f is $I_{\pi g\alpha^\star}$-irresolute, if g is $J_{\pi g\alpha^\star}$-irresolute and f is $I_{\pi g\alpha^\star}$-irresolute.*

Proof. (1) Let V be closed in Z. Then $g^{-1}(V)$ is closed in Y, since g is continuous. $I_{\pi g\alpha^\star}$-continuity of f implies that $f^{-1}(g^{-1}(V))$ is $I_{\pi g\alpha^\star}$-closed in X. Hence g ∘ f is $I_{\pi g\alpha^\star}$-continuous.
(2) Let V be closed in Z. Since g is $J_{\pi g\alpha^\star}$-continuous, $g^{-1}(V)$ is $J_{\pi g\alpha^\star}$-closed in Y. As f is $I_{\pi g\alpha^\star}$-irresolute, $f^{-1}(g^{-1}(V))$ is $I_{\pi g\alpha^\star}$-closed in X. Hence g ∘ f is $I_{\pi g\alpha^\star}$-continuous. (3) Let V be $K_{\pi g\alpha^\star}$-closed in Z. Then $g^{-1}(V)$ is $J_{\pi g\alpha^\star}$-closed in Y, since g is $J_{\pi g\alpha^\star}$-irresolute. Because f is $I_{\pi g\alpha^\star}$-irresolute, $f^{-1}(g^{-1}(V))$ is $I_{\pi g\alpha^\star}$-closed in X. Hence g ∘ f is $I_{\pi g\alpha^\star}$-irresolute.

Remark 3.7. *The following Examples show that:*

1. *every $I_{\pi g\alpha^\star}$-continuous function is not $\pi g\alpha$-continuous,*
2. *every $I_{\pi g\alpha^\star}$-continuous function is not $I_{\pi g}$-continuous.*

Example 3.8. *Let (X, τ, I) be the same ideal topological space as in Example 2.8. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \emptyset, \{y, z\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma)$ as follows: $f(a) = x$, $f(b) = f(c) = y$ and $f(d) = z$. Then f is $I_{\pi g\alpha^\star}$-continuous function but it is not $\pi g\alpha$-continuous.*

Example 3.9. *Let (X, τ, I) be the same ideal topological space as in Example 2.8. Let $Y = \{x, y, z\}$ and $\sigma = \{Y, \emptyset, \{y, z\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma)$ as follows: $f(a) = f(b) = z$, $f(c) = x$ and $f(d) = y$. Then f is $I_{\pi g\alpha^\star}$-continuous function but it is not $I_{\pi g}$-continuous.*

Theorem 3.10. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties hold:*

$$\begin{array}{ccc} \pi g\text{-continuous} & \longrightarrow & \pi g\alpha\text{-continuous} \\ \downarrow & & \downarrow \\ I_{\pi g}\text{-continuous} & \longrightarrow & I_{\pi g\alpha^\star}\text{-continuous} \end{array}$$

Proof. The proof is obvious by Remark 2.14.

4 Quasi-$\alpha^\star$-I-normal Spaces

Definition 4.1. *A space (X, τ) is said to be quasi-α-normal if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint α-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Definition 4.2. *An ideal topological space (X, τ, I) is said to be quasi-$\alpha^\star$-I-normal if for every pair of disjoint π-closed subsets A, B of X, there exist disjoint $\alpha^\star_I$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$.*

Proposition 4.3. *If X is a quasi-α-normal space, then X is quasi-$\alpha^\star$-I-normal.*

Proof. It is obtained from Proposition 2.7.

Theorem 4.4. *The following properties are equivalent for a space X:*

1. *X is quasi-$\alpha^\star$-I-normal,*
2. *for any disjoint π-closed sets A and B, there exist disjoint $I_{\pi g\alpha^\star}$-open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$,*
3. *for any π-closed set A and any π-open set B containing A, there exists an $I_{\pi g\alpha^\star}$-open set U such that $A \subseteq U \subseteq \alpha^\star_I cl(U) \subseteq B$.*

Proof. (1) ⇒ (2) The proof is obvious.
(2) ⇒ (3) Let A be any π-closed set of X and B any π-open set of X such that A ⊆ B. Then A and X\B are disjoint π-closed subsets of X. Therefore, there exist disjoint $I_{\pi g\alpha^\star}$-open sets U and V such that A ⊆ U and X\B ⊆ V. By the definition of $I_{\pi g\alpha^\star}$-open set, We have that X\B ⊆ $\alpha_I^\star$int(V) and U ∩ $\alpha_I^\star$int(V) = ∅. Therefore, we obtain $\alpha_I^\star$cl(U) ⊆ $\alpha_I^\star$cl(X\V) and hence A ⊆ U ⊂ $\alpha_I^\star$cl(U) ⊆ B.
(3) ⇒ (1) Let A and B be any disjoint π-closed sets of X. Then A ⊆ X\B and X\B is π-open and hence there exists an $I_{\pi g\alpha^\star}$-open set G of X such that A ⊆ G ⊆ $\alpha_I^\star$cl(G) ⊆ X\B. Put U = $\alpha_I^\star$int(G) and V = X\ $\alpha_I^\star$cl(G). Then U and V are disjoint $\alpha_I^\star$-open sets of X such that A ⊆ U and B ⊆ V. Therefore, X is quasi-$\alpha^\star$-I-normal.

Theorem 4.5. *Let $f : X \to Y$ be an $I_{\pi g\alpha^\star}$-continuous m-π-closed injection. If Y is quasi-normal, then X is quasi-$\alpha^\star$-I-normal.*

Proof. Let A and B be disjoint π-closed sets of Y. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By the quasi-normality of X, there exist disjoint open sets U and V such that f(A) ⊆ U and f(B) ⊆ V. Since f is $I_{\pi g\alpha^\star}$-continuous, then f^{-1}(U) and f^{-1}(V) are disjoint $I_{\pi g\alpha^\star}$-open sets such that A ⊆ f^{-1}(U) and B ⊆ f^{-1}(V). Therefore X is quasi-$\alpha^\star$-I-normal by Theorem 4.4.

Theorem 4.6. *Let $f : X \to Y$ be an $I_{\pi g\alpha^\star}$-irresolute m-π-closed injection. If Y is quasi-$\alpha^\star$-I-normal, then X is quasi-$\alpha^\star$-I-normal.*

Proof. Let A and B be disjoint π-closed sets of Y. Since f is m-π-closed injection, f(A) and f(B) are disjoint π-closed sets of Y. By quasi-$\alpha^\star$-I-normality of Y, there exist disjoint $I_{\pi g\alpha^\star}$-open sets U and V such that f(A) ⊆ U and f(B) ⊆ V. Since f is $I_{\pi g\alpha^\star}$-irresolute, then f^{-1}(U) and f^{-1}(V) are disjoint $I_{\pi g\alpha^\star}$-open sets such that A ⊆ f^{-1}(U) and B ⊆ f^{-1}(V). Therefore X is quasi-$\alpha^\star$-I-normal.

Theorem 4.7. *Let (X, τ, I) be an ideal topological space where I is codense. Then X is quasi-$\alpha^\star$-I-normal if and only if it is quasi-α-normal.*

5 Conclusion

Topology is an area of Mathematics concerned with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing. By the middle of the 20th century, topology had become a major branch of Mathematics.

Topology as a branch of Mathematics can be formally defined as the study of qualitative properties of certain objects that are invariant under a certain kind of transformation especially those properties that are invariant under a certain kind of equivalence and it is the study of those properties of geometric configurations which remain invariant when these configurations are subjected to one-to-one bicontinuous transformations or homeomorphisms. Topology operates with more general concepts than analysis. Differential properties of a given transformation are nonessential for topology but bicontinuity is essential. As a consequence, topology is often suitable for the solution of problems to which analysis cannot give the answer.

Though the concept of topology has been identified as a difficult territory in Mathematics, we have taken it up as a challenge and cherishingly worked out this research study. Ideal Topology is a generalization of topology in classical mathematics, but it also has its own unique characteristics. It can also further up the understanding of basic structure of classical mathematics and offers new methods and results in obtaining significant results of classical mathematics. Moreover it also has applications in some important fields of Science and Technology.

A new class of sets called $I_{\pi g\alpha^\star}$-closed sets is introduced and its properties are studied in ideal topological space. Moreover $I_{\pi g\alpha^\star}$-continuity and the notion of quasi-$\alpha^\star$-I-normal spaces are introduced.

References

[1] Arockiarani, I., Balachandran, K. and Janaki, C.: On contra-$\pi g\alpha$-continuous functions, Kochi J. Math., 3(2008), 201-209.

[2] Dontchev, J. and Noiri, T.: Quasi-normal spaces and πg-closed sets, Acta Math. Hungar., 89(3)(2000), 211-219.

[3] Dontchev, J., Ganster, M. and Rose, D.: Ideal resolvability, Topology and its Applications, 93(1999), 1-16.

[4] Ekici, E. and Baker, C. W.: On πg-closed sets and continuity, Kochi J. Math., 2(2007), 35-42.

[5] Ekici, E. and Noiri, T.: $\star$-extremally disconnected ideal topological spaces, Acta Math. Hungar., 122(1-2)(2009), 81-90.

[6] Hatir, E. and Noiri, T.: On decompositions of continuity via idealization, Acta Math. Hungar., 96(4)(2002), 341-349.

[7] Hayashi, E.: Topologies defined by local properties, Math. Ann., 156(1964), 205-215.

[8] Jankovic, D. and Hamlett, T. R.: New topologies from old via ideals, Amer. Math. Monthly, 97(4)(1990), 295-310.

[9] Kuratowski, K.: Topology, Vol. 1, Academic Press, New York (1966).

[10] Njastad, O.: On some classes of nearly open sets, Pacific J. Math., 15(1965), 961-970.

[11] Rajamani, M., Inthumathi, V. and Krishnaprakash, S.: $\mathcal{I}_{\pi g}$-closed sets and $\mathcal{I}_{\pi g}$-continuity, Journal of Advanced Research in Pure Mathematics, 2(4)(2010), 63-72.

[12] Renuka Devi, V., Sivaraj, D. and Tamizh Chelvam, T.: Codense and Completely codense ideals, Acta Math. Hungar., 108(3)(2005), 197-205.

[13] Stone, M. H.: Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41(1937), 375-481.

[14] Vaidyanathaswamy, R.: The localization theory in set topology, Proc. Indian Acad. Sci., Sect A, 20(1944), 51-61.

[15] Viswanathan, K. and Jayasudha, J.: Some new sets and decompositions of $\alpha^\star$-$\mathcal{I}$-continuity and A_{IR}-continuity via idealization, Malaya J. Matematik, S(1) (2013), 17-23.

[16] Zaitsev, V.: On certain classes of topological spaces and their bicompactifications, Dokl. Akad. Nauk. SSSR, 178(1968), 778-779.