http://www.newtheory.org

Journal of New Theory

Received: 08.12.2014 *Year*: **2015,** *Number*: **1**, *Pages*: **24-37**
Accepted: 16.12.2014 *Original Article*[**]

New Operations on Interval Neutrosophic Sets

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Abstract - An interval neutrosophic set is an instance of a neutrosophic set, which can be used in real scientific and engineering. In this paper, three new operations on interval neutrosophic sets based on the arithmetic mean, geometrical mean, and respectively harmonic mean are defined on interval neutrosophic set.

Keywords - *Neutrosophic Sets, Interval Valued Neutrosophic Sets.*

1. Introduction

In recent decades, several types of sets, such as fuzzy sets [1], interval-valued fuzzy sets [2], intuitionistic fuzzy sets [3, 4], interval-valued intuitionistic fuzzy sets [5], type 2 fuzzy sets [6, 7], type n fuzzy sets [6], and hesitant fuzzy sets [8], neutrosophic set theory [9], interval valued neutrosophic set [10] have been introduced and investigated widely. The concept of neutrosophic sets introduced by Smarandache [6, 9] is interesting and useful in modeling several real life problems.

The neutrosophic set theory (NS for short) which is a generalization of intuitionistic fuzzy set has three associated defining functions, namely the membership function, the non-membership function and the indeterminacy function which are completely independent. After the pioneering work of Smarandache [9], Wang et al.[10] introduced the notion of interval neutrosophic sets theory (INS for short) which is a special set of neutrosophic sets. This concept is characterized by a membership function, a non-membership function and indeterminacy function whose values are intervals rather than real number, INS is more powerful in dealing with vagueness and uncertainty than NS, also INS is regarded as a

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[**]*Edited by Irfan Deli (Area Editor) and* Naim Çağman *(Editor-in- Chief).*
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useful and practical tool for dealing with indeterminate and inconsistent information in real world .

The theories of both neutrosophic set (NS) and interval neutrosophic set (INS) have achieved great success in various areas such as medical diagnosis [11], database [12,13], topology [14], image processing [15,16,17], and decision making problem [18].

Recently, Ye [19] defined the similarity measures between INSs on the basis of the hamming and Euclidean distances, and a multicriteria decision–making method based on the similarity degree is proposed. Some set theoretic operations such as union, intersection and complement on interval neutrosophic sets have also been proposed by Wang et al. [10]. Later on, Broumi and Smarandache [20] also defined correlation coefficient of interval neutrosophic set. In 2013, Peide Liu [21] have presented some new operational laws for interval neutrosophic sets (INSs) and studied their properties and proposed some aggregation operators, include interval neutrosophic power generalized weighted aggregation (INPGWA) operator and interval neutrosophic power generalized ordered weighted aggregation (INPGOWA) operator and gave a decision making method based on these operators.

In this paper, our aim is to propose three new operations on interval neutrosophic sets (INSs) and study their properties.

Therefore, the rest of the paper is set out as follows: In Section 2, some basic definitions related to neutrosophic sets, and interval valued neutrosophic set are briefly discussed. In Section 3, three new operations on interval neutrosophic sets have been proposed and some properties of the proposed operations on interval neutrosophic sets are proved. In section 4 we concludes the paper.

2. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, and interval neutrosophic sets relevant to the present work. See especially [9, 10, 21] for further details and background.

Definition 2.1 ([9]). Let U be an universe of discourse; then the neutrosophic set A is an object having the form $A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in U\}$, where the functions T, I, F $: U \rightarrow]^-0,1^+[$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in U$ to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x) \leq 3^+. \qquad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$.So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.2 [10]. Let X be a space of points (objects) with generic elements in X denoted by x. An interval neutrosophic set (for short INS) A in X is characterized by truthmembership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that

$$T_A(x), I_A(x), F_A(x) \in [0,1]$$

For convenience, we can use $x =([T^L, T^U], [I^L, I^U], [F^L, F^U])$ to represent an element in INS.

Remark 1. An INS is clearly a NS.

Definition 2.3 [10]. Let A ={([T_A^L , T_A^U] , [I_A^L , I_A^U] , [F_A^L , F_A^U])}

i. An INS A is empty if $T_A^L = T_A^U = 0$, $I_A^L = I_A^U = 1$, $F_A^L = F_A^U = 1$, for all x in A.
ii. Let $\underline{0}$ = <0, 1 ,1> and $\underline{1}$ = <1, 0 ,0>

In the following, we introduce some basic concepts related to INSs.

Definition 2.4 [21] Let $\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])} and $\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])} be two INSs.

i. $\tilde{n}_1 \cup \tilde{n}_2$ =[max(T_1^L, T_2^L), max(T_1^U, T_2^U)], [min (I_1^L, I_2^L),min(I_1^U, I_2^U)] , [min (F_1^L, F_2^L),min(F_1^U, F_2^U)] }
ii. $\tilde{n}_1 \cap \tilde{n}_2$ =[min(T_1^L, T_2^L), min(T_1^U, T_2^U)], [max (I_1^L, I_2^L)max(I_1^U, I_2^U)] , [max (F_1^L, F_2^L),max(F_1^U, F_2^U)] }

Definition 2.5. Let $\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])} and $\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])} be two INSs, then the operational laws are defined as follows.

i. $\tilde{n}^c = [F^L, F^U], [1-I^L, 1-I^U], [T^L, T^U]$
ii. $\tilde{n}_1 \oplus \tilde{n}_2 = ([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U])$
iii. $\tilde{n}_1 \otimes \tilde{h}_2 = [T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]$
iv. $\lambda \tilde{n} = \{([1-(1-T^L)^\lambda, 1-(1-T^U)^\lambda], [(I^L)^\lambda, (I^U)^\lambda], [(F^L)^\lambda, (F^U)^\lambda])\}$.

3. Three New Operations on INSs

Definition 3.1 Let $\tilde{n}_1$ and $\tilde{n}_2$ two interval neutrosophic set, we propose the following operations on INSs as follows:

$$\tilde{n}_1 \ @ \ \tilde{n}_2 = \{([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}],$$

where

$< T_1, I_1, F_1 > \in \tilde{n}_1, < T_2, I_2, F_2 > \in \tilde{n}_2$ }

$\tilde{n}_1 \ \$ \ \tilde{n}_2 = \{([\sqrt{T_1^L\, T_2^L}, \sqrt{T_1^U\, T_2^U}\,], [\sqrt{I_1^L\, I_2^L}, \sqrt{I_1^U\, I_2^U}\,], [\sqrt{F_1^L\, F_2^L}, \sqrt{F_1^U\, F_2^U}\,] \}$

where

$< T_1, I_1, F_1 > \in \tilde{n}_1$,$< T_2, I_2, F_2 > \in \tilde{n}_2$ }

$\tilde{n}_1 \ \# \ \tilde{n}_2 = \{([\frac{2\, T_1^L\, T_2^L}{T_1^L + T_2^L}, \frac{2\, T_1^U\, T_2^U}{T_1^U + T_2^U}], [\frac{2\, I_1^L\, I_2^L}{I_1^L + I_2^L}, \frac{2\, I_1^U\, I_2^U}{I_1^U + I_2^U}], [\frac{2\, F_1^L\, F_2^L}{F_1^L + F_2^L}, \frac{2\, F_1^U\, F_2^U}{F_1^U + F_2^U}]\}$,

where

$< T_1, I_1, F_1 > \in \tilde{n}_1$,$< T_2, I_2, F_2 > \in \tilde{n}_2$ }

With

$T_1 = [T_1^L\ , T_1^U]$, $I_1 = [I_1^L\ , I_1^U]$, $F_1 = [F_1^L\ , F_1^U]$ and $T_2 = [T_2^L\ , T_2^U]$, $I_2 = [I_2^L\ , I_2^U]$, $F_2 = [F_2^L, F_2^U]$

Obviously, for every two $\tilde{n}_1$ and $\tilde{n}_2$,($\tilde{n}_1 @ \tilde{n}_2$), ($\tilde{n}_1 \$ \tilde{n}_2$) and ($\tilde{n}_1 \# \tilde{n}_2$) are also INSs.

Example 3.2 Let $\tilde{n}_1$(x)= {([0.2, 0.3], [0.5, 0.6] , [0.2 , 0.4]),([0.5, 0.8], [0.1, 0.2], [0.6 ,0.1])} and $\tilde{n}_2$ (x)= {([0.4, 0.6],[0.3, 0.4], [0.3 , 0.5]), ([0.3, 0.5], [0.1, 0.2], [0.5 ,0.1]) be two interval neutrosophic sets. Then we have

($\tilde{n}_1$ @ $\tilde{n}_2$) = {([0.3, 0.45], [0.4, 0.5],[0.25 ,0.45]),(b, [0.4, 0.65], [0.1, 0.2], [0.55 , 0.1])}

($\tilde{n}_1$ \$ $\tilde{n}_2$) ={(a, [0.28, 0.42], [0.38, 0.48], [0.24 ,0.44]),(b, [0.38, 0.63], [0.1, 0.2], [0.55 ,0.1])}

($\tilde{n}_1 \# \tilde{n}_2$) = {(a, [0.26, 0.4], [0.37, 0.48] , [0.24 , 0.44]), (b, [0.37, 0.61], [0.1, 0.2], [0.54 , 0.1])}

With these operations, several results follow.

Theorem 3.4 For $\tilde{n}_1, \tilde{n}_2 \in$ INSs(X),

(i) $\tilde{n}_1 @ \ \tilde{n}_2 = \tilde{n}_2 \ @ \tilde{n}_1$;
(ii) $\tilde{n}_1 \$ \ \tilde{n}_2 = \tilde{n}_2 \ \$ \ \tilde{n}_1$;
(iii) $\tilde{n}_1 \# \tilde{n}_2 = \tilde{n}_2 \# \tilde{n}_1$;

Proof. These also follow from definitions.

Theorem 3.5 For $\tilde{n}_1, \tilde{n}_2 \in$ INSs(X), ($\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c$) ^{c} = $\tilde{n}_1 @ \tilde{n}_2$.

Proof. $\tilde{n}_1 \ @ \ \tilde{n}_2 = \{([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}] ,[\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]$

where

$< T_1, I_1, F_1 > \in \tilde{n}_1$,$< T_2, I_2, F_2 > \in \tilde{n}_2$ }

$\tilde{n}_1{}^c$ ={([F_1^L , F_1^U] , [1- I_1^L , $1 - I_1^U$] , [T_1^L , T_1^U])}

$\tilde{n}_2{}^c$ = {([F_2^L , F_2^U] , [1- I_2^L , $1 - I_2^U$] , [T_2^L , T_2^U])}

$\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c$ ={ [$\frac{F_1^L + F_2^L}{2}$, $\frac{F_1^U + F_2^U}{2}$], [$\frac{(1-I_1^L)+(1-I_2^L)}{2}$, $\frac{(1-I_1^U)+(1-I_2^U)}{2}$] ,[$\frac{T_1^L + T_2^L}{2}$, $\frac{T_1^U + T_2^U}{2}$] }

$$(\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c)^c = \left([\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}], [\frac{(1-I_1^L)+(1-I_2^L)}{2}, \frac{(1-I_1^U)+(1-I_2^U)}{2}], [\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}]\right)^c$$

$$= \left([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [1 - \frac{(1-I_1^L)+(1-I_2^L)}{2}, 1 - \frac{(1-I_1^U)+(1-I_2^U)}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]\right)$$

$$= \left([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{2-[2-(I_1^L + I_2^L)]}{2}, \frac{2-[2-(I_1^U + I_2^U)]}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]\right)$$

$$= \left([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{(I_1^L + I_2^L)}{2}, \frac{(I_1^U + I_2^U)}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]\right)$$

Then $(\tilde{n}_1{}^c @ \ \tilde{n}_2{}^c)^c = \tilde{n}_1 @ \ \tilde{n}_2$

This proves the theorem.

Note 1: One can easily verify that

(i) $(\tilde{n}_1{}^c \$ \ \tilde{n}_2{}^c)^c \neq \tilde{n}_1 \$ \ \tilde{n}_2$
(ii) $(\tilde{n}_1{}^c \# \ \tilde{n}_2{}^c)^c \neq \tilde{n}_1 \# \tilde{n}_2$

Theorem 3.6 For $\tilde{n}_1, \tilde{n}_2$ and $\tilde{n}_3$ ∈ INSs(X), we have the following identities:

(i) $(\tilde{n}_1 \cup \tilde{n}_2) @ \tilde{n}_3 = (\tilde{n}_1 @ \tilde{n}_3) \cup (\tilde{n}_2 @ \tilde{n}_3)$;
(ii) $(\tilde{n}_1 \cap \tilde{n}_2) @ \tilde{n}_3 = (\tilde{n}_1 @ \tilde{n}_3) \cap (\tilde{n}_2 @ \tilde{n}_3)$
(iii) $(\tilde{n}_1 \cup \tilde{n}_2) \$ \tilde{n}_3 = (\tilde{n}_1 \$ \tilde{n}_3) \cup (\tilde{n}_2 \$ \tilde{n}_3)$;
(iv) $(\tilde{n}_1 \cap \tilde{n}_2) \$ \tilde{n}_3 = (\tilde{n}_1 \$ \tilde{n}_3) \cap (\tilde{n}_2 \$ \tilde{h}_3)$;
(v) $((\tilde{n}_1 \cup \tilde{n}_2)) \# \tilde{n}_3 = (\tilde{n}_1 \# \tilde{n}_3) \cup (\tilde{n}_2 \# \tilde{n}_3)$;
(vi) $(\tilde{n}_1 \cap \tilde{n}_2) \# \tilde{n}_3 = (\tilde{n}_1 \# \tilde{n}_3) \cap (\tilde{n}_2 \# \tilde{n}_3)$;
(vii) $(\tilde{n}_1 @ \tilde{n}_2) \oplus \tilde{n}_3 = (\tilde{n}_1 \oplus \tilde{n}_3) @ (\tilde{n}_2 \oplus \tilde{n}_3)$;
(viii) $(\tilde{n}_1 @ \tilde{n}_2) \otimes \tilde{n}_3 = (\tilde{n}_1 \otimes \tilde{n}_3) @ (\tilde{n}_2 \otimes \tilde{n}_3)$

Proof . We prove (i), (iii), (v), (vii) and (ix), results (ii), (iv), (vi), (viii) and (x) can be proved analogously

(i) Using definitions in 2.4, 2.5 and 3.1, we have

$\tilde{n}_1$ ={([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])}

$\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])}

$(\tilde{n}_1 \cup \tilde{n}_2) @ \tilde{n}_3$ = {([max(T_1^L, T_2^L) ,max(T_1^U, T_2^U)] , [min (I_1^L, I_2^L) ,min(I_1^U, I_2^U)] ,

[min(F_1^L , F_2^L),min(F_1^U, F_2^U)])} @ {([T_3^L , T_3^U] , [I_3^L , I_3^U] , [F_3^L , F_3^U])}

$=\{[\frac{\max(T_1^L,T_2^L)+T_3^L}{2}, \frac{\max(T_1^U,T_2^U)+T_3^U}{2}], [\frac{\min(I_1^L,I_2^L)+I_3^L}{2}, \frac{\min(I_1^U,I_2^U)+I_3^U}{2}],$
$[\frac{\min(F_1^L,F_2^L)+F_3^L,}{2}, \frac{\min(F_1^U,F_2^U)+F_3^U}{2}]\}$

$=\{[\max(\frac{T_1^L+T_3^L}{2}, \frac{T_2^L+T_3^U}{2}), \max(\frac{T_1^U+T_3^U}{2}, \frac{T_2^U+T_3^U}{2})], [\min(\frac{I_1^L+I_3^L}{2}, \frac{I_2^L+I_3^L}{2}),$
$\min(\frac{I_1^U+I_3^U}{2}, \frac{I_2^U+I_3^U}{2})], [\min(\frac{F_1^L+F_3^L}{2}, \frac{F_2^L+F_3^L}{2}), \min(\frac{F_1^U+F_3^U}{2}, \frac{F_2^U+F_3^U}{2}]\}$

$= (\tilde{n}_1 @ \tilde{n}_3) \cup (\tilde{n}_2 @ \tilde{n}_3)$

This proves (i)

(iii) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \cup \tilde{n}_2) \$ \tilde{n}_3 = \{([\max(T_1^L, T_2^L), \max(T_1^U, T_2^U)], [\min(I_1^L, I_2^L), \min(I_1^U, I_2^U)],$
$[\min(F_1^L, F_2^L), \min(F_1^U, F_2^U)])\} \$ \{([T_3^L, T_3^U], [I_3^L, I_3^U], [F_3^L, F_3^U])\}$

$= \{[\sqrt{max(T_1^L, T_2^L)\, T_3^L}, \sqrt{\max(T_1^U, T_2^U)\, T_3^U}], [\sqrt{\min(I_1^L, I_2^L)\, I_3^L},$
$\sqrt{\min(I_1^U, I_2^U)\, I_3^U}], [\sqrt{\min(F_1^L, F_2^L)\, F_3^L}, \sqrt{\min(F_1^U, F_2^U)\, F_3^U}]\}.$

$=\{[\max(\sqrt{T_1^L T_3^L}, \sqrt{T_2^L T_3^L}), \max(\sqrt{T_1^U T_3^U}, \sqrt{T_2^U T_3^U})], [\min(\sqrt{I_1^L I_3^L}, \sqrt{I_2^L I_3^L}),$
$\min(\sqrt{I_1^U I_3^U}, \sqrt{I_2^U I_3^U})], [\min(\sqrt{F_1^L F_3^L}, \sqrt{F_2^L F_3^L}), \min(\sqrt{F_1^U F_3^U}, \sqrt{F_2^U F_3^U})]\}$

$=(\tilde{n}_1 \$ \tilde{n}_3) \cup (\tilde{n}_2 \$ \tilde{n}_3)$

This proves (iii).

(v) Using definitions in 2.4, 2.5 and 3.1, we have

$((\tilde{n}_1 \cup \tilde{n}_2)) \# \tilde{n}_3 = \{([\max(T_1^L, T_2^L), \max(T_1^U, T_2^U)], [\min(I_1^L, I_2^L), \min(I_1^U, I_2^U)],$
$[\min(F_1^L, F_2^L), \min(F_1^U, F_2^U)])\} \# \{([T_3^L, T_3^U], [I_3^L, I_3^U], [F_3^L, F_3^U])\}$

$= \{[\frac{2\max(T_1^L,T_2^L)\, T_3^L}{\max(T_1^L,T_2^L)+T_3^L}, \frac{2\max(T_1^U,T_2^U)\, T_3^U}{\max(T_1^U,T_2^U)+T_3^U}], [\frac{2\min(I_1^L,I_2^L)\, I_3^L}{\min(I_1^L,I_2^L)+I_3^L}, \frac{2\min(I_1^U,I_2^U)\, I_3^U}{\min(I_1^U,I_2^U)+I_3^U}],$
$[\frac{2\min(F_1^L,F_2^L)\, F_3^L}{\min(F_1^L,F_2^L)+F_3^L}, \frac{2\min(F_1^U,F_2^U)\, F_3^U}{\min(F_1^U,F_2^U)+F_3^U}]\}$

$=\{[\max(\frac{2\,T_1^L\, T_3^L}{T_1^L+T_3^L}, \frac{2\,T_2^L\, T_3^L}{T_2^L+T_3^L}), \max(\frac{2\,T_1^U\, T_3^U}{T_1^U+T_3^U}, \frac{2\,T_2^U\, T_3^U}{T_2^U+T_3^U})],$
$[\min(\frac{2\,I_1^L I_3^L}{I_1^L+I_3^L}, \frac{2\,I_2^L I_3^L}{I_2^L+I_3^L}), \min(\frac{2\,I_1^U I_3^U}{I_1^U+I_3^U}, \frac{2\,I_2^U I_3^U}{I_2^U+I_3^U})], [\min(\frac{2\,F_1^L F_3^L}{F_1^L+F_3^L}, \frac{2\,F_2^L F_3^L}{F_2^L+F_3^L}),$
$\min(\frac{2\,F_1^U F_3^U}{F_1^U+F_3^U}, \frac{2\,F_2^U F_3^U}{F_2^U+F_3^U})]\}$

$=(\tilde{n}_1 \# \tilde{n}_3) \cup (\tilde{n}_2 \# \tilde{n}_3)$

This proves (v)

(vii) Using definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 @ \tilde{n}_2) \oplus \tilde{n}_3 = (\tilde{n}_1 \oplus \tilde{n}_3) @ (\tilde{n}_2 \oplus \tilde{n}_3)$;

$\tilde{n}_1 = \{([T_1^L , T_1^U] , [I_1^L , I_1^U] , [F_1^L , F_1^U])\}$

$\tilde{n}_2 = \{([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])\}$

$\tilde{n}_3 = \{([T_3^L , T_3^U] , [I_3^L , I_3^U] , [F_3^L , F_3^U])\}$

$= \{([\frac{T_1^L + T_2^L}{2} , \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2} , \frac{I_1^U + I_2^U}{2}] ,[\frac{F_1^L + F_2^L}{2} , \frac{F_1^U + F_2^U}{2}])\}$
$\oplus \{([T_3^L , T_3^U] , [I_3^L , I_3^U] , [F_3^L , F_3^U])\}$

$=\{[\frac{T_1^L + T_2^L}{2} + T_3^L - \frac{T_1^L + T_2^L}{2} T_3^L, \frac{T_1^U + T_2^U}{2} + T_3^U - \frac{T_1^U + T_2^U}{2} T_3^U] , [\frac{I_1^L + I_2^L}{2} I_3^L, \frac{I_1^U + I_2^U}{2} I_3^U] ,$
$[\frac{F_1^L + F_2^L}{2} F_3^L, \frac{F_1^U + F_2^U}{2} F_3^U] \}$

$=\{ [\frac{(T_1^L + T_3^L - T_1^L T_3^L) + (T_2^L + T_3^L - T_2^L T_3^L)}{2} , \frac{(T_1^U + T_3^U - T_1^U T_3^U) + (T_2^U + T_3^U - T_2^U T_3^U)}{2}], [\frac{I_1^L + I_2^L}{2} I_3^L, \frac{I_1^U + I_2^U}{2} I_3^U] ,$
$[\frac{F_1^L + F_2^L}{2} F_3^L, \frac{F_1^U + F_2^U}{2} F_3^U] \}$

$= (\tilde{n}_1 \oplus \tilde{n}_3) @ (\tilde{n}_2 \oplus \tilde{n}_3)$

This proves (vi)

Theorem 3.7. For $\tilde{n}_1$ and $\tilde{n}_2 \in$ INSs(X), we have the following identities:

(i) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(ii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(iii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2$;
(iv) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(v) $(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(vi) $(\tilde{n}_1 \otimes \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2$;
(vii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \$ \tilde{n}_2$;
(viii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(ix) $(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(x) $(\tilde{n}_1 \otimes \tilde{n}_2) \cup (\tilde{n}_1 \$ \tilde{n}_2) = \tilde{n}_1 \$ \tilde{n}_2$;
(xi) $(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2$;
(xii) $(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2$;
(xiii) $(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2$;
(xiv) $(\tilde{n}_1 \otimes \tilde{n}_2) \cup (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2$

Proof .We prove (i), (iii), (v), (vii), (ix), (xi) and (xii), other results can be proved analogously.

(i) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)$

$\tilde{n}_1 = \{([T_1^L, T_1^U], [I_1^L, I_1^U], [F_1^L, F_1^U])\}$

$\tilde{n}_2 = \{([T_2^L, T_2^U], [I_2^L, I_2^U], [F_2^L, F_2^U])\}$

$=([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U]) \cap \{[T_1^L T_2^L, T_1^U T_2^U],$ $[I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\}$

$=\{[\min(T_1^L + T_2^L - T_1^L T_2^L, T_1^L T_2^L), \min(T_1^U + T_2^U - T_1^U T_2^U, T_1^U T_2^U)],$
$[\max(I_1^L I_2^L, I_1^L + I_2^L - I_2^L I_2^L), \max(I_1^U I_2^U, I_1^U + I_2^U - I_1^U I_2^U)],$
$[\max(F_1^L F_2^L, F_1^L + F_2^L - F_2^L F_2^L), \max(F_1^U F_2^U, F_1^U + F_2^U - F_1^U F_2^U)]\}$

$=[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]$

$= \tilde{n}_1 \otimes \tilde{n}_2$

This proves (i)

(iii) Using definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2;$

$=\{([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U]) \cap ([\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}],$ $[\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}])$

$=\{[\min(T_1^L + T_2^L - T_1^L T_2^L, \frac{T_1^L + T_2^L}{2}), \min(T_1^U + T_2^U - T_1^U T_2^U, \frac{T_1^U + T_2^U}{2}])],$
$[\max(I_1^L I_2^L, \frac{I_1^L + I_2^L}{2}), \max(I_1^U I_2^U, \frac{I_1^U + I_2^U}{2})], [\max(F_1^L F_2^L, \frac{F_1^L + F_2^L}{2}), \max(F_1^U F_2^U, \frac{F_1^U + F_2^U}{2})]\}.$

$=\{[\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]$

$= \tilde{n}_1 @ \tilde{n}_2$

This proves (iii).

(v) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$

$\{[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\} \cap \{[\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}],$
$[\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]\}.$

$=\{[\min(T_1^L T_2^L, \frac{T_1^L + T_2^L}{2}), \min(T_1^U T_2^U, \frac{T_1^U + T_2^U}{2})], [\max(I_1^L + I_2^L - I_2^L I_2^L, \frac{I_1^L + I_2^L}{2}),$
$\max(I_1^U + I_2^U - I_1^U I_2^U, \frac{I_1^U + I_2^U}{2})], [\max(F_1^L + F_2^L - F_2^L F_2^L, \frac{F_1^L + F_2^L}{2}),$
$\max(F_1^U + F_2^U - F_2^U F_2^U, \frac{F_1^U + F_2^U}{2})]\}.$

$= \{[\ T_1^L\,T_2^L\ ,\ T_1^U\,T_2^U\ ], [I_1^L + I_2^L - I_2^L I_2^L\ ,\ I_1^U + I_2^U - I_1^U I_2^U],$
$[\ F_1^L + F_2^L - F_2^L F_2^L\ ,\ F_1^U + F_2^U - F_2^U F_2^U\ ]\ \}$

$= \tilde{n}_1 \otimes \tilde{n}_2$

This proves (v).

(vii) Using definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \$\, \tilde{n}_2) = \tilde{n}_1 \$\, \tilde{n}_2$

$= ([T_1^L + T_2^L - T_1^L T_2^L\ , T_1^U + T_2^U - T_1^U T_2^U], [\ I_1^L\,I_2^L\ , I_1^U\,I_2^U], [F_1^L F_2^L\ , F_1^U F_2^U])$
$\cap \{([\sqrt{T_1^L\,T_2^L}, \sqrt{T_1^U\,T_2^U}\ ], [\sqrt{I_1^L\,I_2^L},\ \sqrt{I_1^U\,I_2^U}\ ], [\sqrt{F_1^L\,F_2^L},\ \sqrt{F_1^U\,F_2^U}\ ]\}$

$= \{\ [\ \min(T_1^L + T_2^L - T_1^L T_2^L\ , \sqrt{T_1^L\,T_2^L}\ ), \min(T_1^U + T_2^U - T_1^U T_2^U\ , \sqrt{T_1^U\,T_2^U}\ )],$
$[\max(\ I_1^L\,I_2^L, \sqrt{I_1^L\,I_2^L}\ ), \max(I_1^U\,I_2^U\ , \sqrt{I_1^U\,I_2^U}\ )\ ], [\max(\ F_1^L\,F_2^L, \sqrt{F_1^L\,F_2^L}\ ),$
$\max(F_1^U\,F_2^U\ , \sqrt{F_1^U\,F_2^U}\ )\ ]\}$

$= \{[\sqrt{T_1^L\,T_2^L}\ , \sqrt{T_1^U\,T_2^U}\ ], [\ \sqrt{I_1^L\,I_2^L}\ , \sqrt{I_1^U\,I_2^U}\ ], [\ \sqrt{F_1^L\,F_2^L}\ , \sqrt{F_1^U\,F_2^U}\ ]\ \}$

$= \tilde{n}_1 \$\, \tilde{n}_2$

This proves (vii)

(ix) From definitions in 2.4, 2.5 and 3.1, we have

$(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \$\, \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$

$= \{[T_1^L\,T_2^L\ ,\ T_1^U\,T_2^U)], [\ I_1^L + I_2^L - I_2^L I_2^L\ , I_1^U + I_2^U - I_1^U I_2^U], [F_1^L + F_2^L - F_2^L F_2^L\ ,\ F_1^U + F_2^U - F_1^U F_2^U\ ]\}$
$\cap \{[\ \sqrt{T_1^L\,T_2^L}\ , \sqrt{T_1^U\,T_2^U}\ ], [\ \sqrt{I_1^L\,I_2^L}\ , \sqrt{I_1^U\,I_2^U}\ ], [\ \sqrt{F_1^L\,F_2^L}\ , \sqrt{F_1^U\,F_2^U}\ ]\}$

$= \{[\ \min(T_1^L\,T_2^L\ , \sqrt{T_1^L\,T_2^L}), \min(\ T_1^U\,T_2^U, \sqrt{T_1^U\,T_2^U}\ )], [\max(I_1^L + I_2^L - I_2^L I_2^L\ , \sqrt{I_1^L\,I_2^L}),$
$\max(I_1^U + I_2^U - I_1^U I_2^U, \sqrt{I_1^U\,I_2^U})], [\max(F_1^L + F_2^L - F_2^L F_2^L\ , \sqrt{F_1^L\,F_2^L}),$
$\max(F_1^U + F_2^U - F_1^U F_2^U, \sqrt{F_1^U\,F_2^U})]\}$

$= \{[\ T_1^L\,T_2^L\ , T_1^U\,T_2^U\ ], [\ I_1^L + I_2^L - I_2^L I_2^L\ , I_1^U + I_2^U - I_1^U I_2^U\ ],$
$[\ F_1^L + F_2^L - F_2^L F_2^L\ , F_1^U + F_2^U - F_1^U F_2^U\ ]\}$

$= \tilde{n}_1 \otimes \tilde{n}_2$

This proves (ix)

(xiii) From definitions in 2.3, 2.5 and 3.1, we have

$(\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$

$= \{[T_1^L\,T_2^L\ ,\ T_1^U\,T_2^U)], [\ I_1^L + I_2^L - I_2^L I_2^L\ , I_1^U + I_2^U - I_1^U I_2^U],$

$[F_1^L + F_2^L - F_2^L F_2^L \ , F_1^U + F_2^U - F_1^U F_2^U \ ]\} \ \cap \{(\ [\frac{2\,T_1^L\,T_2^L}{T_1^L + T_2^L}, \frac{2\,T_1^U\,T_2^U}{T_1^U + T_2^U}], [\frac{2\,I_1^L\,I_2^L}{I_1^L + I_2^L}, \frac{2\,I_1^U\,I_2^U}{I_1^U + I_2^U}]\ ,$
$[\frac{2\,F_1^L\,F_2^L}{F_1^L + F_2^L}, \frac{2\,F_1^U\,F_2^U}{F_1^U + F_2^U}]\}$

$=\{ \ [\ \min (T_1^L\, T_2^L \ , \frac{2\,T_1^L\,T_2^L}{T_1^L + T_2^L}) \ , \min (T_1^U\, T_2^U \ , \frac{2\,T_1^U\,T_2^U}{T_1^U + T_2^U} \ ) \ ] \ , [\ \max (I_1^L + I_2^L - I_2^L I_2^L \ , \frac{2\,I_1^L\,I_2^L}{I_1^L + I_2^L}) \ ,$
$\max(I_1^U + I_2^U - I_1^U I_2^U \ , \frac{2\,I_1^U\,I_2^U}{I_1^U + I_2^U} \ ], [\ \max (F_1^L + F_2^L - F_2^L F_2^L \ \ , \frac{2\,F_1^L\,F_2^L}{F_1^L + F_2^L}) \ ,$
$\max(F_1^U + F_2^U - F_1^U F_2^U \ \ , \frac{2\,F_1^U\,F_2^U}{F_1^U + F_2^U} \ ]\}$

$=\{ \ [T_1^L\, T_2^L, T_1^U\, T_2^U], [\ I_1^L + I_2^L - I_2^L I_2^L \ , I_1^U + I_2^U - I_1^U I_2^U] \ ,$
$[\ F_1^L + F_2^L - F_2^L F_2^L \ \ , F_1^U + F_2^U - F_1^U F_2^U]\}$

$=\tilde{n}_1 \otimes \tilde{n}_2$

This proves (xiii). This proves the theorem.

Theorem 3.8 . For $\tilde{n}_1$ and $\tilde{n}_2 \in$ INSs(X), then following relations are valid:

(i) $(\tilde{n}_1 \# \tilde{n}_2) \, \$ \, (\tilde{n}_1 \# \tilde{n}_2) = \tilde{n}_1 \# \, \tilde{n}_2;$
(ii) $(\tilde{n}_1 \oplus \tilde{n}_2) \, \$ \, (\tilde{n}_1 \oplus \quad \tilde{n}_2) = \tilde{n}_1 \oplus \tilde{n}_2;$
(iii) $(\tilde{n}_1 \otimes \tilde{n}_2) \, \$ \, (\tilde{n}_1 \otimes \tilde{h}_2) = \tilde{n}_1 \otimes \tilde{n}_2;$
(iv) $(\tilde{n}_1 @ \tilde{n}_2) \, \$ \, (\tilde{n}_1 @ \tilde{n}_2) = \tilde{n}_1 @ \, \tilde{n}_2;$
(v) $(\tilde{n}_1 \# \tilde{n}_2) \, @ \, (\tilde{n}_1 \, \# \tilde{n}_2) = \tilde{n}_1 \, \# \tilde{n}_2;$
(vi) $(\tilde{n}_1 \oplus \tilde{n}_2) \, @ \, (\tilde{n}_1 \otimes \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2;$
(vii) $(\tilde{n}_1 \cup \tilde{n}_2) \, @ \, (\tilde{n}_1 \cap \tilde{n}_2) = \tilde{n}_1 @ \tilde{n}_2;$
(viii) $(\tilde{n}_1 \cup \tilde{n}_2) \, \$ \, (\tilde{n}_1 \cap \tilde{n}_2) = \tilde{n}_1 \$ \, \tilde{n}_2;$
(ix) $(\tilde{n}_1 \cup \tilde{n}_2) \, \# \, (\tilde{n}_1 \cap \tilde{n}_2) = \tilde{n}_1 \# \tilde{n}_2;$

Proof. The proofs of these results are the same as in the above proof

Theorem 3.9 For every two $\tilde{n}_1$ and $\tilde{n}_2 \in$ INSs(X), we have:

(i) $((\tilde{n}_1 \cup \tilde{n}_2) \oplus (\tilde{n}_1 \cap \tilde{n}_2)) \, @ \, ((\tilde{n}_1 \cup \tilde{n}_2) \otimes (\tilde{n}_1 \cap \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2;$
(ii) $((\tilde{n}_1 \cup \tilde{n}_2) \#(\tilde{n}_1 \cap \tilde{n}_2)) \, \$ \, ((\tilde{n}_1 \cup \tilde{n}_2) \, @ \, (\tilde{n}_1 \cap \tilde{n}_2)) = \tilde{n}_1 \$ \tilde{n}_2$
(iii) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2)) \, @ \, ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2))= \tilde{n}_1 @ \tilde{n}_2;$
(iv) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2)) \, @ \, ((\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 @ \tilde{n}_2))= \tilde{n}_1 @ \tilde{n}_2;$
(v) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \# \tilde{n}_2)) \, @ \, ((\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2;$
(vi) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \$ \tilde{n}_2)) \, @ \, ((\tilde{n}_1 \otimes \tilde{n}_2) \cap (\tilde{n}_1 \$ \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2;$
(vii) $((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 @ \tilde{n}_2)) \, @ \, ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \# \tilde{n}_2)) = \tilde{n}_1 \$ \tilde{n}_2.$

Proof . In the following, we prove (i) and (iii), other results can be proved analogously.

(i) From definitions in 2.4, 2.5 and 3.1, we have

$((\tilde{n}_1 \cup \tilde{n}_2) \oplus (\tilde{n}_1 \cap \tilde{n}_2)) \ @ \ ((\tilde{n}_1 \cup \tilde{n}_2) \otimes (\tilde{n}_1 \cap \tilde{n}_2))$

$\tilde{n}_1 =\{(\ [\ T_1^L \ , T_1^U] \ , \ \ [I_1^L \ , I_1^U] \ , \ \ [\ F_1^L \ , F_1^U] \ )\}$

$\tilde{n}_2$ = {([T_2^L , T_2^U] , [I_2^L , I_2^U] , [F_2^L , F_2^U])}

$\tilde{n}_3$= {([T_3^L , T_3^U] , [I_3^L , I_3^U] , [F_3^L , F_3^U])}

(($\tilde{n}_1 \cup \tilde{n}_2$) $\oplus$ ($\tilde{n}_1 \cap \tilde{n}_2$))

={ [max(T_1^L, T_2^L), max(T_1^U ,T_2^U)], [min (I_1^L, I_2^L),min(I_1^U, I_2^U)] ,
[min (F_1^L, F_2^L), min(F_1^U, F_2^U)] } $\oplus$ { [min(T_1^L, T_2^L), min(T_1^U ,T_2^U)],
[max (I_1^L, I_2^L), max(I_1^U, I_2^U)] , [max (F_1^L, F_2^L), max(F_1^U, F_2^U)] }.

={[max(T_1^L, T_2^L)+ min(T_1^L, T_2^L) - max(T_1^L, T_2^L) min(T_1^L, T_2^L), max(T_1^U ,T_2^U) +
min(T_1^U ,T_2^U) - max(T_1^U ,T_2^U min(T_1^U ,T_2^U)] ,[min(I_1^L, I_2^L) max (I_1^L, I_2^L),
min(I_1^U, I_2^U) max(I_1^U, I_2^U)], [min(F_1^L, F_2^L) max (F_1^L, F_2^L), min(F_1^U, F_2^U) max(F_1^U, F_2^U)]}.

($\tilde{n}_1 \cup \tilde{n}_2$) $\otimes$ ($\tilde{n}_1 \cap \tilde{n}_2$)

={ [max(T_1^L, T_2^L), max(T_1^U ,T_2^U)], [min (I_1^L, I_2^L),min(I_1^U, I_2^U)] , [min (F_1^L, F_2^L), min(F_1^U, F_2^U)
] } $\otimes$ { [min(T_1^L, T_2^L), min(T_1^U ,T_2^U)], [max (I_1^L, I_2^L), max(I_1^U, I_2^U)] ,
[max (F_1^L, F_2^L), max(F_1^U, F_2^U)] }.

= {[max(T_1^L, T_2^L) min(T_1^L, T_2^L) , max(T_1^U ,T_2^U) min(T_1^U ,T_2^U)] ,
[min (I_1^L, I_2^L) + max (I_1^L, I_2^L) - min (I_1^L, I_2^L) max (I_1^L, I_2^L), min (I_1^U, I_2^U) +
max (I_1^U, I_2^U) - min (I_1^U, I_2^U) max (I_1^U, I_2^U)] , [min (F_1^L, F_2^L) + max (F_1^L, F_2^L) - min (F_1^L, F_2^L)
max (F_1^L, F_2^L), min (F_1^U, F_2^U) + max (F_1^U, F_2^U) - min (F_1^U, FI_2^U) max (F_1^U, F_2^U)]}.

(($\tilde{n}_1 \cup \tilde{n}_2$) $\oplus$ ($\tilde{n}_1 \cap \tilde{n}_2$)) @ (($\tilde{n}_1 \cup \tilde{n}_2$) $\otimes$ ($\tilde{n}_1 \cap \tilde{n}_2$))

$$=\{[\frac{\max(T_1^L,T_2^L)+ \min(T_1^L,T_2^L)- \max(T_1^L,T_2^L)\min(T_1^L,T_2^L)+\max(T_1^L,T_2^L) \min(T_1^L,T_2^L)}{2},$$
$$\frac{\max(T_1^U ,T_2^U) + \min(T_1^U ,T_2^U) - \max(T_1^U ,T_2^U)\min(T_1^U ,T_2^U) +\max(T_1^U ,T_2^U) \min(T_1^U ,T_2^U)}{2}],$$
$$[\frac{[\min(I_1^L,I_2^L)\max (I_1^L,I_2^L) +\min (I_1^L,I_2^L) + \max (I_1^L,I_2^L) - \min (I_1^L,I_2^L) \max (I_1^L,I_2^L))}{2},$$
$$\frac{\min(I_1^U,I_2^U) \max(I_1^U,I_2^U)+\min (I_1^U,I_2^U) + \max (I_1^U,I_2^U) - \min (I_1^U,I_2^U) \max (I_1^U,I_2^U)}{2}]$$

$$[\frac{[\min(F_1^L,F_2^L)\max (F_1^L,F_2^L),+\min (F_1^L,F_2^L) + \max (F_1^L,F_2^L) - \min (F_1^L,F_2^L) \max (F_1^L,F_2^L))}{2},$$
$$\frac{\min(F_1^U,F_2^U) \max(F_1^U,F_2^U)+\min (F_1^U,F_2^U) + \max (F_1^U,F_2^U) - \min (F_1^U,F_2^U) \max (F_1^U,F_2^U)}{2}]\}$$
$$= [\frac{\max(T_1^L,T_2^L)+ \min(T_1^L,T_2^L)}{2}, \frac{\max(T_1^U ,T_2^U) + \min(T_1^U ,T_2^U)}{2}],$$
$$[\frac{\min (I_1^L,I_2^L) + \max (I_1^L,I_2^L)}{2}, \frac{\min (I_1^U,I_2^U) + \max (I_1^U,I_2^U)}{2}], [\frac{\min (F_1^L,F_2^L) + \max (F_1^L,F_2^L)}{2},$$
$$\frac{\min (I_1^U,I_2^U) + \max (I_1^U,I_2^U)}{2}]$$

$$=\{[\frac{T_1^L+ T_2^L}{2}, \frac{T_1^U+ T_2^U}{2}], [\frac{I_1^L+I_2^L}{2}, \frac{I_1^L+I_2^L}{2}], [\frac{F_1^L+F_2^L}{2}, \frac{F_1^L+F_2^L}{2}]\}$$

= $\tilde{n}_1 @ \tilde{n}_2$

This proves (i).

(iii) From definitions in 2.4, 2.5 and 3.1, we have

$((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))\ @\ ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2))$

$= \tilde{n}_1 @ \tilde{n}_2; (\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)$

$= \{([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U])\}$
$\cap \{[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\}$

$= \{ [\min(T_1^L + T_2^L - T_1^L T_2^L, T_1^L T_2^L), \min(T_1^U + T_2^U - T_1^U T_2^U, T_1^U T_2^U)],$
$[\max(I_1^L I_2^L, I_1^L + I_2^L - I_2^L I_2^L), \max(I_1^U I_2^U, I_1^U + I_2^U - I_1^U I_2^U)],$
$[\max(F_1^L F_2^L, F_1^L + F_2^L - F_2^L F_2^L), \max(F_1^U F_2^U, F_1^U + F_2^U - F_1^U F_2^U)]\}.$

$= \{[T_1^L T_2^L, T_1^U T_2^U], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]\}$

$(\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2)$

$= \{([T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U])\}$
$\cup \{[T_1^L T_2^L, T_1^U T_2^U)], [I_1^L + I_2^L - I_2^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U],$
$[F_1^L + F_2^L - F_2^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U]$

$= \{ [\max(T_1^L + T_2^L - T_1^L T_2^L, T_1^L T_2^L), \max(T_1^U + T_2^U - T_1^U T_2^U, T_1^U T_2^U)],$
$[\min(I_1^L I_2^L, I_1^L + I_2^L - I_2^L I_2^L), \min(I_1^U I_2^U, I_1^U + I_2^U - I_1^U I_2^U)],$
$[\min(F_1^L F_2^L, F_1^L + F_2^L - F_2^L F_2^L), \min(F_1^U F_2^U, F_1^U + F_2^U - F_1^U F_2^U)]\}$

$= \{ [T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U], [I_1^L I_2^L, I_1^U I_2^U], [F_1^L F_2^L, F_1^U F_2^U] \}$

$((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))\ @\ ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2))$
$= \{[\frac{T_1^L T_2^L + T_1^L + T_2^L - T_1^L T_2^L}{2}, \frac{T_1^U T_2^U + T_1^U + T_2^U - T_1^U T_2^U}{2}], [\frac{I_1^L + I_2^L - I_2^L I_2^L + I_1^L I_2^L}{2},$
$\frac{I_1^U + I_2^U - I_1^U I_2^U + I_1^U I_2^U}{2}], [\frac{F_1^L + F_2^L - F_2^L F_2^L + F_1^L F_2^L,}{2}, \frac{F_1^U + F_2^U - F_1^U F_2^U + F_1^U F_2^U}{2}]\}$

$= \{[\frac{T_1^L + T_2^L}{2}, \frac{T_1^U + T_2^U}{2}], [\frac{I_1^L + I_2^L}{2}, \frac{I_1^U + I_2^U}{2}], [\frac{F_1^L + F_2^L}{2}, \frac{F_1^U + F_2^U}{2}]$

Hence ,

$((\tilde{n}_1 \oplus \tilde{n}_2) \cup (\tilde{n}_1 \otimes \tilde{n}_2))\ @\ ((\tilde{n}_1 \oplus \tilde{n}_2) \cap (\tilde{n}_1 \otimes \tilde{n}_2)) = \tilde{n}_1 @ \tilde{n}_2$

This proves (iii).

4. Conclusion

In this paper we have defined three new operations on interval neutrosophic sets based on the arithmetic mean, geometrical mean, and respectively harmonic mean, which involve different defining functions. Some related results have been proved and bring out the characteristics of the interval neutrosophic sets.

Acknowledgements

The authors are very grateful to the anonymous referees for their insightful and constructive comments and suggestions, which have been very helpful in improving the paper.

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