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Novel Methods for Solving the Conformable Wave Equation

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Abstract − In this paper, a two-dimensional conformable fractional wave equation describing a circular membrane undergoing axisymmetric vibrations is formulated. It was found that the analytical solutions of the fractional wave equation using the conformable fractional formulation can be easily and efficiently obtained using separation of variables and double Laplace transform methods. These solutions are compared with the approximate solution obtained using the differential transform method for certain cases.

Keywords − *conformable derivative, wave equation, double Laplace transform, differential transform method*

1. Introduction

The fractional formulation of differential equations is an extension of the fractional calculus that was first introduced in 1695 when L'Hôpital and Leibniz discussed the extension of the integer order derivative to the derivative of order 1/2. Both Euler and Lacroix studied the fractional order derivative and defined the fractional derivative using the expression for the nth derivative of the power function [1]. Several physical and mechanical systems can be modeled more accurately using fractional derivative formulations due to the fact that many systems contain internal damping, which implies that it is impossible to derive equations describing the physical behavior of a non-conservative system using the classical energy based approach. The fractional derivative formulations can be successfully obtained in non-conservative systems by minimizing certain functionals with fractional derivative terms using some techniques from calculus of variations [2]. Several fractional formulations for derivatives and integrals such as Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, and Grünwald-Letnikov have been introduced with applications in science and engineering (refer to [1–6]).

While the classical definitions of fractional derivatives such as Riemann-Liouville and Caputo try to satisfy the fundamental properties of standard derivatives such as the derivatives of constant, product rule, quotient rule, and chain rule. None of the definitions are successful in their attempts other than the shared linear property between all the definitions of fractional derivatives [7]. Khalil et al. [8] put forward a new definition of fractional derivative named conformable fractional derivative as follows:

Definition 1.1. For $0 < \beta \leq 1$, given a function $f : [0, \infty) \to \Re$ such that for all $t > 0$ and $\beta \in (0, 1)$, the βth order conformable fractional derivative (CFD) of f, denoted by $G_\beta(f)(t)$, can be written as:

$$G_\beta(f)(t) = f^{(\beta)}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\beta}) - f(t)}{\varepsilon}. \quad (1)$$

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If f is β-differentiable in some $(0, b)$, $b > 0$, and the limit of $f^{(\beta)}(t)$ exists as t approaches 0^+, then by CFD definition:

$$f^{(\beta)}(0) = \lim_{t \to 0^+} f^{(\beta)}(t). \tag{2}$$

The CFD definition is an extension of the classical derivative that happens naturally and satisfies the properties of standard derivative. The conformable derivative of constant, the product rule, the quotient rule, and the chain rule all satisfy the standard formula of standard limit-based derivative [9]. Various conformable fractional forms have been introduced to many mathematical notions such as Norther's symmetry theorem and Action Principle for particles under frictional forces and have been shown to be much simpler than the ones with classical fractional derivative formulations such as Riemann-Liouville and Caputo [9]. For more applications of conformable fractional derivative, see also [10, 11]. G_β satisfies all the standard derivative properties in the following theorem [7, 8]:

Theorem 1.2. Assume that $0 < \beta \leq 1$, and f , h be β-differentiable at a point t, then:

(i) $G_\beta(mf + wh) = mG_\beta(f) + wG_\beta(h)$, for all $m, w \in \Re$.

(ii) $G_\beta(t^s) = st^{s-\beta}$, for all $s \in \Re$.

(iii) $G_\beta(fh) = fG_\beta(h) + hG_\beta(f)$.

(iv) $G_\beta(\frac{f}{h}) = \frac{hG_\beta(f) - fG_\beta(h)}{h^2}$.

(v) $G_\beta(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(vi) If f is a differentiable function, then $G_\beta(f)(t) = t^{1-\beta}\frac{df}{dt}$.

For more mathematical examples about each property in theorem 1, we refer to [7, 12]. Çenesiz and Kurt [10] discussed the possibility of applying the CFD definition for solving the two-dimensional and three-dimensional time fractional wave equation in rectangular domain. As a result, Çenesiz and Kurt [10] showed how the conformable fractional derivatives can easily and efficiently transform fractional differential equations into classical usual differential equations without the need for complicated methods to find the analytical solutions for partial fractional differential equations of higher dimensional systems. On the other hand, Tasbozanet et al. [11] discussed how to find the analytical traveling wave solutions in the sense of the conformable derivatives for nonlinear partial differential equations such as Nizhnik-Novikov-Veselov and Klein-Gordon equations by introducing a method consisting of a series of exponential functions, known as exp-function method, to study nonlinear evolution equations.

Recently, numerical and analytical solution methods to the conformable fractional differential equations are attracting attention from all over the world. Yavuz discussed in [13] some novel methods such as Adomian decomposition method and modified homotopy perturbation method for solving the initial boundary value problems in the sense of conformable fractional differentiation. Yavuz and Yaşkıran applied in [14] comfortable derivatives in modeling neuronal dynamics using methods of modified homotopy perturbation and reduced differential transform to solve the conformable fractional cable equation (CFCE). In addition, CFCE has also been solved in [15] using Adomian decomposition method and variational iteration method. In [16], conformable derivative has been successfully applied to solve the Black-Scholes equation of the European call option pricing models using Adomian decomposition method and modified homotopy perturbation method.

The CFD is a type of the local fractional derivative (LFD) [17]. The LFD has been successfully applied in modeling several applications in engineering such as the entropy (function of state) analysis of thermodynamic systems and the control theory of dynamic systems [18]. A new mathematical branch, known as fractal calculus, have been recently introduced in modeling various mathematical and engineering phenomena in hierarchical structures or porous media such as fractal kinetics [19], heat conduction in fractal medium [19], and the porous hairs of polar bear [20]. Research studies showed that there is a relation between the fractional order and the fractional dimension [19]. Several definitions of fractal derivatives have been proposed by researchers such as Chen's fractal derivative and Ji-Huan He's fractal derivative (HFD) [20]. However, some fractional derivatives lacks the physical and geometrical interpretation, therefore, the fractal calculus is very helpful in providing a physical interpretation for many fractional models in fractal media [19]. Both LFD and HFD have been applied extensively in science and engineering due to their accurate mathematical properties, physical insights, and geometrical interpretations [18–20]. The fractal derivative with fractal dimensions can be applied in modeling engineering problems and describing their discontinuous media [21] such as the applications of multi-scale fabrics and wool fibers by modeling their water permeation [20]. LFD and HFD have been defined in [18, 20] on a fractal space as follows:

Definition 1.3. For a fractal dimension, β, where $0 < \beta \leq 1$, given a set of non-differentiable functions with fractal dimension, say $C_\beta(a, b)$ such that for $\Phi(x) \in C_\beta(a, b)$, the βth order local fractional derivative (LFD) of $\Phi(x)$ at $x = x_0$, denoted by $D_x^{(\beta)}\Phi(x_0)$, can be written as:

$$D_x^{(\beta)}\Phi(x_0) = \Phi^{(\beta)}(x_0) = \frac{d^\beta \Phi(x)}{dx^\beta}|_{x=x_0} = \lim_{x \to x_0} \frac{\triangle^\beta(\Phi(x) - \Phi(x_0))}{(x - x_0)^\beta}, \tag{3}$$

where $\triangle^\beta(\Phi(x) - \Phi(x_0)) \cong \Gamma(1 + \beta)\triangle(\Phi(x) - \Phi(x_0))$.

Definition 1.4. Using figure 1 in [21], the fractal geometry describes the distance between two points, say x_a and x_b, in a discontinuous media i.e. porous medium such that M is supposed to be the smallest measure (thickness) in the given fractal media where any discontinuity less than this measure is neglected. Given a fractal dimension, say β, and constant, say ξ, the Ji-Huan He's fractal derivative (HFD) can be written [20, 21] as follows:

$$\frac{D\Phi(t)}{Dx^\beta} = \lim_{\Delta x \to M} \frac{\Phi(x_a) - \Phi(x_b)}{\xi M^\beta} = \Gamma(1 + \beta) \lim_{\Delta x \to M} \frac{\Phi(x_a) - \Phi(x_b)}{(x_a - x_b)^\beta}, \tag{4}$$

where $\Delta x = x_a - x_b$; and Δx tends only to M and it does not tend to 0. By using the fractal gradient [19], $\xi M^\beta = \frac{M^\beta}{\Gamma(1+\beta)}$ such that ξM^β is extremely small, but $\xi M^\beta > M$. For more applications using HFD in applied science and engineering, we refer to [19, 20, 22, 27].

In addition, He's fractional derivative (HFcD) has been applied for modeling several scientific phenomena (see [20, 23]). The physical and geometrical interpretations of the HFcD were discussed in [19, 27]. The following is the definition of HFcD [20, 23]:

Definition 1.5. Assume β to be the fractional dimension of the fractal medium, the He's fractional derivative (HFcD), denoted by $\frac{\partial^\beta}{\partial t^\beta}$, can be written as::

$$\frac{\partial^\beta \Psi}{\partial t^\beta} = \frac{1}{\Gamma(m - \beta)} \frac{d^m}{dt^m} \int_{t_0}^{t} (\xi - t)^{m-\beta-1}[\Psi_0(\xi) - \Psi(\xi)]d\xi, \tag{5}$$

where for a fractional-order problem in fractal media, the continuum partner of problem with the same initial and boundary conditions of the fractal partner has the same solution which is $\Psi_0(x, t)$ [20].

The conformable fractional derivative (CFD) is basically a generalized fractal derivative or q-derivative [24]. The q-derivative is very important in quantum calculus where the derivative is expressed using Leibniz's notation and the spacetime is discontinuous in quantum scales [27] (see also [25]). The generalized q-derivative (fractal derivative) using CFD definition 1 can be written [24] as follows:

Definition 1.6. Using definition 1, given a function $\Psi : [0, \infty) \to \Re$ such that for all $t > 0$ and $\beta \in (0, 1)$, and by assuming $q = 1 + \varepsilon t^{-\beta}$ where q tends to 1 and ε tends to 0, the generalized q-derivative (fractal derivative), denoted by $G_\beta(\Psi)(t)$, is written as:

$$G_\beta(\Psi)(t) = \Psi_q^{(\beta)}(t) = \lim_{q \to 1} \frac{\Psi(qt) - \Psi(t)}{qt^\beta - t^\beta} = \lim_{q \to 1} \frac{\Psi(qt) - \Psi(t)}{(q - 1)t^\beta}. \tag{6}$$

This generalized q-derivative coincides with definition 11 of q-derivative in [27].

Weberszpil and Chen in [26] showed that using the method of change of variables in part (vi) of theorem 1 to transform t to $1+\frac{x}{l_0}$, the CFD is simply a Hausdorff derivative (HD) which is valid for differential functions. HD is a kind of fractal derivatives [28] that has been applied in various engineering phenomena to describe the physical behaviors and complex mechanics [29]. HD extends the modeling approach used in the classical continuum mechanics to fractal materials using the Hausdorff calculus [28]. Some examples of HD applications in science and engineering are anomalous diffusion, non-Guassian distribution, creep and relaxation in fractal media, and viscosity [28, 29].

CFD is simply a usual Newton derivative multiplied by the term $t^{1-\beta}$ [17]. The term $t^{1-\beta}$ in the definition 1 is basically a type of fractional conformable function (FCF) (see definition 5 in [17]) [17]. CFD combines the properties of usual derivative with the properties of fractional derivatives [30]. Therefore, CFD can be applied to extend and generalize theorems from the classical calculus such as integration by parts, mean value theorem, power series expansion, and Rolle's theorem [30]. From definition 1, the function is differentiable in the sense of conformable derivatives which implies that the Taylor power series expansion (TPSE) exists for CFD, while the other forms of fractional derivatives where functions are not differentiable, TPSE do not exist, when there are infinitely differentiable functions at some points [30]. As a result, several researchers got motivated to explore the CFD and apply it in modeling phenomena in applied science and engineering [30].

The CFD can be physically interpreted as a modified standard limit-based derivative in magnitude and direction [31]. Therefore, CFD is a special case of the well-known directional derivative (DD). The directional derivative is a kind of G$\widehat{a}$teaux derivative (GD). Zhao and Luo proposed in [17] a new generalized form of CFD named the general conformable fractional derivative (GRCFD) by extending and generalizing the definition of G$\widehat{a}$teaux derivative (see definition 2 in [17]) into Extended G$\widehat{a}$teaux derivative (see definition 3 in [17]) and Linear Extended G$\widehat{a}$teaux derivative (see definition 4 in [17]) together with the definition of CFD. The physical and geometrical interpretations of CFD were also discussed in [17] using GRCFD as a special case of CFD. Using definitions 2, 3, 4, and 5 and using $\Re^+$ as a space in [17] and definition 1 in this paper, GRCFD can be defined [17] as follows:

Definition 1.7. For $0<\beta\leq 1$, given a fractional conformable function, say $\Omega(m,\beta)$, the general conformable fractional derivative (GRCFD) can be written as:

$$D_{\Omega}^{\beta}G_m=\lim_{\varepsilon\to 0}\frac{G(m+\varepsilon\Omega(m,\beta))-G(m)}{\varepsilon}. \tag{7}$$

For the definition of GRCFD of arbitrary order, we refer to definition 7 in [17]. Since CFD is a modified version of Newton derivative, then the geometrical and physical meaning of CFD can be interpreted [17] as the slope of tangent where the value of the given function in the definition of G$\widehat{a}$teaux derivative in [17] changes as m (independent variable) changes ε, and the magnitude and direction of the velocity of particle are obtained from the ratio limit of the changes in function value. In addition, the Extended G$\widehat{a}$teaux derivative can be interpreted [17] as a special case of velocity of particle where the magnitude and direction of this velocity depends only on $\Omega(m,\varepsilon,\beta)$, while the physical meaning of the Linear Extended G$\widehat{a}$teaux derivative is just a modified version of usual velocity (as a multiple of usual velocity of particle) in magnitude and direction where this derivative can be geometrically represented [17] as the gradient of a given function, G , projected onto $\Omega(m,\beta)$ (we also refer to [32] for new proposed multiplicative (geometric) forms of conformable fractional derivatives and integrals).

In addition, Guzmán et al. [33] proposed a new definition of local fractional derivative known as the non-conformable fractional derivative (NCFD) which is also extended naturally from the usual derivative of a function in a point. NCFD can be defined as [33]: Given a function $\psi:[0,+\infty)\to\Re$. The NCFD, denoted by N-derivative of ψ of order β can be written as: $N_1^{\beta}\psi(t)=\lim_{\varepsilon\to 0}\frac{\psi(t+\varepsilon e^{t^{-\beta}})-\psi(t)}{\varepsilon}$, for all $t>0$ and $\beta\in(0,1)$. If the function ψ is β-differentiable in some $(0,b)$, and $\lim_{t\to 0^+}N_1^{(\beta)}\psi(t)$ exists, then we have: $N_1^{(\beta)}\psi(0)=\lim_{t\to 0^+}N_1^{(\beta)}\psi(t)$. By comparing both CFD and NCFD, the angle of the tangent line to the curve in NCFD is not conserved, while in CFD is conserved [33]. For more new results about NCFD definition, we refer to [34]. Recently, several research studies with applications have been done using the definition NCFD such as the oscillatory character of liénard's system [36] (see also [35]), Laplace transform [37], and Hermite-Hadamard inequality [38].

Circular vibrating membrane problem (CVMP) has been applied in several applications in engineering such as industrial dynamic filtration modules and vibratory shear enhanced process (VSEP) for wastewater treatment systems [39,40]. CVMP has been also used extensively in investigating the transverse vibration using a vibrating membrane in a linearly transverse direction and analyzing the modes of transverse vibratory motion [41]. CVMP studies the vibration of membranes (vibration equation) which has many practical applications in industry and bioengineering [42]. Studying the two-dimensional analysis of wave mechanics and propagation in CVMP is very important in building the components of microphones, speakers, and some medical and industrial instruments [42].

In this paper, we formulate the two-dimensional time fractional wave partial differential equation in the sense of conformable fractional derivative for a circular membrane undergoing axisymmetric vibrations, and we solve it using the methods of separation of variables, double Laplace transform, and reduced differential transform. We compare and discuss all obtained approximate solutions using those methods and the error between analytical and approximate solutions.

In Section 2, the conformable fractional wave partial differential equation is solved using the methods of separation of variables, double Laplace transform, and reduced differential transform. In Section 3, we discuss the error between analytical and approximate solutions from section 2, and we compare all results with the classical analytical solution from [43,44]. In Section 4, the conclusion of this study is presented.

2. Conformable fractional wave equation

In this section, we investigate the conformable fractional mixed initial- boundary value problem of a circular membrane [44] of radius R and constant density ρ_o where the initial vibration conditions are radially symmetric or axisymmetric. Under such conditions, polar coordinates (r,θ) can be introduced such that $m(x,y,t) = M(r,t)$ where the displacement is independent of θ, and the initial displacement and velocity functions can be written as $q(r)$ and $n(r)$, respectively. The laplacian in polar coordinates can be written as:

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right). \tag{8}$$

Since the initial vibration conditions are axisymmetric, they are dependent only on the radial distance r from the center of the circle. Hence, $q(r)$ and $p(r)$ do not depend on θ, and instead they depend only on r, and from equation (3), the term $\frac{\partial^2}{\partial\theta^2} = 0$. Consequently, the governing system of equations for a circular membrane undergoing axisymmetric vibrations can be mathematically modeled by the following two-dimensional wave partial differential equation equation in the sense of CFD:

$$\frac{\partial^{2\beta} M}{\partial t^{2\beta}} = c_o^2\left(\frac{\partial^2 M}{\partial r^2} + \frac{1}{r}\frac{\partial M}{\partial r}\right). \tag{9}$$

$0 < r < R$; $t > 0$; $0 < \beta \leq 1$; and $c_o^2{=}(\tau_o/\rho_o)$ where τ_o is the assumed to be the constant value of the elastic membrane stretch-resisting restorative force per unit length or surface tension. Equation (9) is subjected to the following boundary and initial conditions:

$$M(R,t) = 0; \text{ and } M(r,t) \text{ bounded as } r \to 0 \text{ for } t > 0. \tag{10}$$

$$M(r,0) = q(r); \text{ and } \frac{\partial^\beta M}{\partial t^\beta}(r,0) = p(r); \text{ for } 0 < r < R \text{ and } 0 < \beta \leq 1. \tag{11}$$

The problem is divided into two main parts; analytical solution part and approximate solution part:

2.1. The analytical solution by the separation of variables method

By using the separation of variables method, we let $M(r,t) = V(r)G(t)$ to be the solution form of the governing conformable fractional wave partial differential equation and boundary conditions. The following is obtained from substituting the assumed solution form in equation (9):

$$\frac{d^{2\beta}G(t)}{dt^{2\beta}}V(r) = c_o^2\left(\frac{d^2V(r)}{dr^2}G(t) + \frac{1}{r}\frac{dV(r)}{dr}G(t)\right). \tag{12}$$

Dividing both sides of equation (12) by c_o^2, left hand side term of equation (12) by $G(t)$, and the two terms of the right hand side of equation (12) by $V(r)$, we obtain:

$$\frac{d^{2\beta}G(t)}{dt^{2\beta}}\frac{1}{G(t)c_o^2} = c_o^2\left(\frac{d^2V(r)}{dr^2}\frac{1}{V(r)} + \frac{1}{r}\frac{dV(r)}{dr}\frac{1}{V(r)}\right) \equiv -\lambda^2. \tag{13}$$

where λ is the separation constant. As a result, the following two equations are obtained:

$$\frac{d^{2\beta}G(t)}{dt^{2\beta}} + c_o^2 G(t)\lambda^2 = 0. \tag{14}$$

$$\frac{d^2V(r)}{dr^2} + \frac{1}{r}\frac{dV(r)}{dr} + \lambda^2 V(r) = 0. \tag{15}$$

From equation (14), it is necessary to introduce the sequential CFD from [45] as follows:

Definition 2.1. For $0 < \beta < 1$, and $n \in \mathbb{Z}^+$, given a function $f : [0,\infty) \to \Re$, the nth order of sequential CFD can be generally written as:
${}^{(n)}G_\beta f(t) = G_\beta G_\beta G_\beta ... G_\beta f(t)$. Let's consider the $f : [0,\infty) \to \Re$ to be a second continuously differentiable function [45] and $\beta \in (0, 0.5]$, then the 2^{nd} order of sequential CFD is written as:

$$^{(2)}G_\beta f(t) = G_\beta G_\beta f(t) = \begin{cases} (1-\beta)t^{1-2\beta}f^{(1)}(t) + t^{2-2\beta}f^{(2)}(t) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases} \tag{16}$$

By using the sequential CFD definition and property (vi) from theorem (1), equation (14) can be re-written as:

$$(1-\beta)t^{1-2\beta}G^{(1)}(t) + t^{2-2\beta}G^{(2)}(t) + c_o^2 G(t)\lambda^2 = 0. \tag{17}$$

Multiplying both sides equation (15) by r^2 to make calculations simple, we obtain:

$$r^2\frac{d^2V(r)}{dr^2} + r\frac{dV(r)}{dr} + r^2\lambda^2 V(r) = 0. \tag{18}$$

Let's now introduce the change of variables [44]: $s = \lambda r$ for $V(r) = \psi(s)$ such that for $\frac{dV(r)}{dr}$, it is transformed into the following:

$$\frac{dV(r)}{dr} = \frac{d\psi(s)}{dr}(s) = \frac{d\psi(s)}{ds}\frac{ds}{dr}(s) = \lambda\frac{d\psi(s)}{ds}. \tag{19}$$

Similarly, for $\frac{d^2V(r)}{dr^2}$, it is transformed into the following:

$$\frac{d^2V(r)}{dr^2} = \frac{d}{dr}\left(\lambda\frac{d\psi(s)}{ds}(s)\right) = \lambda^2\frac{d^2\psi}{ds^2}(s). \tag{20}$$

Substituting $r = (\frac{s}{\lambda})$ and results from (19) and (20) in equation (18), we obtain the following equation:

$$s^2\frac{d^2\psi(s)}{ds^2} + s\frac{d\psi(s)}{ds} + s^2\psi(s) = 0; \text{ for } 0 < s < \lambda R. \tag{21}$$

From the boundary condition in (10), $\psi(s)$ in equation (21) is also bounded as $s \to 0$, and $\psi(\lambda R) = 0$. By using the results from the eigenvalue problem involving the Bessel function of the first kind of order zero in [43, 44], we have: $V(R) = 0 \to J_o(\lambda R) = 0$ where λR is the root of Bessel function J_o, and $V(r) = J_o(\lambda r)$. Hence, it can be concluded that for $n \in \mathbb{Z}^+$, $\lambda_n = \frac{\xi_n}{R}$, and $V_n(r) = J_o(\frac{\xi_n r}{R})$ is the corresponding solution to equation (10) where J_o has infinitely many positive zeros such that $\xi_1 < \xi_2 < \xi_3 < ... < \xi_n$ where ξ_n is the n^{th} positive zero of the Bessel function J_o.

For equation (17), the WolframAlpha computational intelligence solver is used to obtain the following solution:

$$G_n(t) = E_n cos\left(\frac{c_o\lambda_n t^\beta}{\beta}\right) + K_n sin\left(\frac{c_o\lambda_n t^\beta}{\beta}\right); \text{for } n \in \mathbb{Z}^+. \tag{22}$$

By substituting $\lambda_n = \frac{\xi_n}{R}$ in equation (22), the solution can be re-written as:

$$G_n(t) = E_n cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right) + K_n sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right); \text{for } n \in \mathbb{Z}^+. \tag{23}$$

By using the superposition principle, the general solution for the conformable fractional mixed initial-boundary value can be written as a linear combination of both $V_n(r)$ and $G_n(t)$:

$$\begin{aligned} M(r,t) &= \sum_{n=1}^{\infty} M_n(r,t) = \sum_{n=1}^{\infty} V_n(r)G_n(t) = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) \\ &\times \left[E_n cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right) + K_n sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right]; \text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{24}$$

To find the coefficients, E_n and K_n, from equation (24) so the general solution satisfies the initial conditions in (11), the first condition $M(r,0) = q(r)$ is substituted in equation (24) as follows:

$$\begin{aligned} M(r,0) = q(r) &= \sum_{n=1}^{\infty} M_n(r,0) = \sum_{n=1}^{\infty} V_n(r)G_n(0) = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) \\ &\times \left[E_n cos\left(\frac{c_o\xi_n (0)^\beta}{\beta R}\right) + K_n sin\left(\frac{c_o\xi_n (0)^\beta}{\beta R}\right)\right] \\ = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) &\times \left[E_n cos\,(0) + K_n sin\,(0)\right] = \sum_{n=1}^{\infty} J_o\left(\frac{\xi_n r}{R}\right) E_n; \text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{25}$$

For the second initial condition, $\frac{\partial M}{\partial t}(r,0) = p(r)$, in (11), we first find $\frac{\partial^\beta M}{\partial t^\beta}(r,t)$ from equation (24) using the two examples from [7] where $G_\beta(sin(\frac{t^\beta}{\beta})) = cos(\frac{t^\beta}{\beta})$ and $G_\beta(cos(\frac{t^\beta}{\beta})) = -sin(\frac{t^\beta}{\beta})$, and our previous conclusion $\lambda_n = \frac{\xi_n}{R}$ as follows:

$$\begin{aligned} \frac{\partial^\beta M}{\partial t^\beta}(r,t) = p(r) &= \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)\left(\frac{c_o\xi_n}{R}\right) \\ \times \left[-E_n sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right) + K_n cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right]&; \\ &\text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{26}$$

We now substitute $\frac{\partial M}{\partial t}(r,0) = p(r)$ in equation (26) as follows:

$$\begin{aligned} \frac{\partial^\beta M}{\partial t^\beta}(r,0) = p(r) &= \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)\left(\frac{c_o\xi_n}{R}\right) \\ \times &\left[-E_n sin\left(\frac{c_o\xi_n 0^\beta}{\beta R}\right) + K_n cos\left(\frac{c_o\xi_n 0^\beta}{\beta R}\right)\right] \\ = \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)&\left(\frac{c_o\xi_n}{R}\right)\left[-E_n sin\,(0) + K_n cos\,(0)\right] \\ &= \sum_{n=1}^{\infty} J_0\left(\frac{\xi_n r}{R}\right)\left(\frac{c_o\xi_n}{R}\right) K_n; \text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{27}$$

Using the orthogonality property of Bessel function $J_0(\frac{\xi_n r}{R})$ and representing the normalization constant in terms of $J_1(\xi_n)$, we obtain the following:

$$E_n = \frac{\left\langle q(r), J_0\left(\frac{\xi_n r}{R}\right)\right\rangle_r}{\left\| J_0\left(\frac{\xi_n r}{R}\right)\right\|_r^2} = \frac{2}{R^2 J_1^2(\xi_n)} \int_0^R r q(r) J_0\left(\frac{\xi_n r}{R}\right)\, \mathrm{d}r; \text{for } n \in \mathbb{Z}^+. \tag{28}$$

$$K_n = \left(\frac{R}{c_o\xi_n}\right) \frac{\left\langle p(r), J_0\left(\frac{\xi_n r}{R}\right)\right\rangle_r}{\left\|J_0\left(\frac{\xi_n r}{R}\right)\right\|_r^2} = \frac{2}{Rc_o\xi_n J_1^2(\xi_n)} \int_0^R rp(r) J_0\left(\frac{\xi_n r}{R}\right) \mathrm{d}r; \quad \text{for } n \in \mathbb{Z}^+. \tag{29}$$

By substituting the results from (28) and (29) in equation (24), the most general solution for the conformable fractional mixed initial-boundary value problem emerging from the separation of variables method can be written as follows:

$$\begin{aligned} M(r,t) = &\sum_{n=1}^{\infty}\left[\left(\frac{2}{R^2 J_1^2(\xi_n)}\int_0^R q(r) J_0\left(\frac{\xi_n r}{R}\right) r\,\mathrm{d}r\right) cos\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right] J_0\left(\frac{\xi_n r}{R}\right) \\ &+\sum_{n=1}^{\infty}\left[\left(\frac{2}{Rc_o\xi_n J_1^2(\xi_n)}\int_0^R p(r) J_0\left(\frac{\xi_n r}{R}\right) r\,\mathrm{d}r\right) sin\left(\frac{c_o\xi_n t^\beta}{\beta R}\right)\right] J_0\left(\frac{\xi_n r}{R}\right); \\ &\text{for } n \in \mathbb{Z}^+. \end{aligned} \tag{30}$$

2.2. The analytical solution by the conformable fractional double Laplace transform method

The classical Laplace transform method for a function of single variable has been used extensively in solving ordinary differential equations and partial differential equations. Double Laplace transform and other multiple Laplace transformations were introduced by Estrin and Higgins in [46] to solve partial differential equations. Double Laplace transform (DLT) has been rarely introduced or not at all for certain cases in the literature for solving partial differential equations [47]. Introducing double Laplace transform to solve the fractional differential equations is an open math problem [48]. Eltayeb and Kılıçman in [49] used the DLT and Sumudu transform methods to solve non-fractional one-dimensional wave equation with variable coefficients (see also [50, 51]). There are some recent research studies on solving fractional differential equations such as heat and telegraph equations in the sense of Caputo derivatives [48, 52].

To define the conformable fractional double Laplace transform, let's first define the conformable fractional integral (CFI) [31] as follows:

Definition 2.2. For $0 < \beta \leq 1$, given a function $f : [0, \infty) \to \Re$ such that for all $t \geq 0$, the βth order conformable fractional integral (CFI) of f from 0 to t can be written as:

$$I_\beta(f)(t) = \int_0^t f(\psi) d_\beta \psi = \int_0^t f(\psi)\psi^{\beta-1} d\psi. \tag{31}$$

If $\beta = 1$, then $I_\beta(f)(t) = I_{\beta=1}(t^{\beta-1}f)(t)$ which is the classical improper Riemann integral of a function $f(t)$. For $0 < \beta \leq 1$, given a continuous function f on $(0, \infty)$, then $G_\beta(f)(t)\,[I_\beta(f)(t)] = f(t)$.

Let's now define the conformable fractional double Laplace transform (CFDLT) as follows:

Definition 2.3. For $0 < \beta \leq 1$, given a function $M(r, t) : [0, \infty) \to \Re$ such that for all $r, t > 0$, the βth order conformable fractional double Laplace transform (CFDLT) of $M(r, t)$, denoted by $\ell_\beta^{rt}[M(r, t)]$, starting from 0 can be written as:

$$\begin{aligned} \ell_\beta^{rt}[M(r,t)] = \ell_\beta^{r}\ell_\beta^{t}[M(r,t)] = \underline{\mathrm{M}}_\beta^{rt}(s_a, s_b) &= \int_0^\infty e^{-s_a\frac{r^\beta}{\beta}} \int_0^\infty e^{-s_b\frac{t^\beta}{\beta}} M(r,t)\, d_\beta t\, d_\beta r \\ &= \int_0^\infty \int_0^\infty e^{-(s_a\frac{r^\beta}{\beta}+s_b\frac{t^\beta}{\beta})} M(r,t)\, d_\beta r\, d_\beta t \\ &= \int_0^\infty \int_0^\infty e^{-(s_a\frac{r^\beta}{\beta}+s_b\frac{t^\beta}{\beta})} M(r,t) r^{\beta-1} t^{\beta-1}\, dr\, dt, \end{aligned} \tag{32}$$

where $s_a, s_b \in \mathbb{C}$. The above definition is true provided that the above integral exists. Previously, it is assumed that $M(r,t) = V(r)G(t)$. By using definition (9), the CFDLT can be written [52]:

$$\ell_\beta^{rt}[V(r)G(t)] = \ell_\beta^r \ell_\beta^t [V(r)G(t)] = \underset{\sim}{\mathrm{V}}_\beta(s_a)\underset{\sim}{\mathrm{G}}_\beta(s_b) = \ell_\beta^r[V(r)]\ell_\beta^t[G(t)]. \tag{33}$$

Let's show the CFDLT of the second-order conformable fractional partial derivative (CFPD) with respect to t [53] as follows:

$$\begin{aligned} \ell_\beta^{rt}\left[\frac{\partial^{2\beta}}{\partial t^{2\beta}} M(r,t)\right] &= \int_0^\infty \int_0^\infty e^{-s_a \frac{r^\beta}{\beta}} e^{-s_b \frac{t^\beta}{\beta}} \frac{\partial^{2\beta} M}{\partial t^{2\beta}}(r,t)\, d_\beta r\, d_\beta t \\ &= \int_0^\infty e^{-s_a \frac{r^\beta}{\beta}} \int_0^\infty \left\{ e^{-s_b \frac{t^\beta}{\beta}} \frac{\partial^{2\beta} M}{\partial t^{2\beta}}(r,t)\, d_\beta t \right\} d_\beta r. \end{aligned} \tag{34}$$

To find the above inner integral, let's use the theorem 3.1 of conformable fractional integration by parts and lemma 2.8 in [2] in addition to definition (7) to obtain the following:

$$\begin{aligned} &\ell_\beta^{rt}\left[\frac{\partial^{2\beta}}{\partial t^{2\beta}} M(r,t)\right] \\ &= \int_0^\infty e^{-s_a \frac{r^\beta}{\beta}} \left\{ e^{-s_b \frac{t^\beta}{\beta}} M(r,t)|_{t=0}^\infty - \int_0^\infty \left(\frac{\partial^{2\beta}}{\partial t^{2\beta}} e^{-s_b \frac{t^\beta}{\beta}} \right) M(r,t)\, d_\beta t \right\} d_\beta r \\ &= \int_0^\infty e^{-s_a \frac{r^\beta}{\beta}} M(r,0) r^{\beta-1} dr \\ &+ \int_0^\infty \int_0^\infty \left[s_b^2 \frac{e^{-s_b \frac{t^\beta}{\beta}}}{t^{\beta-1}} - s_b \frac{e^{-s_b \frac{t^\beta}{\beta}}}{t^\beta}(1-\beta) \right] e^{-s_b \frac{t^\beta}{\beta}} e^{-s_a \frac{r^\beta}{\beta}} r^{\beta-1}\, dt\, dr \\ &= s_b^{2\beta} \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b) - s_b^{2\beta-1} \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,0) - s_b^{2\beta-2} (\underset{\sim}{\mathrm{M}}_\beta^{rt})_t(s_a,0). \end{aligned} \tag{35}$$

As a result, The CFDLT of the first-order conformable fractional partial derivative (CFPD) with respect to t can be similarly written as:

$$\ell_\beta^{rt}\left[\frac{\partial^\beta}{\partial t^\beta} M(r,t)\right] = s_b^\beta \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b) - s_b^{\beta-1} \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,0). \tag{36}$$

The CFDLT of the first-order conformable fractional partial derivative (CFPD) with respect to t can be also generally written as:

$$\ell_\beta^{rt}\left[\frac{\partial^\beta}{\partial t^\beta} M(r,t)\right] = s_b^\beta \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b) - \sum_{\gamma=0}^{\zeta-1} s_b^{\beta-1-\gamma} \ell_r \left[\frac{\partial^\gamma M(r,0)}{\partial t^\gamma}\right]. \tag{37}$$

The double Laplace transform in (37) coincides with the general form of the double Laplace transform of the partial fractional derivatives in the sense of Caputo derivatives in [48, 52]. The complex double integral formula in [47, 52] can be used to write the inverse conformable fractional double Laplace transform, denoted by $(\ell_\beta^{rt})^{-1}[\underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b)]$, as follows:

Definition 2.4. For $0 < \beta \leq 1$, given an analytic function $\underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b)$ for all $s_a, s_b \in \mathbb{C}$ such that both s_a and s_b are defined [52] by $Re\{s_a \geq \varrho\}$ and $Re\{s_b \geq \varsigma\}$, where $\varrho, \varsigma \in \Re$, the inverse conformable fractional double Laplace transform (ICFDLT) can be written as follows:

$$\begin{aligned} &(\ell_\beta^{rt})^{-1}[\underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b)] = (\ell_\beta^r)^{-1}\ell_\beta^t)^{-1}[\underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b)] = \\ &M(r,t) = \frac{1}{2\pi i}\int_{\varrho-i\infty}^{\varrho+i\infty} e^{s_a r} ds_a \frac{1}{2\pi i}\int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{s_b t} \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b) ds_b \\ &= \frac{-1}{4\pi^2}\int_{\varrho-i\infty}^{\varrho+i\infty}\int_{\varsigma-i\infty}^{\varsigma+i\infty} e^{s_a r} e^{s_b t} \underset{\sim}{\mathrm{M}}_\beta^{rt}(s_a,s_b)\, ds_a\, ds_b. \end{aligned} \tag{38}$$

Let's prove the existence and uniqueness of CFDLT in the following theorem:

Theorem 2.5. For $0 < \beta \leq 1$, given a continuous exponential-order function $M(r,t) : [0,\infty) \to \Re$ such that for some $\varrho, \varsigma \in \Re$ and $s_a, s_b \in \mathbb{C}$ where $Re\{s_a > \varrho\}$ and $Re\{s_b > \varsigma\}$, then there exists a conformable fractional double Laplace transform of $M(r,t)$, denoted by $\underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b)$, for both s_a and s_b.

Proof. Since $M(r,t)$ is a continuous exponential-order function $M(r,t) : [0,\infty) \to \Re$ such that for some $\varrho, \varsigma \in \Re$ and $s_a, s_b \in \mathbb{C}$ on the interval $[0,\infty) = \{r, t | 0 \leq r, t < \infty\}$, then $\exists L \in \mathbb{Z}_+$ such that $\forall s_a > S_a$ and $s_b > S_b$ [47,48] as follows:

$$|M(r,t)| \leq L\, e^{\varrho \frac{r^\beta}{\beta} + \varsigma \frac{t^\beta}{\beta}}, \tag{39}$$

$$\text{Examine: } \sup_{r,t>0} \left| \frac{M(r,t)}{e^{\omega \frac{r^\beta}{\beta} + \mu \frac{t^\beta}{\beta}}} \right| < 0, \text{then we have the following:}$$

$$\begin{aligned}
&\lim_{(r,t)\to\infty} e^{-\omega \frac{r^\beta}{\beta} - \mu \frac{t^\beta}{\beta}} |M(r,t)| = L\, e^{-(\omega-\varrho) r^{\frac{r^\beta}{\beta}}} e^{-(\mu-\varsigma) r^{\frac{t^\beta}{\beta}}} = 0;\ \forall\, \omega > \varrho;\ \mu > \varsigma \\
&\text{Similarly,} |\underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b)| = \left| \int_0^\infty \int_0^\infty e^{-(s_a \frac{r^\beta}{\beta} + s_b \frac{t^\beta}{\beta})} M(r,t)\, d_\beta r\, d_\beta t \right| \\
&= \left| \int_0^\infty \int_0^\infty e^{-(s_a \frac{r^\beta}{\beta} + s_b \frac{t^\beta}{\beta})} M(r,t) r^{\beta-1} t^{\beta-1}\, dr\, dt \right| \\
&\leq L \int_0^\infty \int_0^\infty e^{-((s_a-\varrho)\frac{r^\beta}{\beta} + (s_b-\varsigma)\frac{t^\beta}{\beta})} M(r,t) r^{\beta-1} t^{\beta-1}\, dr\, dt \\
&= \int_0^\infty e^{-(s_a-\varrho)\frac{r^\beta}{\beta}} r^{\beta-1}\, dr \int_0^\infty e^{-(s_b-\varsigma)\frac{t^\beta}{\beta}} t^{\beta-1}\, dt \\
&= \frac{L}{(s_a-\varrho)(s_b-\varsigma)};\ \forall\, Re\{s_a > \varrho\},\ Re\{s_b > \varsigma\}.
\end{aligned} \tag{40}$$

Since the $\lim_{(s_a,s_b)\to\infty} |\underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b)| = \lim_{(s_a,s_b)\to\infty} \underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b) = 0$ [47], then the conformable fractional double Laplace transform (CFLT) of $M(r,t)$ exists and can be written as (32) $\forall\, s_a > \varrho,\ s_b > \varsigma$. □

Numerical Experiment 1:

By using the above definitions and theorems of the CFDLT, let's solve the mixed initial-boundary value problem (equation (9)) subject to the following boundary and initial conditions:

$$M(R,t) = 0; \text{ and } M(r,t) \text{ bounded as } r \to 0 \text{ for } t > 0. \tag{41}$$

$$M(r,0) = 0; \text{ and } \frac{\partial^\beta M}{\partial t^\beta}(r,0) = cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right); \text{ for } 0 \leq r < R \text{ and } 0 < \beta \leq 1. \tag{42}$$

Let's apply the CFDLT method to equation (9), the following is obtained:

$$\begin{aligned}
&s_b^{2\beta} \underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b) - s_b^{2\beta-1} \underline{\mathrm{M}}^{rt}_{\beta}(s_a, 0) - s_b^{2\beta-2} (\underline{\mathrm{M}}^{rt}_{\beta})_t(s_a, 0) \\
&= c_o^2 \left(\frac{\partial^2 \underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b)}{\partial r^2} + \frac{1}{r} \frac{\partial \underline{\mathrm{M}}^{rt}_{\beta}(s_a, s_b)}{\partial r} \right).
\end{aligned} \tag{43}$$

Similarly, let's apply the conformable fractional single Laplace transform of the initial conditions in (42):

$$\underline{\mathrm{M}}^{rt}_{\beta}(s_a, 0) = 0; \text{ and } (\underline{\mathrm{M}}^{rt}_{\beta})_t(s_a, 0) = \left(\frac{s_b}{s_b^2 + \left(\frac{1}{\beta}\right)^2} + \left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2 + \left(\frac{1}{\beta}\right)^2} \right) \right). \tag{44}$$

By substituting the initial conditions of (44) in equation (43), we obtain:

$$s_b^{2\beta}\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)-s_b^{2\beta-1}(0)-s_b^{2\beta-2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right) = c_o^2\left(\frac{\partial^2\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)}{\partial r^2}+\frac{1}{r}\frac{\partial\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)}{\partial r}\right). \tag{45}$$

Let's simplify (45) to obtain the following:

$$s_b^{2\beta}\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)-\frac{s_b^{2\beta}}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right) = c_o^2\left(\frac{\partial^2\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)}{\partial r^2}+\frac{1}{r}\frac{\partial\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)}{\partial r}\right). \tag{46}$$

By taking $s_b^{2\beta}$ as a common factor on the left side of (46) and dividing both sides by c_o^2, we obtain the following:

$$\frac{s_b^{2\beta}}{c_o^2}\left(\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)-\frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right) = \left(\frac{\partial^2\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)}{\partial r^2}+\frac{1}{r}\frac{\partial\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)}{\partial r}\right). \tag{47}$$

Assume that $\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)=\underline{\mathrm{M}}_{\beta}^{rt}(s_a,s_b)-\frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right).$ (48)

By applying the assumption in (48) on (47) and combine the left-hand side term with the right-hand side terms together, the following is obtained:

$$\frac{\partial^2\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r^2}+\frac{1}{r}\frac{\partial\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r}-\frac{s_b^{2\beta}}{c_o^2}\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)=0. \tag{49}$$

Multiplying all terms in (49) on both sides by r^2, we obtain:

$$r^2\frac{\partial^2\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r^2}+r\frac{\partial\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)}{\partial r}-\frac{s_b^{2\beta}}{c_o^2}r^2\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)=0. \tag{50}$$

The WolframAlpha computational intelligence solver is used to obtain the following solution of (50):

$$\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)=\psi\, J_0\left(\frac{is_b^{\beta}r}{c_o}\right)+\varphi\, Y_0\left(\frac{-is_b^{\beta}r}{c_o}\right); \tag{51}$$

where $J_0\left(\frac{is_b^{\beta}r}{c_o}\right)$ and $Y_0\left(\frac{-is_b^{\beta}r}{c_o}\right)$.

are the zeroth order Bessel functions of $1st$ and $2nd$ kind, respectively.

From the boundary conditions in (42), $M(R,t)=0$ and $M(r,t)$ remains bounded as $r\to 0$ for $t>0$ which means that $\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)$ has a finite value. As a result, $\underline{\mathrm{M}}_{\beta*}^{rt}(R,s_b)$ has a finite value, and from the physical point of view for wave equation solution in [43], φ is set to be zero so that the whole term, $Y_0\left(\frac{-is_b^{\beta}r}{c_o}\right)$, is terminated. The solution of (51) becomes as follows:

$$\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)=\psi\, J_0\left(\frac{is_b^{\beta}r}{c_o}\right); \quad \text{where } J_0\left(\frac{is_b^{\beta}r}{c_o}\right) \text{ is the zeroth order Bessel functions of } 1st \text{ kind.} \tag{52}$$

Similarly, since $M(R,t)=0$ from (42), then $\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)=0$. let's substitute $\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)=0$ and (52) in equation (48) to obtain the following:

$$\underline{\mathrm{M}}_{\beta*}^{rt}(R,s_b)=\psi\, J_0\left(\frac{is_b^{\beta}R}{c_o}\right)=\underline{\mathrm{M}}_{\beta}^{rt}(R,s_b)-\frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right). \tag{53}$$

$$\underline{\mathrm{M}}_{\beta*}^{rt}(R,s_b)=\psi\, J_0\left(\frac{is_b^{\beta}R}{c_o}\right)=-\frac{1}{s_b^2}\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right). \tag{54}$$

As a result, ψ can be written as follows:

$$\begin{aligned}\psi &= \left(\frac{-\frac{1}{s_b^2}\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right)\\ &=-\left(\frac{\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{s_b^2 J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right).\end{aligned} \tag{55}$$

By substituting (55) in equation (52), the following is obtained:

$$\underline{\mathrm{M}}_{\beta*}^{rt}(s_a,s_b)=-\left(\frac{\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{s_b^2 J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right) J_0\left(\frac{is_b^{\beta}r}{c_o}\right). \tag{56}$$

By substituting (56) in equation (48), we obtain the following:

$$\underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b) = -\left(\frac{\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right)}{s_b^2 J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\right) J_0\left(\frac{is_b^{\beta}r}{c_o}\right) \tag{57}$$
$$+\frac{\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)}{s_b^2}.$$

After simplifications, we obtain:

$$\underline{\mathrm{M}}_{\beta}^{rt}(s_a, s_b) = \frac{J_0\left(\frac{is_b^{\beta}R}{c_o}\right) - J_0\left(\frac{is_b^{\beta}r}{c_o}\right)}{s_b^2 J_0\left(\frac{is_b^{\beta}R}{c_o}\right)}\left(\left(\left(\frac{s_b}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)+\left(\frac{\left(\frac{1}{\beta}\right)}{s_b^2+\left(\frac{1}{\beta}\right)^2}\right)\right)\right). \tag{58}$$

By using the residue theorem of the complex inversion formula and the solution in [54] with a few mathematical simplifications, it is easy to obtain the ICFDLT of equation (58) which is the following approximate analytical solution for equation (9) subject to the boundary and initial conditions in (41) and (42), respectively, using the method of CFDLT:

$$M(r,t) = \sum_{\xi=1}^{\infty}\frac{iJ_0\left(\lambda_{\xi}\frac{r}{R}\right)\left\{cos\left(-\lambda_{\xi}\frac{c_o t^{\beta}}{R\beta}\right)+i\,sin\left(-\lambda_{\xi}\frac{c_o t^{\beta}}{R\beta}\right)\right\}}{\lambda_{\xi}^2\left(\frac{c_o}{R}\right)J_1(\lambda_{\xi})}; \tag{59}$$
$$\text{where } \frac{i\,R\,s_{b_{\xi}}^{\beta}}{c_o} = \lambda_{\xi};\ \frac{i\,r\,s_{b_{\xi}}^{\beta}}{c_o} = \lambda_{\xi}\left(\frac{r}{R}\right) \text{ [54]; and } 0<\beta\leq 1.$$

Figures (1), (2), and (3) show the numerical simulation of the approximate analytical solution (59) for $\beta = 1; 0.75; 0.50$, respectively.

2.3. The approximate analytical solution by the conformable reduced differential transform method

To show the efficiency of CFD, let's obtain an approximate analytical solution to the two-dimensional conformable fractional wave equation. A fractional differential equation (FDE) approximate method is the conformable fractional differential transform method (CFDTM) in the sense of CFD [55]. The differential transform method (DTM) was introduced by Zhou [57] for solving ordinary differential equations by formulating Taylor series [55, 57]. With the introduction of fractional differential equations (FDEs), the fractional differential transform method (FDTM) was developed by Arikoglu and Ozkol in [56] to solve FDEs by formulating power series. Similarly, CFDTM can be used to solve CFD by formulating conformable fractional power series, and can be defined as [55]:

Definition 2.6. For some $0 < \beta \leq 1$, given a function $f(t)$ is infinitely β-differentiable function. Then, the conformable fractional differential transform of $f(t)$ can be written as:

$$F_{\beta}(k) = \frac{1}{\beta^k k!}\left[\left(G_{\beta}^{t_o} f\right)^{(k)}(t)\right]_{t=t_o}, \tag{60}$$

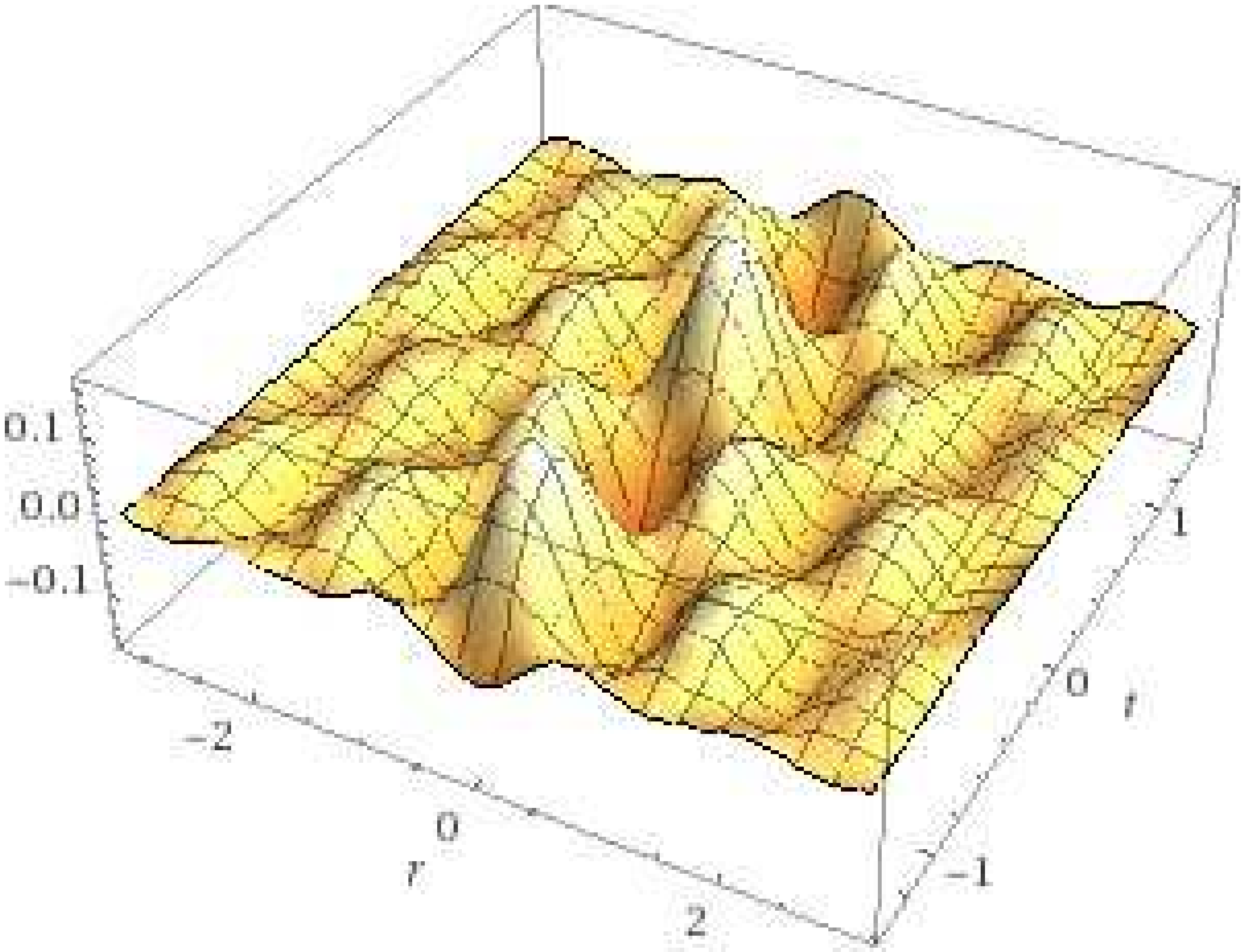


Fig. 1. Approximate Analytical Solution in (59) for $\beta = 1$

where $(G_\beta^{t_o} f)^{(k)}(t)$ is the kth number of CFD application's times, and the conformable fractional differential transform of initial conditions for integer order derivatives can be also written as [55]:

$$F_\beta(k) = \begin{cases} \frac{1}{(\beta k)!}\left[\left(\frac{d^{\beta k} f(t)}{dt^{\beta k}}\right)\right]_{t=t_o} \text{ for } k\text{=0,1,...,}\left(\frac{n}{\beta}-1\right) & \text{if } \beta k \in \mathbb{Z}^+ \\ 0 & \text{if } \beta k \notin \mathbb{Z}^+ \end{cases} \tag{61}$$

Definition 2.7. Suppose that $F_\beta(k)$ is the conformable fractional differential transform for $f(t)$ such that the inverse conformable fractional differential transform of $F_\beta(k)$ can be written as [55]:

$$f(t) = \sum_{k=0}^{\infty} F_\beta(k)(t-t_o)^{\beta k} = \sum_{k=0}^{\infty} \frac{1}{\beta^k k!}\left[\left(G_\beta^{t_o} f\right)^{(k)}(t)\right]_{t=t_o}(t-t_o)^{\beta k}. \tag{62}$$

Recently, Acan et al. [58] introduced the reduced differential transform method (RDTM) for solving partial differential equation, and Acan and Baleanu [59] developed a new definition for the conformable reduced differential transform method (CRDTM) as follows:

Definition 2.8. For some $0 < \beta \leq 1$, given a function $m(x,t)$ is analytic continuously β-differentiable function with respect to time t and space x. Then, the conformable reduced differential transform of $m(x,t)$ can be written as:

$$M_k^\beta(x) = \frac{1}{\beta^k k!}\left[\left({}_tG_\beta^{(k)} m\right)\right]_{t=t_o}, \tag{63}$$

where ${}_tG_\beta{}^{(k)} m = (({}_tG_\beta)({}_tG_\beta)...({}_tG_\beta))m(x,t)$, and the conformable reduced differential transform of

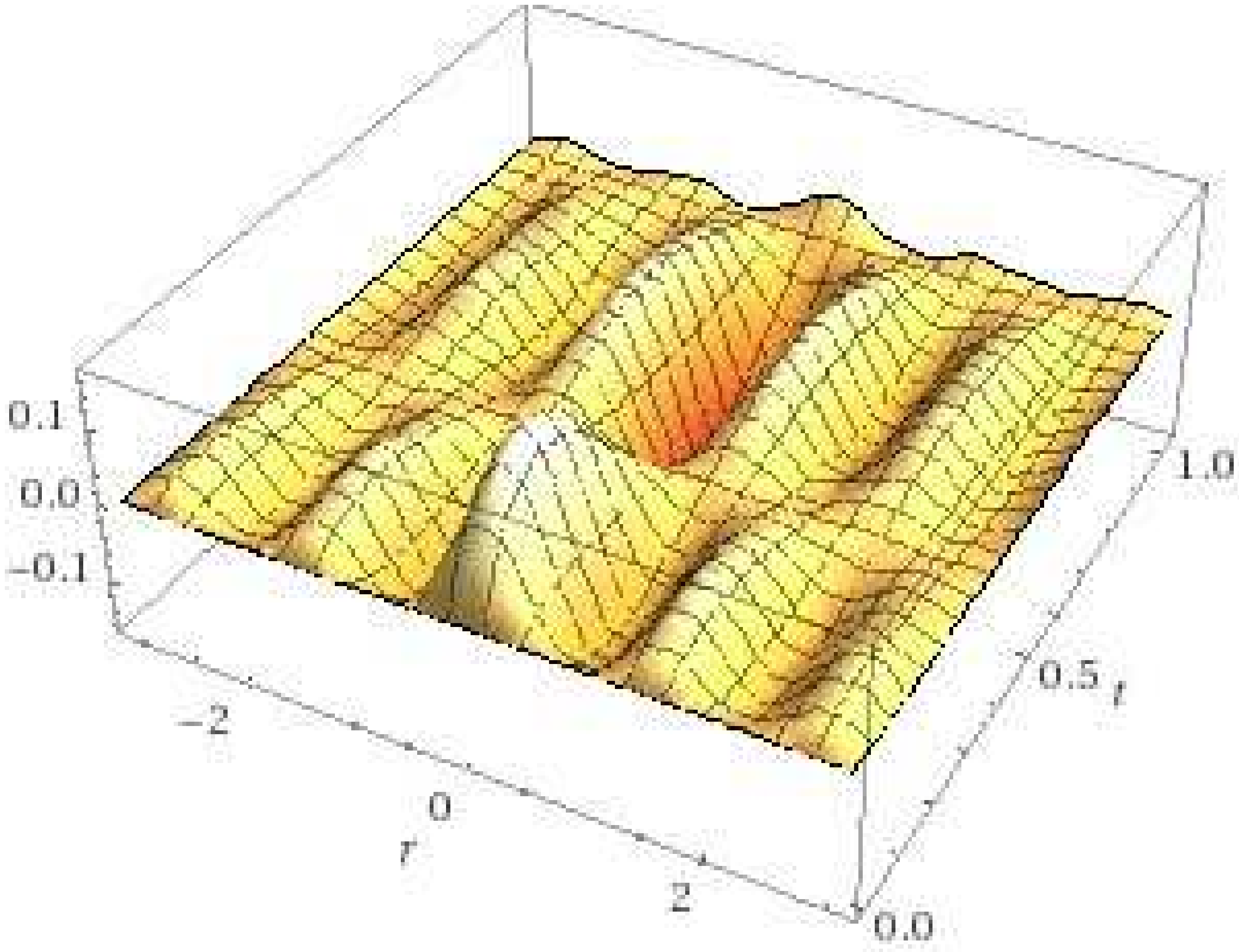


Fig. 2. Approximate Analytical Solution in (59) for $\beta = 0.75$

initial conditions for integer order derivatives can be also written as [58,59]:

$$F_\beta(k) = \begin{cases} \frac{1}{(\beta k)!}\left[\left(\frac{\partial^{\beta k}}{\partial t^{\beta k}} m(x,t)\right)\right]_{t=t_o} \text{ for } k=0,1,...,\left(\frac{n}{\beta}-1\right) & \text{if } \beta k \in \mathbb{Z}^+ \\ 0 & \text{if } \beta k \notin \mathbb{Z}^+ \end{cases} \tag{64}$$

For (61) and (64), n is the order of conformable differential operator for ordinary differential equation and partial differential equation, respectively.

Definition 2.9. Suppose that $M_k^\beta(x)$ is the conformable reduced differential transform for $m(x,t)$ such that the inverse conformable reduced differential transform of $M_k^\beta(x)$ can be written as [58,59]:

$$m(x,t) = \sum_{k=0}^{\infty} M_k^\beta(x)(t-t_o)^{\beta k} = \sum_{k=0}^{\infty} \frac{1}{\beta^k k!}\left[\left({}_tG_\beta^{(k)} m\right)\right]_{t=t_o}(t-t_o)^{\beta k}. \tag{65}$$

For theorems and basic operations about both DTM and CRDTM, we refer to [55,59].

Numerical Experiment 2:

By using the basic operations of CRDTM in [59], CRDTM is applied to solve the mixed initial-boundary value problem (see equation (9)) as follows:

$$\beta(k+1)(k+2)M_{k+2}^\beta(r) = \left[\frac{\partial^2}{\partial r^2}M_k^\beta(r) + \sum_{j=0}^{k} M_{k-j}^\beta(r)\frac{\partial}{\partial r}M_j^\beta(r)\right]; \tag{66}$$

c_o^2 is assumed to be equal 1 for simplicity

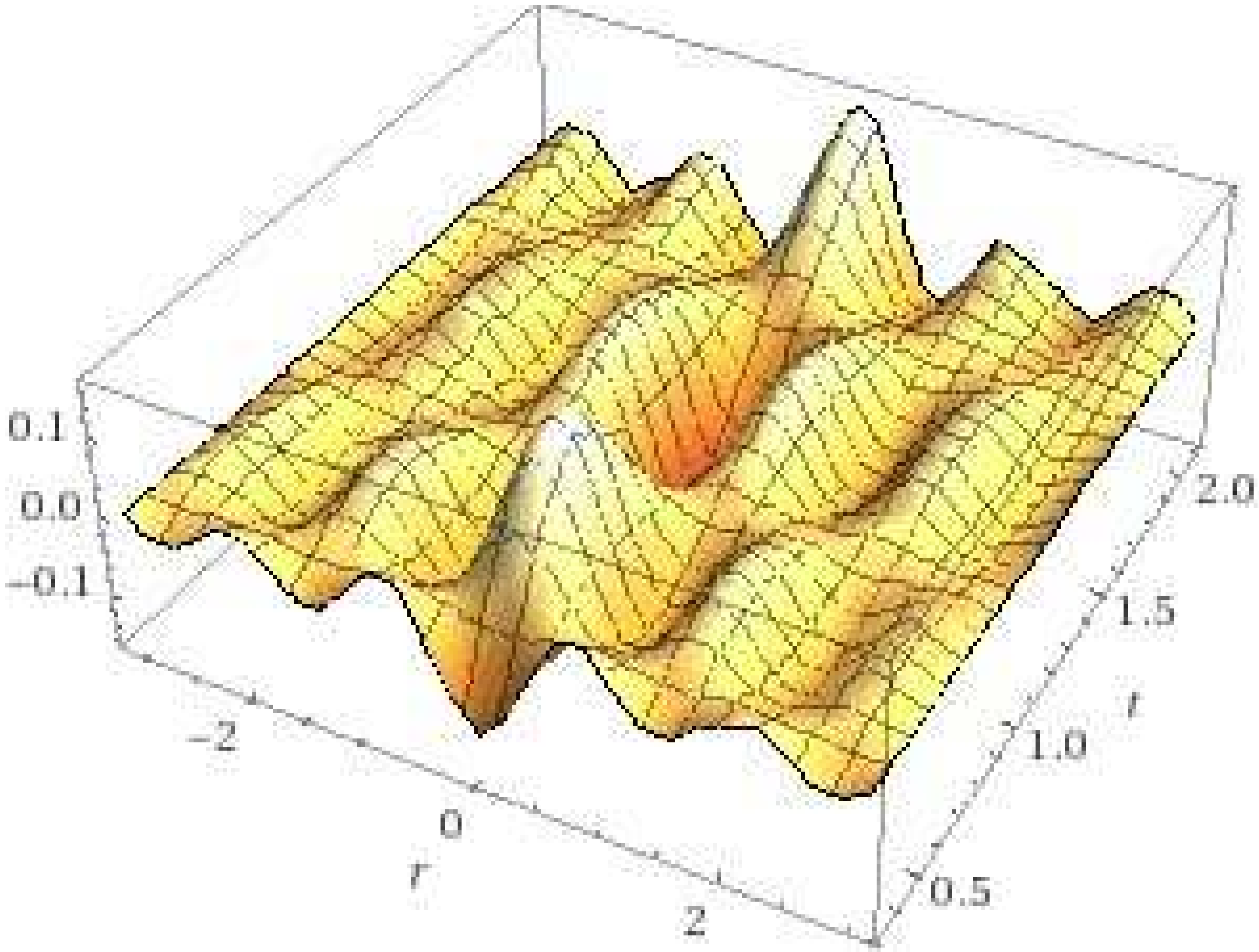


Fig. 3. Approximate Analytical Solution in (59) for $\beta = 0.50$

Hence, the recurrence relation of equation (66) can be written as follows:

$$M_{k+2}^{\beta}(r) = \left[\frac{\frac{\partial^2}{\partial r^2} M_k^{\beta}(r) + \sum_{j=0}^{k} M_{k-j}^{\beta}(r) \frac{\partial}{\partial r} M_j^{\beta}(r)}{\beta(k+1)(k+2)} \right], \tag{67}$$

where $M_k^{\beta}(r)$ is the conformable reduced differential function. For the initial conditions in (11), we assume that $q(r) = cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right)$ and $p(r) = 2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right)$. By applying CRDTM to the assumed initial conditions, we obtain the following:

$$\begin{aligned} M_0^{\beta}(r) &= cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right). \\ M_1^{\beta}(r) &= 2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right). \end{aligned} \tag{68}$$

By substituting (68) in equation (67), the following $M_k^\beta(r)$ values are obtained as follows:

$$\begin{aligned} M_2^\beta(r) &= \frac{-cos\left(\frac{r}{\beta}\right) - sin\left(\frac{r}{\beta}\right)}{2!\beta^2}, \\ M_3^\beta(r) &= \frac{-2cos\left(\frac{r}{\beta}\right) - 2sin\left(\frac{r}{\beta}\right)}{3!\beta^3}, \\ M_4^\beta(r) &= \frac{cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right)}{4!\beta^4}, \\ M_5^\beta(r) &= \frac{2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right)}{5!\beta^5}, \\ M_6^\beta(r) &= \frac{-cos\left(\frac{r}{\beta}\right) - sin\left(\frac{r}{\beta}\right)}{6!\beta^6}. \end{aligned} \tag{69}$$

Consequently, the set of values $\{M_k^\beta(r)\}_{k=0}^n$ provides the following approximate solution:

$$\begin{aligned} \widetilde{m}_w(r,t) &= \sum_{k=0}^{w} M_k^\beta(r) t^{\beta k} \\ &= \begin{cases} \sum_{k=0}^{w} \frac{(-1)^{\frac{3k}{2}}}{k!\beta^k} \left[cos\left(\frac{rt^{\beta k}}{\beta}\right) + sin\left(\frac{rt^{\beta k}}{\beta}\right)\right]; & \text{if } k \text{ is even} \\ \\ 2\left[cos\left(\frac{rt^{\beta k}}{\beta}\right) + sin\left(\frac{rt^{\beta k}}{\beta}\right)\right] \\ \\ +\sum_{k=3}^{w} \frac{(-1)^{\frac{2k}{3}+(k-(k-3))}}{k!\beta^k} \left[2cos\left(\frac{rt^{\beta k}}{\beta}\right) + 2sin\left(\frac{rt^{\beta k}}{\beta}\right)\right]; & \text{if } k \text{ is odd} \end{cases} \end{aligned} \tag{70}$$

3. Comparison of results and Discussion

Consequently, after trying to solve this particular two-dimensional wave equation using the classical definitions of fractional derivatives such as Riemann-Liouville, Caputo, Riesz, Riesz-Caputo, and Grünwald-Letnikov, the analytical solution is very complicated to obtain or even may impossible to obtain due to the fact that the classical fractional derivatives are nonlocal differential operators represented using convolution integrals with a weakly singular kernels [60]. To show a simple example of how complicated to obtain analytical solution using classical fractional derivatives for this particular problem, we refer to the general solution obtained in [61] for the time fractional wave equation for a vibrating string. We also refer to a method used in solving classical fractional differential equations (FDEs) in [62], but it cannot find analytical solutions to some examples and cases of FDEs. As a result, the conformable fractional derivatives (CFD) are local operator and can be implemented successfully and easily in various case studies arising from science and engineering in comparison to classical fractional derivatives. CFD can also be used very efficiently in constructing mathematical models for complex problems in physics and engineering.

Due to the difficulty of analytical solutions using classical fractional derivatives, several research studies have developed approximate methods to approximate analytical solutions for the fractional differential equations in the calculus of variations. Approximate methods for FDEs have been introduced successfully in [63–66].

To discuss the error between the analytical and approximate solutions from using all three methods in sections 2.1, 2.2, and 2.3, let's do a numerical test for various values of β and t with various initial conditions from the suggested numerical experiments in this paper and using example 2 in section 4.2 of [43] to discuss the accuracy, reliability, and applicability of the three proposed methods in sections 2. All numerical data of the obtained approximate solutions in table 1, 3, and 3 have been calculated and approximated for the first three terms using an online computer software, known as Keisan Online Calculator service, developed by CASIO COMPUTER CO., LTD.

Numerical Example 1:

By using the mixed initial-boundary value problem in (9) and (10), and example 2 in section 4.2 of [43], the initial conditions in (11) can be written as:

$$M(r,0) = q(r) = 1 - r^2; \text{ and } \frac{\partial^\beta M}{\partial t^\beta}(r,0) = p(r) = 0; \text{ for } 0 < r < R \text{ and } 0 < \beta \leq 1, \quad R = c_o = 1. \tag{71}$$

The above example represents a circular membrane with axisymmetric initial shape [43]. By using the conformable separation of variables method (CSVM) in section 2.1, the approximate analytical solution can be written as:

$$M_{approximate}(r,t) = \sum_{n=1}^{\infty} \left[\frac{8}{\xi_n^3 J_1(\xi_n)} cos\left(\frac{\xi_n t^\beta}{\beta}\right) J_0(\xi_n r) \right]. \tag{72}$$

Similarly, the analytical solution in [43] using the separation of variables method (SVM) can be also written as:

$$M_{analytical}(r,t) = \sum_{n=1}^{\infty} \left[\frac{8}{\xi_n^3 J_1(\xi_n)} cos\left(\xi_n t\right) J_0(\xi_n r) \right]. \tag{73}$$

Table 1 shows the numerical data for both analytical and approximate analytical solutions from using CSVM and SVM for different values of r, t, and β. The absolute error between the analytical and approximate analytical solutions, written as $Error = |M_{approximate}(r,t) - M_{analytical}(r,t)|$, has also been recorded in table 1. From Table 1, it is obvious that at various values of r and t, when β values are getting close to 1, absolute error values become very small. At $\beta = 1$, the obtained approximate analytical solution from CSVM becomes equivalent to the analytical solution from SVM. Figure 4 shows the approximate solutions for different values of t and β at a fixed $r = 0.5$. From Figure 4, at $\beta = 0.75$ the obtained approximate solution by CSVM are closer to the analytical solution using the SVM for integer-order derivatives. Therefore, the behavior of membrane's displacement with respect to time at various β values at a fixed value of membrane radii [42] in Figure 4 can be described as the value of β increases in the conformable formulation (CSVM), the approximate solution from CSVM becomes closer to the analytical solution using the integer-order SVM, and the absolute error value between analytical and approximate solutions becomes small.

Table 1. Comparison of the Analytical and Approximate Solutions from using SVM and CSVM

(r,t)	SVM	β	CSVM	Error
(0.1,0.1)	1.0003	0.25	0.9963	4E-3
		0.75	1.0002	1E-4
		1	1.0003	0
(0.3,0.3)	0.9051	0.25	0.9005	4.6E-3
		0.75	0.9050	1E-4
		1	0.9051	0
(0.5,0.5)	0.7496	0.25	0.7432	6.4E-3
		0.75	0.7494	2E-4
		1	0.7496	0
(0.7,0.7)	0.5152	0.25	0.5056	9.6E-3
		0.75	0.5148	4E-4
		1	0.5152	0
(0.9,0.9)	0.1803	0.25	0.1752	5.1E-3
		0.75	0.1800	3E-4
		1	0.1803	0

Numerical Example 2:

By using the numerical experiment 1 in section 2.2, the approximate solution in (59) can be written with only the real part which satisfies the mixed initial-boundary value problem in (9), (10), and (11) as follows: [54]

$$M(r,t) = \sum_{\xi=1}^{\infty} \frac{J_0\left(\lambda_\xi \frac{r}{R}\right) sin\left(\lambda_\xi \frac{c_o t^\beta}{R\beta}\right)}{\lambda_\xi^2 \left(\frac{c_o}{R}\right) J_1(\lambda_\xi)}; \quad \text{where } 0 < \beta \leq 1. \tag{74}$$

The above equation (74) represents the approximate solution with a real part only using the conformable double Laplace transform method (DLTM). Let's also assume $R = c_o = 1$. So, equation (74) can be simplified as follows:

$$M(r,t) = \sum_{\xi=1}^{\infty} \frac{J_0\left(\lambda_\xi r\right) sin\left(\lambda_\xi \frac{t^\beta}{\beta}\right)}{\lambda_\xi^2 J_1(\lambda_\xi)}; \quad \text{where } 0 < \beta \leq 1. \tag{75}$$

To compare the above approximate solution with approximate analytical solution, let's use the proposed mixed initial-boundary value problem in (41) and (42) to find the approximate analytical solution in the sense of conformable derivative. Since $M(r,0) = q(r) = 0$, then $E_n = 0$ in (28). K_n in (29) can be found as follows:

$$\begin{aligned} K_n &= \frac{2}{Rc_o\xi_n J_1^2(\xi_n)} \int_0^R p(r) J_0\left(\frac{\xi_n r}{R}\right) r \, \mathrm{d}r \\ &= \frac{2}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \, \mathrm{d}r. \end{aligned} \tag{76}$$

By using integration by parts for (76) and the identity (11) in [43], we have the following: $u = cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right)$; $du = \left(-sin\left(\frac{r}{\beta}\right) + cos\left(\frac{r}{\beta}\right)\right) dr$; $dv = J_0\left(\xi_n r\right) r \, \mathrm{d}r$; and $v = \frac{1}{\xi_n} J_1\left(\xi_n\right)$. Let

$\left(-sin\left(\frac{r}{\beta}\right)+cos\left(\frac{r}{\beta}\right)\right)=\omega$; we have the following:

$$\begin{aligned}
&\int_0^1 cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r\,\mathrm{d}r \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)-\int_0^1 \frac{J_1\left(\xi_n\right)}{\xi_n}\omega\,\mathrm{d}r \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\int_0^1 \frac{\left(-cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)\right)}{\xi_n} J_1\left(\xi_n\right)\,\mathrm{d}r \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\frac{J_1\left(\xi_n\right)}{\xi_n}\left[\left(-\frac{sin\left(\frac{r}{\beta}\right)}{\frac{1}{\beta}}-\frac{cos\left(\frac{r}{\beta}\right)}{\frac{1}{\beta}}\right)\right]_{r=0}^{r=1} \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\frac{J_1\left(\xi_n\right)}{\xi_n}\left[-\beta sin\left(\frac{1}{\beta}\right)-\beta cos\left(\frac{1}{\beta}\right)+\frac{1}{\beta}\right] \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)}{\xi_n} J_1\left(\xi_n\right)+\frac{J_1\left(\xi_n\right)}{\xi_n}\left[-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)\right] \\
&=\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)}{\xi_n} J_1\left(\xi_n\right).
\end{aligned} \tag{77}$$

By substituting (77) in (76), we obtain K_n as follows:

$$\begin{aligned}
&\frac{2}{\xi_n J_1^2(\xi_n)}\int_0^1 cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r\,\mathrm{d}r \\
&=\frac{2}{\xi_n J_1^2(\xi_n)}\frac{cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)}{\xi_n} J_1\left(\xi_n\right) \\
&=\frac{2}{\xi_n^2 J_1(\xi_n)}\left[cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)\right].
\end{aligned} \tag{78}$$

To obtain the approximate analytical solution using CSVM, let's substitute (78) in (30) as follows:

$$\begin{aligned}
&M(r,t)=\sum_{n=1}^{\infty}\frac{2}{\xi_n^2 J_1(\xi_n)}\left[cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)-\beta\left(sin\left(\frac{1}{\beta}\right)+cos\left(\frac{1}{\beta}\right)-1\right)\right] \\
&\times\left[sin\left(\frac{\xi_n t^{\beta}}{\beta}\right) J_0\left(\xi_n r\right)\right]; \\
&\text{for } n\in\mathbb{Z}^+ \text{ and } c_o=R=1.
\end{aligned} \tag{79}$$

Table 2 shows the numerical data for approximate solutions from using CSVM and DLTM for different values of r, t and β. The absolute error between approximate solutions using CSVM and DLTM has been recorded in table 2. At $r=t=0.9$ and $\beta=0.50$, both approximate solutions using CSVM and DLTM in Table 2 are equivalent to each other with no absolute error between them. When $\beta=0.75$ or $\beta=1$ for $r=t=0.1; 0.3; 0.5; 0.7; 0.9$, the absolute error values become very small. Figure 5 shows the approximate solutions for different values of t and β at a fixed $r=0.5$. Between $t=0.1$ and $t=0.2$ at a fixed value ($r=0.5$) of membrane radii in figure 5, the behavior of membrane's displacement with respect to time shows that the numerical values of approximate solution DLTM and CSVM at various β values are very close to each other and the absolute error values between them small. In table 2, it is also clear that numerical value of approximate solutions using CSVM at $\beta=1$ and DLTM at $\beta=0.50$ are close to each other and the error between them is very small. When the time is very small i.e. $t = 0.1$, both approximate solutions from using CSVM and DLTM are very close to each other in value and the absolute error values between them become smaller than other numerical values of the same approximate solutions at larger time periods.

Table 2. Comparison of the Analytical and Approximate Solutions from using CSVM and DLTM

(r,t)	β	CSVM	DLTM	Error
(0.1,0.1)	0.50	0.0017	0.0071	5.4E-3
	0.75	6.275E-4	0.0027	2.073E-3
	1	2.643E-4	0.0011	8.357E-4
(0.3,0.3)	0.50	0.0190	0.0083	0.0107
	0.75	0.0093	0.0041	5.2E-3
	1	0.0052	0.0023	2.9E-3
(0.5,0.5)	0.50	0.0356	0.0117	0.0239
	0.75	0.0198	0.0066	0.0132
	1	0.0125	0.0041	8.4E-3
(0.7,0.7)	0.50	0.0291	0.0178	0.0113
	0.75	0.0176	0.0109	6.7E-3
	1	0.0121	0.0075	4.6E-3
(0.9,0.9)	0.50	0.0098	0.0098	0
	0.75	0.0063	0.0064	1E-4
	1	0.0046	0.0047	1E-4

Numerical Example 3:

To compare the approximate solutions from using CSVM and conformable reduced differential transform method (CRDTM), let's use the approximate solution in (70) from the numerical experiment 2 in section 2.3. Similarly, we need to find the approximate analytical solution in the sense of conformable derivative using CSVM for the mixed initial-boundary value problem in the numerical experiment 2 as we did in numerical example 2.We choose $R = c_o = 1$ in this example. Since $q(r) = cos\left(\frac{r}{\beta}\right)+sin\left(\frac{r}{\beta}\right)$ and $p(r) = 2cos\left(\frac{r}{\beta}\right) + 2sin\left(\frac{r}{\beta}\right)$, then let's find E_n in (28) and K_n in (29) as follows:

$$\begin{aligned} E_n &= \frac{2}{R^2 J_1^2(\xi_n)} \int_0^R rq(r)J_0\left(\frac{\xi_n r}{R}\right) \mathrm{d}r \\ &= \frac{2}{J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \,\mathrm{d}r. \end{aligned} \tag{80}$$

$$\begin{aligned} K_n &= \frac{2}{Rc_o\xi_n J_1^2(\xi_n)} \int_0^R p(r)J_0\left(\frac{\xi_n r}{R}\right) r \,\mathrm{d}r \\ &= \frac{4}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \,\mathrm{d}r. \end{aligned} \tag{81}$$

From the result in (77), E_n and K_n can be written as follows:

$$\begin{aligned} E_n &= \frac{2}{J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \,\mathrm{d}r \\ &= \frac{2}{\xi_n J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right]. \end{aligned} \tag{82}$$

$$\begin{aligned} K_n &= \frac{4}{\xi_n J_1^2(\xi_n)} \int_0^1 cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) J_0\left(\xi_n r\right) r \,\mathrm{d}r \\ &= \frac{4}{\xi_n^2 J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right]. \end{aligned} \tag{83}$$

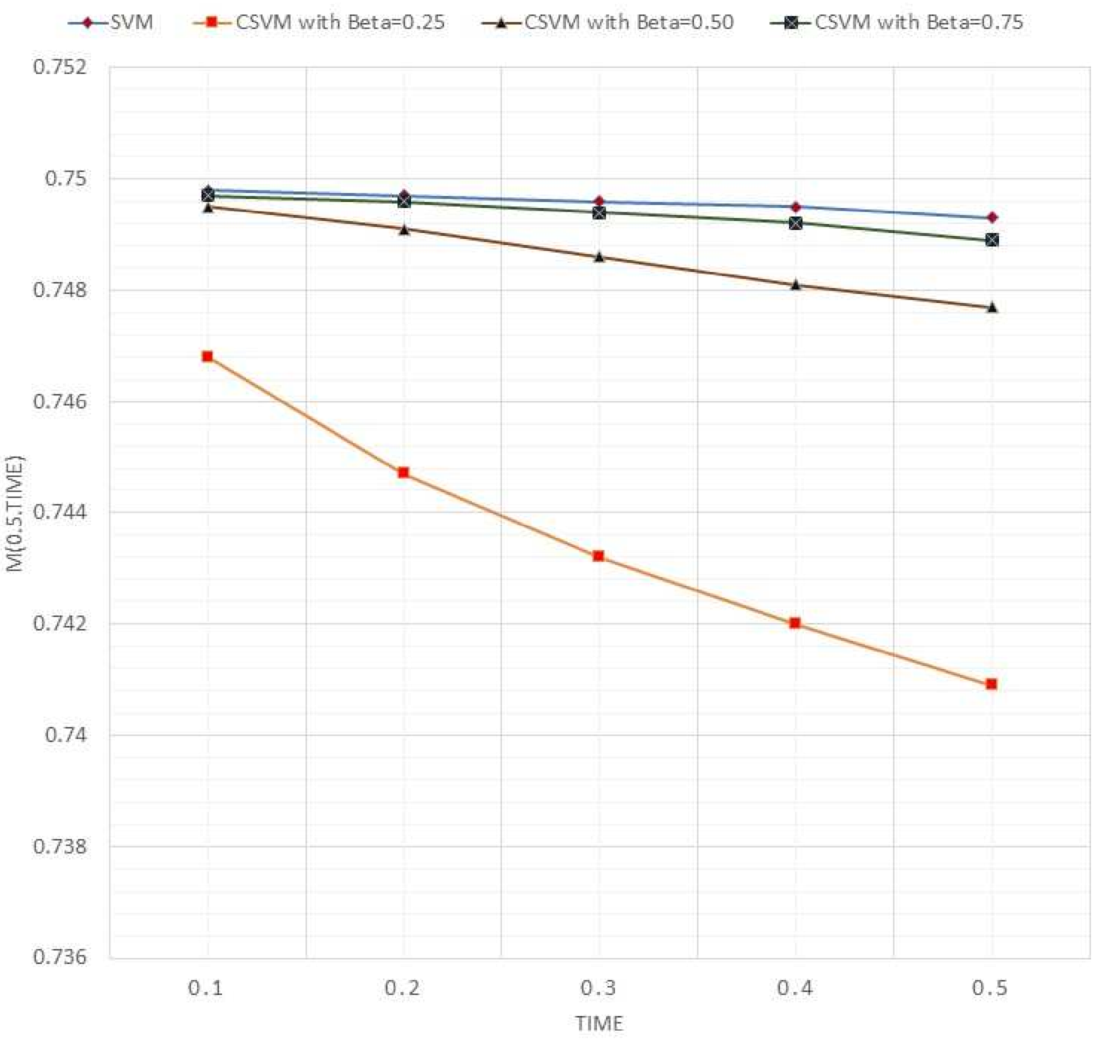


Fig. 4. Comparison of Analytical and Approximate Solutions in (73) and (72) for different values of β at a fixed $r = 0.5$

By substituting both (82) and (83) in (30), we obtain the following approximate analytical solution using CSVM:

$$\begin{aligned} M(r,t) &= \sum_{n=1}^{\infty} \frac{2}{\xi_n J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right] \\ &\times \left[cos\left(\frac{\xi_n t^\beta}{\beta}\right) J_0\left(\xi_n r\right)\right] \\ &+ \sum_{n=1}^{\infty} \frac{4}{\xi_n^2 J_1(\xi_n)} \left[cos\left(\frac{r}{\beta}\right) + sin\left(\frac{r}{\beta}\right) - \beta\left(sin\left(\frac{1}{\beta}\right) + cos\left(\frac{1}{\beta}\right) - 1\right)\right] \\ &\times \left[sin\left(\frac{\xi_n t^\beta}{\beta}\right) J_0\left(\xi_n r\right)\right]; \\ &\text{for } n \in \mathbb{Z}^+ \text{ and } c_o = R = 1. \end{aligned} \tag{84}$$

The numerical data for approximate solutions from using CSVM and CRDTM have been recorded in Table 3 for various values of r, t and β. Table 2 shows also the absolute error between approximate solutions using CSVM and CRDTM. The absolute error value is the smallest at $r = t = 0.7$ and $\beta = 1$ in Table 3 which implies that both approximate solutions using CSVM and CRDTM are very close in numerical value to each other. From Table 3, it is very clear that at $\beta = 0.85$ and $\beta = 1$ at various values of r and t, most of the absolute error values between approximate solutions from using CSVM and CRDTM are smaller than absolute error values at $\beta = 0.75$. Figure 6 shows the approximate solutions for different values of t and β at a fixed value ($r = 0.5$) of membrane radii. The behavior of

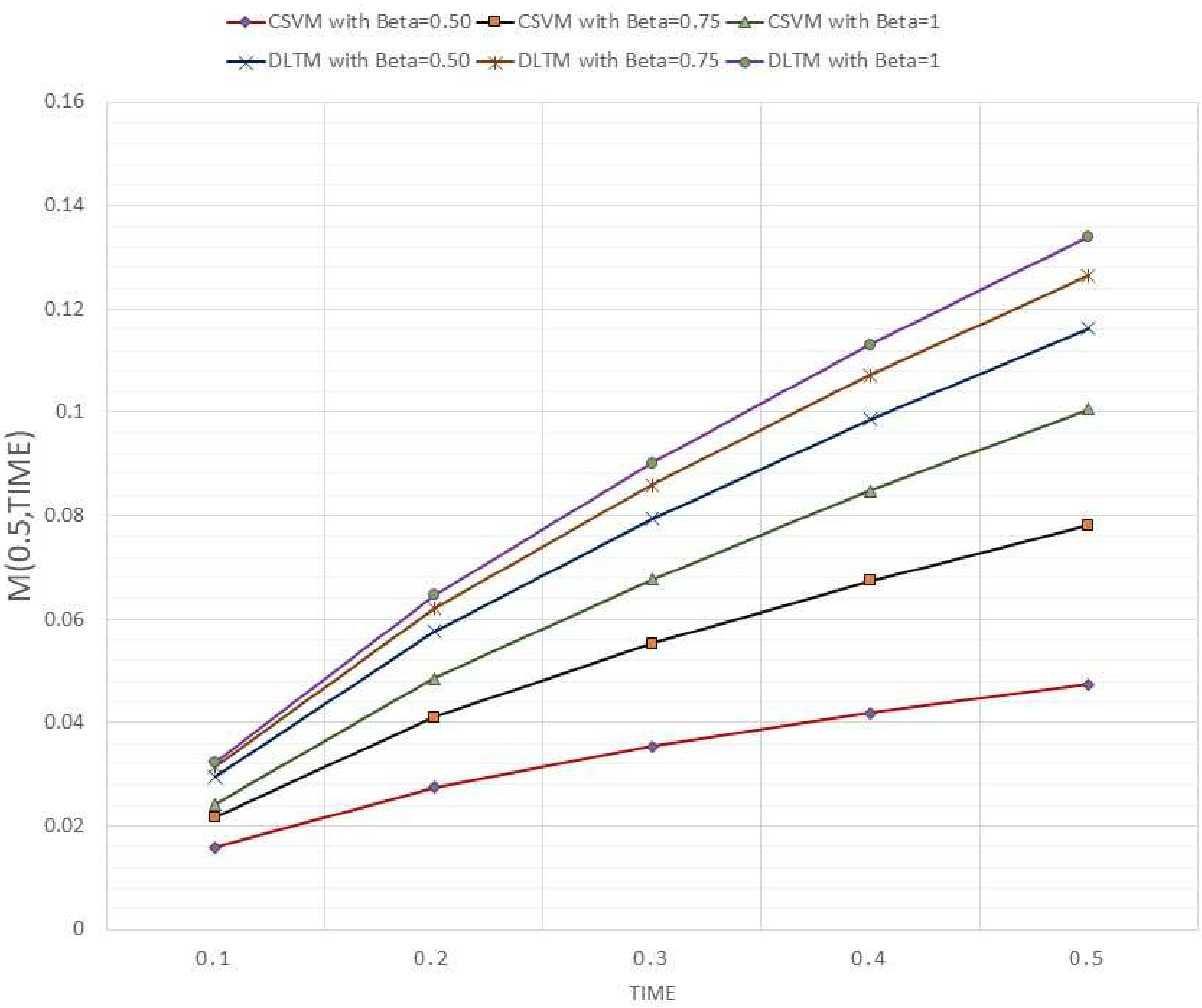


Fig. 5. Comparison of Approximate Solutions in (75) and (79) for different values of β at a fixed $r = 0.5$

membrane's displacement with respect to time in figure 6 shows that using CSVM at $\beta = 1$ is closer in numerical value to the numerical values using CRDTM at $\beta = 0.75; 0.85; 1$. Using CSVM at $\beta = 0.75$ and $\beta = 1$, the numerical values of approximate solutions are close to each other, but both solutions farther in value comparing to the approximate solution using CSVM at $\beta = 1$.

Table 3. Comparison of the Analytical and Approximate Solutions from using CSVM and CRDTM

(r,t)	β	CSVM	CRDTM	Error
(0.1,0.1)	0.75	1.2840	0.3207	0.9633
	0.85	1.2806	0.7651	0.5155
	1	1.2828	1.1667	0.1161
(0.3,0.3)	0.75	0.8764	0.3232	0.5532
	0.85	0.8719	0.7677	0.1042
	1	0.8672	1.1689	0.3017
(0.5,0.5)	0.75	0.9710	0.3292	0.6418
	0.85	0.9647	0.7734	0.1913
	1	1.4054	1.1739	0.2315
(0.7,0.7)	0.75	1.2609	0.3380	0.9229
	0.85	1.2533	0.7819	0.4714
	1	1.2445	1.1817	0.0628
(0.9,0.9)	0.75	0.6170	0.3495	0.2675
	0.85	0.6135	0.7931	0.1796
	1	0.6092	1.1922	0.5830

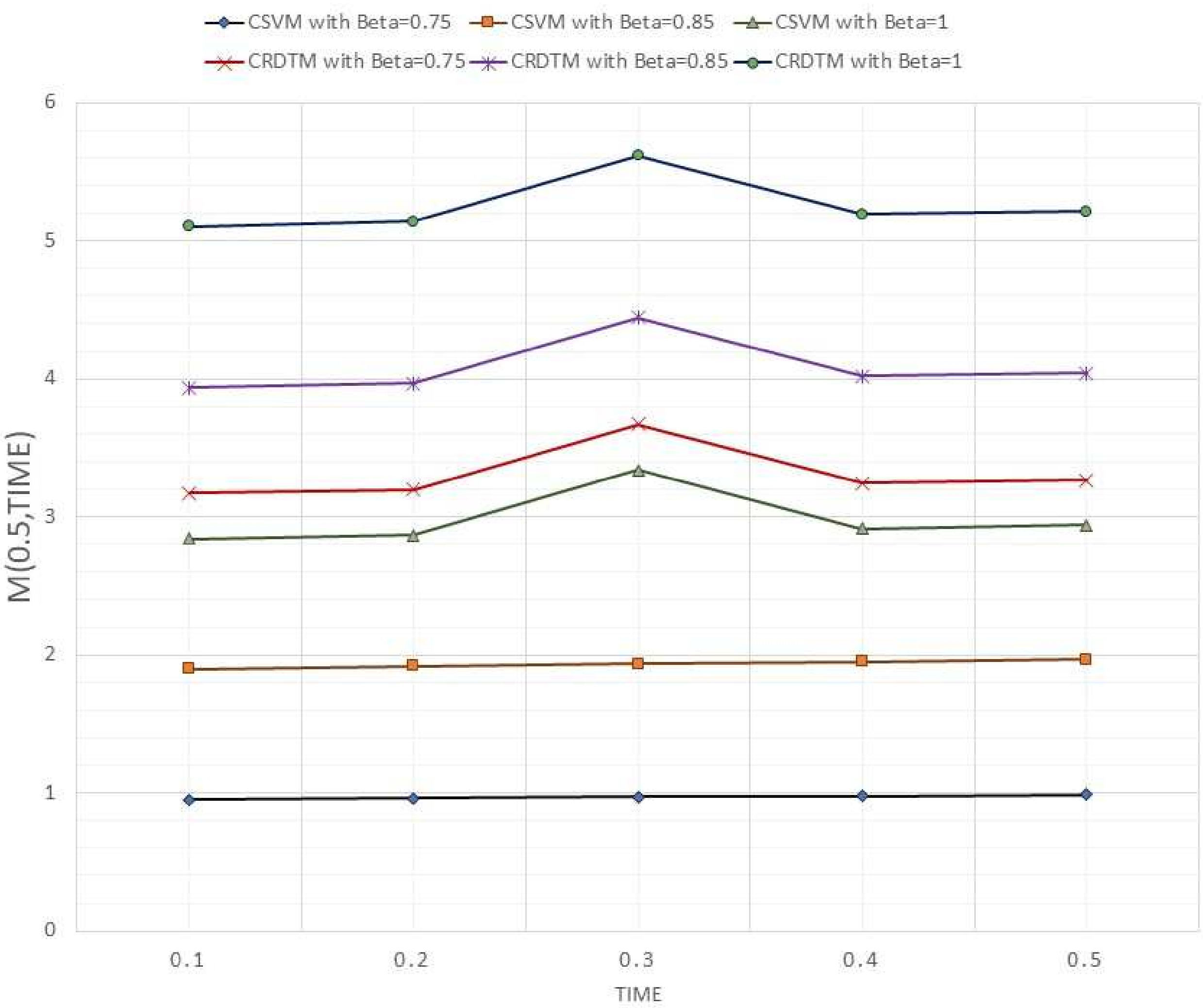


Fig. 6. Comparison of Approximate Solutions in (84) and (70) for different values of β at a fixed $r = 0.5$

By comparing the analytical and approximate solutions in (30), (59) and (70), with the classical non fractional standard analytical solution in [43, 44], we obtain the same analytical solution provided by [43, 44] by substituting $\beta = 1$ in equations (30), (59) and (70) since $0 < \beta \leq 1$. Figures (7), (8), and (9) show the comparison of analytical and approximate solutions in (30) and (70) graphically for various values of $\beta = 1; 0.75; 0.25$.

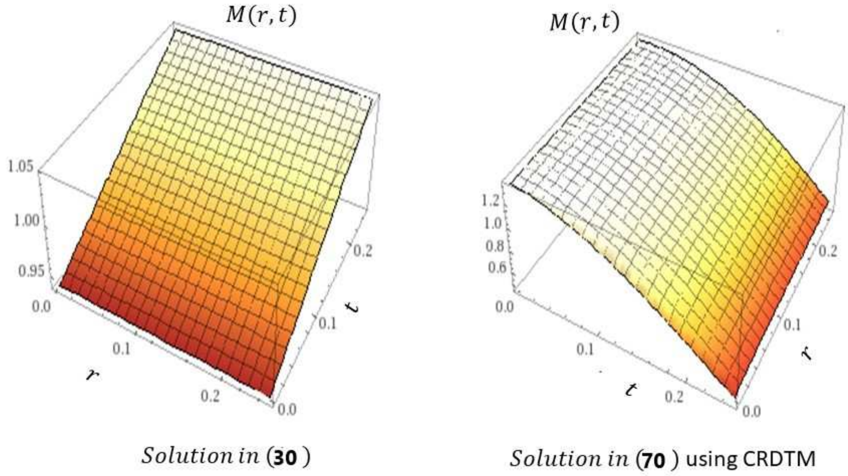


Fig. 7. Comparison of Solutions in (30) and (70) for $\beta = 1$

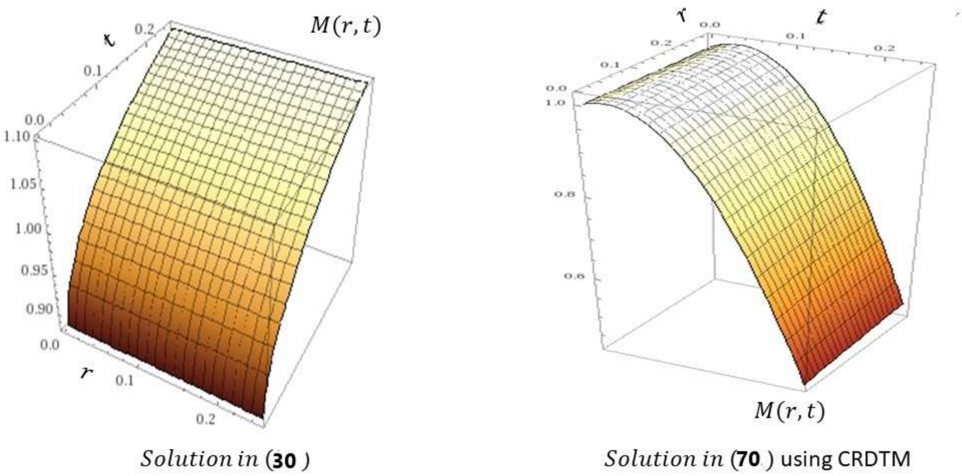
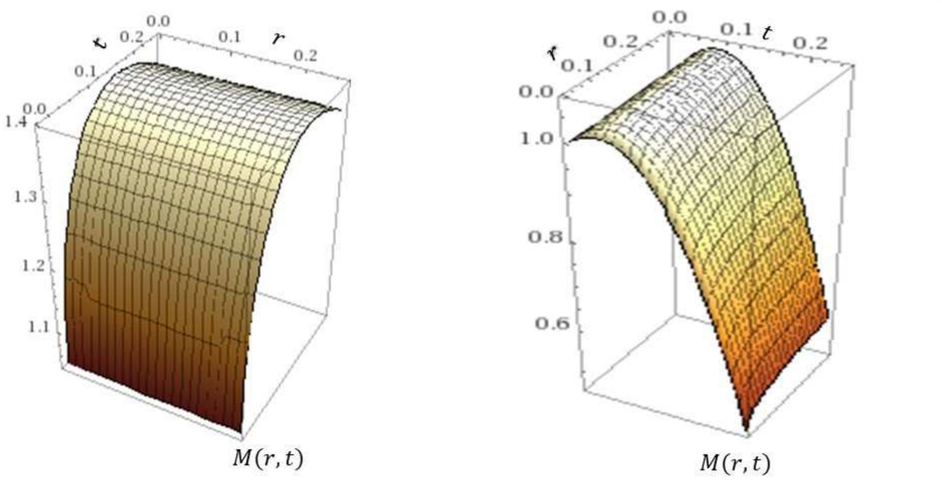


Fig. 8. Comparison of Solutions in (30) and (70) for $\beta = 0.75$

Thus, the CFD formulation is a simple fractional definition to obtain analytical solutions for fractional partial differential equations in comparison to the complicated classical fractional formulations that require various theorems, generalizations, or mathematical extensions to obtain analytical solution or even in some cases can not be obtained at all without introducing numerical and approximate



Solution in (**30**)   *Solution in* (**70**) using CRDTM

Fig. 9. Comparison of Solutions in (30) and (70) for $\beta = 0.25$

methods. The analytical solutions provided in this paper can be extended to solve higher order fractional PDEs more efficiently than nonlocal classical fractional derivatives formulations.

4. Conclusion

Fractional differential equations have been undergoing major developments due to the importance of understanding the physical and dynamical behavior of problems arising from physics and engineering applications. This article sheds the light on the importance of the conformable fractional derivatives (CFD) and the fact that the CFD can provide efficient analytical and approximate analytical solutions for the two-dimensional fractional wave equation using novel methods such as conformable separation of variables, conformable double Laplace transform, and conformable reduced differential transform methods. We believe that the conformable fractional formulation can be applied effectively in modeling various PDEs problems.

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