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TOPOLOGY SPECTRUM OF A KU-ALGEBRA

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Abstract – The aim of this paper is to study the Zariski topology of a commutative KU-algebra. Firstly, we introduce new concepts of a KU-algebra, such as KU-lattice, involutory ideal and prime ideal and investigate some basic properties of these concepts. Secondly, the notion of the topology spectrum of a commutative KU-algebra is studied and several properties of this topology are provided. Also, we study the continuous map of this topological space.

Keywords *– KU-algebra, KU-lattice, involutory ideal, prime ideal, topology spectrum.*

1. Introduction

The Zariski topology on the spectrum of prime ideals of a commutative ring is one of the main tools in Algebraic Geometry. Atiyah and Macdonald [1] introduced the spectrum $Spc(R)$ of a ring R as the following: for each ideal I of R, $V(I)=\{P\in Spec(R): I\subseteq P\}$, then the set $V(I)$ satisfy the axioms for the closed sets of a topology on $Spc(R)$, called the Zariski topology. Also, the notion of a spectrum of modules has been introduced by many authors see [2, 5, 6 and 7]. Prabpayak and Leerawat [11] introduced a new algebraic structure which is called KU-algebras. They introduced the concept of homomorphisms of KU-algebras and investigated some related properties. In [3, 4, 12 and 13], the authors introduced topologies on the set of all prime ideals by different way. In this paper, we study the relationship between a KU-algebra and topological space by the notion of the Zariski topology. We give the new concept of KU-lattice, involutory ideal and prime ideal of a KU-algebra X and discuss some properties which related to these concepts. Consequently,

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we show that $Spc(X)$ of a KU-algebra X is a compact and disconnected space. Also, we study some of separation axioms and continuous map of this topological space.

2. Preliminaries

Now we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1 **[11].** Let X be a nonempty set with a binary operation $*$ and a constant 0. The triple $(X,*,0)$ is called a KU-algebra, if the following axioms are satisfied. For all $x,y,z\in X$.

(ku_1) $(x*y)*[(y*z))*(x*z)]=0$.

(ku_2) $x*0=0$.

(ku_3) $0*x=x$.

(ku_4) $x*y=0$ and $y*x=0$ implies $x=y$.

(ku_5) $x*x=0$.

On a KU-algebra X, we can define a binary relation $\leq$ on X by putting $x\leq y\Leftrightarrow y*x=0$. Then $(X,\leq)$ is a partially ordered set and 0 is its smallest element. Thus $(X,*,0)$ satisfies the following conditions. For all $x,y,z\in X$, we that

$(ku_{1'})$ $(y*z)*(x*z)\leq(x*y)$

$(ku_{2'})$ $0\leq x$

$(ku_{3'})$ $x\leq y, y\leq x$ implies $x=y$,

$(ku_{4'})$ $y*x\leq x$.

Theorem 2.2 **[8].** In a KU-algebra X. The following axioms are satisfied.
For all $x,y,z\in X$,

(1) $x\leq y$ imply $y*z\leq x*z$,

(2) $x*(y*z)=y*(x*z)$,

(3) $((y*x)*x)\leq y$.

Definition 2.3 **[11].** A non-empty subset I of a KU-algebra X is called an ideal of X if for any $x,y\in X$, then

(i) $\mathbf{0}\in I$ and

(ii) $x*y, x\in I$ imply $y\in I$.

Definition 2.4 **[9].** A KU-algebra X is said to be KU-commutative if it satisfies $(y*x)*x=(x*y)*y$, for all x,y in X.

Lemma 2.5 **[9].** If X is KU-commutative algebra, then for any distinct elements $x,y,z\in X$, $x\dot{\wedge}(y*z)=(x\dot{\wedge}y)*(x\dot{\wedge}z)$.

Definition 2.6. If there is an element E of a KU-algebra X satisfying $x \le \mathrm{E}$ for all $x \in X$, then the element E is called unit of X. A KU-algebra with unit is called bounded. In a bounded KU-algebra X, we denote $x * \mathrm{E}$ by N_x. It is easy to see that $N_E = 0, N_0 = E$.

Example 2.7. Let $X = \{0, a, b, c, d\}$ be a set with a binary operation $*$ defined by the following table.

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	0	0	b	c	b	c
b	0	a	0	b	a	d
c	0	a	0	0	a	a
d	0	0	0	b	0	b
e	0	0	0	0	0	0

Using the algorithms in Appendix, we can prove that (X, *, 0) is a KU-algebra and by routine calculations, we can see that X is a bounded KU-algebra with unit "d".

Theorem 2.8. For a bounded KU-commutative algebra X, we denote $x \dot{\vee} y = N_{(N_x \dot{\wedge} N_y)}$ and for all $x, y \in X$, we have

(a) $N_{N_x} = x$,

(b) $N_x \dot{\wedge} N_y = N_{(x \dot{\vee} y)}$, $N_x \dot{\vee} N_y = N_{(x \dot{\wedge} y)}$,

(c) $x \le y$ implies $N_y \le N_x$.

(d) $E \dot{\wedge} x = x$,

(e) $x \dot{\wedge} E = E$.

Proof. The proof is straightforward. □

Definition 2.9. A partially ordered set $(L, \le)$ is said to be a lower semilattice if every pair of elements in L has a greatest lower bound and it is called to be an upper semilattice if every pair of elements in L has a least upper bound. If L is a lattice, then we define $x \wedge y = \mathbf{glb}\{x, y\}$ and $x \vee y = \mathbf{lub}\{x, y\}$. A lattice L is said to be distributive if it satisfies the following conditions. For all $x, y, z \in L$

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Theorem 2.10. Every KU-commutative algebra X is a KU-lower semilattice with respect to $(X, \le)$.

Proof. Suppose X is a KU-commutative algebra. We know that $x \dot{\wedge} y \le x$ and $x \dot{\wedge} y \le y$. Let z be any element of X such that $z \le x$ and $z \le y$, then $x * z = y * z = 0$ (by Definition of $\le$), so we have that $z = 0 * z = \overbrace{(x * z) * z = (z * x) * x}^{commutativity}$.

By the same reason we have $z = (z * y) * y$, and hence $z = (z * x) * x = (((z * y) * y) * x) * x \le (y * x) * x = x \dot\wedge y$, thus $x \dot\wedge y$ is the greatest KU-lower bound and so $(X, \le)$ is a KU-lower semilattice. $\square$

The converse of this theorem may not be true. For example, in Example 2.7 we have that X is a lower semilattice, but $(a * c) * c = c * c = 0 \ne a = 0 * a = (c * a) * a$.

Theorem 2.11. Any bounded KU-commutative algebra X with respect to $(X, \le)$ is a KU-lattice.

Proof. Since $N_x \dot\wedge N_y \le N_x$ and $N_x \dot\wedge N_y \le N_y$, from Theorem 2.8 we have that $x = N_{N_x} \le N_{(N_x \dot\wedge N_y)} = x \dot\vee y$ and $y = N_{N_y} \le N_{(N_x \dot\wedge N_y)} = x \dot\vee y$.
This shows that $x \dot\vee y$ is a common upper bound of x and y. Now, by Theorem 2.8 if $x \le z$ and $y \le z$, then $N_z \le N_x$ and $N_z \le N_y$. It follows that $N_z \le N_x \dot\wedge N_y$, therefore $N_{(N_x \dot\wedge N_y)} \le N_{N_z}$ and $x \dot\vee y \le z$. Hence $x \dot\vee y$ is a least upper bound of x and y, i.e. $(X, \le)$ is a KU-upper semilattice. By using Theorem 2.10 and this Theorem, we obtain $(X, \le)$ is a KU-lattice. $\square$

Definition 2.12. Let X be a KU-algebra and A a nonempty subset of X. The ideal of X generated by A is denoted by $\langle A \rangle = \{x \in X : \exists a_1, ..., a_n \in A$ such that $(a_1 * (... * (a_n * x) = 0\}$, if $A \ne \phi$. We have that $\langle \phi \rangle = \{0\}$.

Definition 2.13. Let X be KU-commutative algebra and A a subset of X. Then we define $A^* = \{x \in X : a \dot\wedge x = \mathbf{0}$ for all $a \in A\}$ and call it the KU-annihilator of A.

We write A^{**} in place of $(A^*)^*$. Note that A^* is a nonempty since $\mathbf{0} \in A^*$. Obviously we have $X^* = \{\mathbf{0}\}$ and $\{\mathbf{0}\}^* = X$. If A is an ideal it is easy to see that $A \cap A^* = \{\mathbf{0}\}$. We observe that if $x \in A^*$ then $a \dot\wedge x = \mathbf{0}$ for all $a \in A$. It follows that $(x * a) * a = 0$ then $a \le x * a$ and $a * (x * a) = 0$, hence $x * a \le a$ which implies that $a = x * a$. Thus $x \in A^*$ if and only if $a = x * a$ for all $a \in A$. Moreover if X is commutative, then $x \in A^*$ if and only if $a = x * a$ for all $a \in A$.

If $A = \{a\}$, then we write $(a)^*$ instead of $(\{a\})^*$.

Example 2.14. Let $X = \{0, a, b, c, d, e\}$ be a set with a binary operation $*$ defined by the following table.

It is easy to show that X is a bounded KU-commutative algebra. If $A = \{b, c\}$, then $A^* = \{\mathbf{0}, a\}$.

Definition 2.15. An ideal A of a KU-commutative algebra X is said to be involutory if $A = A^{**}$. Moreover a KU-commutative algebra X is said to be involutory if every ideal of X is involutory.

Clearly $\{0\}$ and X are involutory ideals.

Remark 2.16. In involutory KU-commutative algebra X, for any two ideals A, B of X, we have that $(A \cap B)^* = \langle A^* \cup B^* \rangle$.

Lemma 2.17. Let X be involutory KU-commutative algebra. Then $X = \langle A \cup A^* \rangle$ for any ideal A of X.

Proof. Note that $A \cap A^* = \{0\}$. By Remark 2.16 and note X is involutory, we have $\langle A \cup A^* \rangle = \langle A^{**} \cup A^* \rangle = (A^* \cap A)^* = (0)^* = X$.

Definition 2.18. A KU-algebra X is said to be KU-positive implicative if it satisfies that $(z*x)*(z*y) = z*(x*y)$, for all x, y, z in X.

Definition 2.19. A nonempty subset I of a KU-algebra X is said to be a KU-positive implicative ideal if for all x, y, z in X, then
(1) $0 \in I$ and
(2) $z*(x*y) \in I$ and $z*x \in I$ imply $z*y \in I$.

Theorem 2.20. If we are given an ideal I of a KU-algebra X, then I is a KU-positive implicative if and only if, for any $a \in X$ the set $A_a = \{x \in X : a * x \in I\}$ is an ideal of X.

Proof. ($\Rightarrow$) Suppose that I is positive implicative ideal and $(x*y) \in A_a$ and $x \in A_a$. Then $a*(x*y) \in I$ and $a*x \in I$. By Definition 2.19 we obtain $(a*y) \in I$ i.e. $y \in A_a$. This says A_a is an ideal.

($\Leftarrow$) Suppose that A_a is an ideal of X, for any $a \in X$. If $z*(x*y) \in I$ and $z*x \in I$, then $(x*y) \in A_z$ and $x \in A_z$. Since A_z is an ideal of X then $y \in A_z$ and $z*y \in I$. This means that I is positive implicative ideal. □

Corollary 2.21. If I is a KU-positive implicative ideal of X, then $A_a = \{x \in X : a*x \in I\}$ is the least ideal containing I and a, for any $a \in X$.

Definition 2.22. A nonempty subset I of a KU-algebra X is said to be a KU-implicative ideal if for all x, y, z in X, then
(1) $0 \in I$ and
(2) $z*((x*y)*x) \in I$ and $z \in I$ imply $x \in I$.

Definition 2.23. A proper ideal I of a KU-algebra X is called a maximal ideal if and only if $I \subseteq A \subseteq X$ implies that $I = A$ or $A = X$, for any ideal A of X.

Theorem 2.24. If I is an ideal of a KU-algebra X. Then the following statements are equivalent.
(a) I is maximal and KU-implicative ideals,
(b) I is maximal and KU-positive implicative ideals,
(c) $x, y \notin I$ implies $x*y \in I$ and $y*x \in I$ for all x, y in X.

Proof. (a)$\Rightarrow$(b). Suppose that I is KU-implicative ideal and $z*(x*y)\in I$, $z*x\in I$. Since $(z*x)*(z*(z*y))\le x*(z*y)=z*(x*y)\in I$ then $(z*x)*(z*(z*y))\in I$ and $z*x\in I$. I is an ideal, we have that $(z*(z*y))\in I$. It follows that $((z*y)*y)*(z*y)=z*(z*y)\in I$ and $0*(((z*y)*y))*(z*y))\in I$. Combining $0\in I$ we obtain $z*y\in I$. Hence I is KU-positive implicative ideal.

(b) $\Rightarrow$(c). Let $x,y\notin I$. Since I is KU-positive implicative. By Corollary 2.21 $A_y=\{u\in X: u*y\in I\}$ is the least ideal containing I and y. Using maximality of I we have that $A_y=X$. Hence $x\in A_y$, that is $x*y\in I$. Likewise for $y*x\in I$.

(c) $\Rightarrow$(a) At first we prove that I is KU-implicative. Suppose I does not KU implicative, then there are x,y in X such that $(x*y)*x\in I$ but $x\notin I$. If $x*y\in I$, combining $(x*y)*x\in I$ we get $x\in I$. This contradicts to $x\notin I$. If $x*y\notin I$, by (c) we have $y\in I$ as $x\notin I$. By $ku_{4^\backslash}$, we have $x*y\le y$, we get $x*y\in I$. This contradicts to $x*y\notin I$. Hence I is KU-implicative. Next we prove that I is maximal. Note that I is also KU-positive implicative. Hence it is sufficient to prove that for any $a\notin I$ we have $A_a=\{x\in X: x*a\in I\}=X$. By Corollary 2.21, A_a is the least ideal containing I and a. For all x in X, when $x\in I$ then $x\in A_a$ and when $x\notin I$, by $a\in I$ and (c) we have that $x*a\in I$ i.e. $x\in A_a$. This means that $A_a=X$. Therefore I is maximal ideal of X. □

Definition 2.25. Let X be a KU-lower semilattice and P a proper ideal of X. Then P is said to be a prime ideal if $a\dot{\wedge}b\in P$ implies $a\in P$ or $b\in P$, for any a,b in X.

Theorem 2.26. In a KU-lower semilattice X, a proper ideal P of X is said to be a prime if $A\bigcap B\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$, for any ideals A,B in X.

Proof. Suppose that $A\bigcap B\subseteq P$, $A\not\subset P$ and $B\not\subset P$ for some two ideals A,B in X. Thus there exist a and b such that $a\in A-P$ and $b\in B-P$. From $a\dot{\wedge}b\le a$ and $a\dot{\wedge}b\le b$ it follows that $a\dot{\wedge}b\in A,B$ and $a\dot{\wedge}b\in A\bigcap B\subseteq P$. This contradicts to primness of P. Hence $A\subseteq P$ or $B\subseteq P$. □

Theorem 2.27. If X is a KU-implicative algebra, then each prime ideal of X is maximal.

Proof. Suppose that P is prime ideal and $a,b\notin P$. Since X is KU-implicative, then $a\dot{\wedge}(a*b)=((a*b)*a)*a=a*a=0\in P$. Noticing $a\notin P$, we have $a*b\in P$. By the same way we get $b*a\in P$. Hence P is maximal ideal by Theorem 2.24. □

Lemma 2.28. Let X be a KU-lower semilattice. If $a\le x^n$ and $a\le x^m$ for natural numbers n and m, then there exists a natural number p such that $a\le(x\dot{\wedge}y)^p$, for any $x,y,a\in X$.

Proof. Since for $m\le n$, $a\le x^m$ implies $a\le x^n$, it suffices to verify that when $x^n*a=y^n*a=0$, there exist a natural number p such that $(x\wedge y)^p*a=0$. We proceed by induction on n. When $n=1$, we have $x*a=y*a=0$, $a\le x$ and $a\le y$. Hence $a\le x\dot{\wedge}y$, i.e., $(x\dot{\wedge}y)*a=0$.

Now suppose the assertion holds for natural number n, that is, $x^n * a = y^n * a = 0$ implies that there exists a natural number p such that $(x \dot\wedge y)^p * a = 0$.

If $x^{n+1} * a = y^{n+1} * a = 0$, then $0 = x^{n+1} * a = (x * (y^n * (x^n * a))$.

By the same argument we have $0 = y * (y^n * (x^n * a)$. In view of the first step of induction we get

$(x \dot\wedge y) * (y^n * (x^n * a) = 0$, $(y^n * (x^n * ((x \dot\wedge y) * a) = 0$, $(x * (x^{n-1} * (y^n * ((x \dot\wedge y) * a)) = 0$.

From $y^{n+1} * a = 0$. It easily follows that $(y * (x^{n-1} * (y^n * ((x \dot\wedge y) * a)) = 0$. Hence $x^{n-1} * (y^n * ((x \dot\wedge y)^2 * a) = 0$. Repeating the above procedure n times we obtain $y^n * ((x \dot\wedge y)^{n+1} * a) = 0$...........(1). By an entirely similar way we have that

$x^n * ((x \dot\wedge y)^{n+1} * a) = 0$...........(2). By the induction hypothesis and (1), (2), we know that there is a natural number p such that $(x \dot\wedge y)^p * ((x \dot\wedge y)^{n+1} * a) = 0$, $(x \dot\wedge y)^{n+p+1} * a = 0$. □

Corollary 2.29. Let X be a KU-lower semilattice and P an ideal in X. Then for any $x, y \in X$ if $x \dot\wedge y \in P$, then $\langle P \bigcup \{x\} \rangle \bigcap \langle P \bigcup \{y\} \rangle = P$.

Definition 2.30. Let X be a KU-lower semilattice. A nonempty subset S of X is said to be $\dot\wedge$-closed if $x \dot\wedge y \in S$ whenever $x, y \in S$.

Theorem 2.31. Let X be a KU-lower semilattice and S a nonempty $\dot\wedge$-closed subset of X such that $\mathbf{0} \notin S$, $I(X)$ denotes the set of all ideals of X then $\{I \in I(X) : I \bigcap S = \phi\}$ have a maximal ideal P such that $P \bigcap S = \phi$. Moreover P is a prime ideal.

Proof. The existence of an ideal P easily follows from Zorn's lemma. We will prove that P is a prime ideal. Let us suppose it is not the case, i.e., there exist $x, y \in X$ such that $x \dot\wedge y \in P$, $x \notin P$ and $y \notin P$. Then P is properly contained in both $\langle P \bigcup \{x\} \rangle = P_1$ and $\langle P \bigcup \{y\} \rangle = P_2$. Because of maximality of P, $P_1 \bigcap S \neq \phi$ and $P_2 \bigcap S \neq \phi$. Let $s_i \in P_i \bigcap S, i = 1,2$. We known $s_1 \dot\wedge s_2 \leq s_i, i = 1,2$ implies $s_1 \dot\wedge s_2 \in P_1 \bigcap P_2 = P$ (by Corollary 2.29). On the other hand $s_1 \dot\wedge s_2 \in S$. This is a contradiction. Hence P is a prime ideal. □

Theorem 2.32. In a KU-lower semilattice X. Any maximal ideal must be prime.

Proof. By using Theorem 2.31 and Corollary 2.29, we obtain the result.

Definition 2.33. Let I be an ideal of a KU-algebra X. We will call an ideal J of X a minimal prime ideal associated with the ideal I if J is a minimal element in the set of all prime ideals containing I.

Lemma 2.34. Let I be a proper ideal of a KU-lower semilattice X .Then
(a) I is contained in a prime ideal,
(b) Any prime ideal containing I contains a minimal prime ideal associated with the ideal I .

Proof. If I is a prime ideal, then the Lemma is true. Let us suppose that I is not a prime ideal and $a \in X - I$. Obviously, $S = \{x \in X : a \leq x\}$ is a nonempty, $\dot{\wedge}$ -closed and $\mathbf{0} \notin S$. By Theorem 2.31, there exists a prime ideal P such that $P \cap S = \phi$. (a) holds.

To show (b) it is sufficient to show that the intersection of any chain of prime ideals is a prime ideal. Let $\{P_i : i \in \omega\}$ be a chain of prime ideals of X and $P = \bigcap\{P_i : i \in \omega\}$. Suppose that P is not a prime ideal, that is, there are $x, y \in X$ such that $x \dot{\wedge} y \in P, x \notin P, y \notin P$. Thus, there are $i, j \in \omega$ such that $x \notin P_i, y \notin P_j$. Without loss of generality we can assume that $P_i \subseteq P_j$, $x \notin P_i, y \notin P_i$ and $x \dot{\wedge} y \in P \subseteq P_i$. This contradicts to P_i being a prime.

3. Topology Spectrum of KU-commutative algebra X

In this section, we define the notion of a spectrum of KU-commutative algebra X and study some of its properties.

Definition 3.1. Let X be KU-commutative algebra and $Spec(X)$ the set of all prime ideals of X . Then for any ideal A of X , we define $W(A) = \{P \in Spec(X) \mid A \not\subset P\}$.

Proposition 3.2. Let X be KU-commutative semilattice algebra. Then
(i) $A \subseteq B$ implies that $W(A) \subseteq W(B)$, for any ideals A, B of X ,
(ii) $W(A) = W(\langle A \rangle)$.

Proof. (i) Let $L \in W(A) \Rightarrow A \not\subset L$. Since $A \subseteq B \Rightarrow L \in W(B)$. Hence $W(A) \subseteq W(B)$.
(ii) Since $A \subseteq \langle A \rangle$ from (i) we get that $W(A) \subseteq W(\langle A \rangle)$. Let $P \in W(\langle A \rangle) \Rightarrow \langle A \rangle \not\subset P$ and since $A \subseteq \langle A \rangle$ then $A \not\subset P$, $P \in W(A)$ it follows that
$W(\langle A \rangle) \subseteq W(A)$. Hence $W(A) = W(\langle A \rangle)$. □

Theorem 3.3. Let X be KU-commutative algebra. Then the family $T(X) = \{W(A)\}_{A \in I(X)}$ forms a topology on $Spec(X)$.

Proof. $W(0) = \{P \in Spec(X) : (0) \not\subset P\} = \phi$ and
$W(X) = \{P \in Spec(X) : X \not\subset P\} = Spec(X)$. For any family $\{W(A_i)\}_{i \in I}$
$\bigcup_{i \in I} W(A_i) = \{P \in Spec(X) : A_i \not\subset P$ for some $A_i\} = \{P \in Spec(X) : \bigcup_{i \in I} A_i \not\subset P\}$
$= \{P \in Spec(X) : \langle \bigcup_{i \in I} A_i \rangle \not\subset P\} = W(\langle \bigcup_{i \in I} A_i \rangle)$ implies that $\bigcup_{i \in I} W(A_i) \in T(X)$.
Finally, $W(A) \cap W(B) = \{P \in Spec(X) : A \not\subset P\} \cap \{P \in Spec(X) : B \not\subset P\}$

$$= \{P \in Spec(X) : A \not\subset P \text{ and } B \not\subset P\}.$$

Since P is a prime ideal, therefore can be written as

$W(A)\bigcap W(B)=\{P\in Spec(X): A\bigcap B\not\subset P\}=W(A\bigcap B)$, i.e., $W(A)\bigcap W(B)\in T(X)$. Hence $T(X)$ is a topology on $Spec(X)$, this topology will be called the spectrum topology.

Example 3.4. In Example 2.14. By using the algorithms in Appendix A, we can found that $\{X,\{0\},\{0,a\},\{0,b,c\}\}$ is the set of all ideals. Note that $\{\{0,a\}$, $\{0,b,c\}\}$ is the set of all prime ideals of X and $Spec(X)=\{\{0,a\},\{0,b,c\}\}$. Therefore $T(X)=\{\phi, Spec(X)\}$ this is the indiscrete topology.

Definition 3.5. For any $A\in I(X)$ we denote the complement of $W(A)$ by $V(A)$. Hence $V(A)=\{P\in spec(X)\mid A\subseteq P\}$, it follows that the set $\{V(A)\}_{A\in I(X)}$ is the family of the closed sets of a topological space $Spec(X)$.

Remark 3.6. For any $x\in A$ we denote $V(\{x\})$ by $V(x)$ and $W(\{x\})$ by $W(x)$, i.e.
$V(x)=\{P\in spec(X)\mid x\in P\}$ and $W(x)=\{P\in spec(X)\mid x\notin P\}$.

Now, we give some properties of the topological space $Spec(X)$.

Theorem 3.7. Let X be a KU-commutative semilattice. The family $\{W(x)\}_{x\in A}$ is a basis for the topology of $Spec(X)$.

Proof. Let $A\subseteq X$ and $W(A)$ an open subset of $Spec(X)$, then $W(A)=W(\bigcup_{x\in A}\{x\})=\bigcup_{x\in A}W(x)$. Hence, any open set of $Spec(X)$ is union of subsets from the family $\{W(x)\}_{x\in A}$.□

Theorem 3.8. Let X be a KU-lower semilattice and A a proper ideal of X. Then A is equal to the intersection of all minimal prime ideals associated with it.

Proof. Denote $J(A)=\bigcap\{P\in I(X): P$ is a prime ideal and associated with $A\}$.

It is clearly $A\subseteq J(A)$. We will show that $J(A)\subseteq A$. Let us suppose that it is not the case, then there is $a\in J(A)$ and $a\notin A$. As in the proof of Lemma 2.34, we can show that if $S=\{x\in X: a\leq x\}$, then there exists a prime ideal P such that $A\subseteq P$ and $P\bigcap S=\phi$. The existence of such a prime ideal P contradicts to the assumptions. Hence $J(A)=A$. □

Lemma 3.9. The mapping $f: I(X)\to T(X)$ given by $f(A)=W(A)$ is a lattice isomorphism.

Proof. By Theorem 3.3 of $W(A)$, it follows that f define a lattice homomorphism. We only show that f is one to one and onto. For any ideals $A,B\in I(X)$. Suppose that $f(A)=f(B)$ then $W(A)=W(B)$ and $Spec(X)-W(A)=Spec(X)-W(B)$. Consequently, $J(A)=J(B)$, hence $A=B$, it follows that f is one to one and onto. Hence $I(X)$ and $T(X)$ are isomorphic. □

Proposition 3.10. If X is a bounded KU-commutative algebra, then $Spec(X)$ is a compact space.

Proof. Let $\{W(A_i)\}_{i\in I}$ be an open cover of $Spec(X)$. Then $Spec(X)=\bigcup_{i\in I}W(A_i)=W(\langle\bigcup_{i\in I}A_i\rangle)$. By injectiveness of W (Lemma3.9) implies that $\langle\bigcup_{i\in I}A_i\rangle = X$. Since X is a bounded $\Rightarrow$ $\mathrm{E}\in\langle\bigcup_{i\in I}A_i\rangle$ and hence $(a_1*(a_2*(...*(a_n*\mathrm{E}))=0$. We may assume that $a_k\in A_i$ for $k=1,2,...,n$, then $a_k\in\bigcup_{k=1}^{n}A_{i_k}$ for all $k=1,2,...,n$. This implies that $\mathrm{E}\in\langle\bigcup_{k=1}^{n}A_{i_k}\rangle$ and hence $\langle\bigcup_{k=1}^{n}A_{i_k}\rangle = X$ (because no proper ideal contains E). This shows that $\bigcup_{k=1}^{n}W(A_{i_k})=W(\bigcup_{k=1}^{n}A_{i_k})=W(\langle\bigcup_{k=1}^{n}A_{i_k}\rangle)=W(X)=Spec(X)$. Thus we obtain a finite Sub cover and consequently, $Spec(X)$ is compact. □

Proposition 3.11. Let X be KU-commutative algebra. Then $Spec(X)$ is T_0 topological space.

Proof. Let P and Q be any two distinct prime ideals in $Spec(X)$. Then either $P\not\subset Q$ or $Q\not\subset P$. If $P\not\subset Q$, there exists $x\in P$ such that $x\notin Q$ *which implies that* $Q\in W(x)$ and $P\notin W(x)$. Therefore exists an open set $W(x)$ containing Q but not P. Similarly, if $Q\not\subset P$. There exists $x\in Q$ such that $x\notin P$, *which implies that* $Q\notin W(x)$ and $P\in W(x)$. Therefore exists an open set $W(x)$ containing P but not Q. Hence $Spec(X)$ is a T_0-space.

Proposition 3.12. If X is a KU-implicative algebra. Then $Spec(X)$ is T_1 topological space.

Proof. If $Spec(X)=\phi$, *then* $Spec(X)$ is trivial space and it is a T_1 space.
If $Spec(X)\neq\phi$, then there exist a prime ideal P of $Spec(X)$. It follows by Theorem 2.27 that P is a maximal ideal. Hence $V(P)=\{i\}$ and $\{i\}$ is closed set in $Spec(X)$, i.e. $Spec(X)$ is a T_1 space. □

Proposition 3.13. If A is an involutory ideal of X and $P\in Spec(X)$, then $P\notin W(A^*)$ if and only if $P\in W(A)$.

Proof. If $P\notin W(A^*)$, then $A^*\subseteq P$. Since A is an involutory ideal of X, therefore by Lemma 2.17 $X=\langle A\bigcup A^*\rangle$ and hence $A\not\subset P$. This implies that $P\in W(A)$.
Conversely, assume that $P\in W(A)$ then $A\not\subset P$. Since $A\bigcap A^*=\{0\}\subseteq P$ and P is a prime ideal. Therefore by Theorem 2.26 $A\subseteq P$ or $A^*\subseteq P$, but $A\not\subset P$. It follows that $A^*\subseteq P$ and consequently we have $P\notin W(A^*)$. □

Proposition 3.14. Let X be an involutory KU-algebra with at least one involutory ideal (proper). Then $Spec(X)$ is a disconnected topological space.

Proof. Let A be an involutory (proper) ideal of X. We claim that $W(A)$ and $W(A^*)$ form disconnection of $Spec(X)$. That $W(A)$ and $W(A^*)$ mutually exclusive, follows from

Proposition 3.13. We show that $Spec(X) = W(A) \bigcup W(A^*)$. Indeed A is an involutory ideal, then $X = \langle A \bigcup A^* \rangle$. This implies that

$$W(X) = W(\langle A \bigcup A^* \rangle) = W(A \bigcup A^*) = W(A) \bigcup W(A^*).$$

This means that $Spec(X) = W(A) \bigcup W(A^*)$ and consequently $Spec(X)$ is a disconnected space. □

Proposition 3.15. If X is an involutory KU-algebra, then $Spec(X)$ is Hausdorff space.

Proof. Let P and Q be any two distinct prime ideals in $Spec(X)$. Then there exists an element x in X such that $x \in P$ and $x \notin Q$. This implies that $\langle x \rangle \subseteq P$ and $\langle x \rangle \not\subset Q$. In other word $P \notin W(\langle x \rangle)$ and $Q \in W(\langle x \rangle)$. By Proposition 3.13, we have $P \in W(\langle x \rangle^*)$. Thus we obtain two open sets $W(\langle x \rangle)$ and $W(\langle x \rangle^*)$ such that $P \in W(\langle x \rangle^*)$ and $Q \in W(\langle x \rangle)$. It follows that $W(\langle x \rangle) \bigcap W(\langle x \rangle^*) = W(\langle x \rangle \bigcap \langle x \rangle^*) = W(\mathbf{0}) = \phi$. Hence $Spec(X)$ is Hausdorff space. □

Corollary 3.16. If X is a bounded involutory KU-algebra, then $Spec(X)$ is normal space.

***Definition 3.17* [4].** Let $(G,*,0)$ and $(H,\bullet,0)$ be KU-algebras. A homomorphism is a map $h: G \to H$ satisfying $h(x*y) = h(x) \bullet h(y)$ for all $x, y \in G$. An injective homomorphism is called monomorphism and a surjective homomorphism is called epimorphism.

Proposition 3.18. Let $(G,*,0)$ and $(H,\bullet,0)$ be KU-algebras and $h: G \to H$ a homomorphism map of KU-algebras, then for any prime ideal P of H. The ideal $h^{-1}(P) = \{x \in G : h(x) \in P\}$ is also a prime ideal of G.

Proof. Let $x \dot\wedge y \in h^{-1}(P)$ for any $x, y \in G$, then
$(y*x)*x \in h^{-1}(P) \Rightarrow h((y*x)*x) \in P(\text{by homomorphism}) \Rightarrow h(y*x) \bullet h(x) \in P \Rightarrow$
$(h(y) \bullet h(x)) \bullet h(x) \in P \Rightarrow h(x) \dot\wedge h(y) \in P.$
Since P is prime $\Rightarrow h(x) \in P$ or $h(y) \in P$
$\Rightarrow x \in h^{-1}(P)$ or $y \in h^{-1}(P)$. Hence $h^{-1}(P)$ is prime ideal of G. □

Theorem 3.19. Let $(G,*,0)$, $(H,\bullet,0)$ be KU-algebras and $h: G \to H$ a homomorphism map of KU-algebras. If $\sigma : SpecH \to SpecG$, define by $\sigma(P) = h^{-1}(P)$ for any $P \in SpecH$, then σ is continuous map.

Proof. Let $W(x)$ be a basic open set in $Spec(G)$, for any $x \in G$. Then

$$\begin{aligned}\sigma^{-1}(W(x)) &= \{P \in SpecH : \sigma(P) \in W(x)\}\\ &= \{P \in SpecH : h^{-1}(P) \in W(x)\}\\ &= \{P \in SpecH : x \notin h^{-1}(P)\}\\ &= \{P \in SpecH : h(x) \notin P\}\end{aligned}$$

, which is open in $Spec(H)$.

Thus the inverse image of any open set in $Spec(G)$ is open in $Spec(H)$ and hence σ is a continuous map. □

4. Conclusion

This work is a study of the relationship between the KU-algebras and topological spaces. We introduced the topology spectrum of a commutative KU-algebra and we obtained some results that were different from the topology spectrum of commutative ring. However, there are differences because KU-algebras are not rings. We proved that the spectrum of KU-algebra is compact, disconnected and Hausdorff space. Also, we studied the continuous map of this topological space. The main purpose of our future work is to investigate the fuzzy topology of KU-algebras, which may have a lot of applications in different branches of mathematics.

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Appendix Algorithms

<u>Algorithm for KU-algebras</u>

Input (X :set, $*$:binary operation)
Output (" X is a KU-algebra or not")
Begin
If $X = \phi$ then go to (1.);
EndIf
If $0 \notin X$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \le |X|$ and not (Stop) do
If $x_i * x_i \ne 0$ then
Stop: = true;
EndIf
$j := 1$
While $j \le |X|$ and not (Stop) do
If $((y_j * x_i) * x_i) \ne 0$ then
Stop: = true;
EndIf
EndIf
$k := 1$
While $k \le |X|$ and not (Stop) do
If $(x_i * y_j) * ((y_j * z_k) * (x_i * z_k)) \ne 0$ then
Stop: = true;
EndIf

EndIf While
EndIf While
EndIf While
If Stop then

(1.) Output (" X is not a KU-algebra")
Else
Output (" X is a KU-algebra")
EndIf
End

<u>Algorithm for ideals</u>

Input (X :KU-algebra, I :subset of X);
Output ("I is an ideal of X or not");
Begin
If $I = \phi$ then go to (1.);
EndIf
If $0 \notin I$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \le |X|$ and not (Stop) do
$j := 1$
While $j \le |X|$ and not (Stop) do
If $(x_i * y_j) \in I$ and $x_i \in I$ then
If $y_j \notin I$ then
Stop: = true;
EndIf
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
Output ("I is an ideal of X ")
Else
(1.) Output ("I is not an ideal of X ")
EndIf
End

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