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## Theory of Generalized Sets in Generalized Topological Spaces

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**Abstract** — Several specific types of generalized sets (briefly,  $\mathbf{g}\mathcal{T}_g$ -sets) in generalized topological spaces (briefly,  $\mathcal{T}_g$ -spaces) have been defined and investigated for various purposes from time to time in the literature of  $\mathcal{T}_g$ -spaces. Our recent research in the field of a new class of  $\mathbf{g}\mathcal{T}_g$ -sets in  $\mathcal{T}_g$ -spaces is reported herein as a starting point for more generalized classes. It is shown that the class of  $\mathbf{g}\mathcal{T}_g$ -sets is a superclass of those whose elements are called open, closed, semi-open, semi-closed, pre-open, pre-closed, semi-pre-open, and semi-pre-closed sets in a  $\mathcal{T}_g$ -space. A subclass of the  $\mathcal{T}_g$ -subspace corresponds to the class of  $\mathbf{g}\mathcal{T}_g$ -sets of a  $\mathcal{T}_g$ -space. A class of  $\mathbf{g}\mathcal{T}_g$ -sets of the Cartesian product of these  $\mathcal{T}_g$ -spaces corresponds to the Cartesian product of a finite number of classes of  $\mathbf{g}\mathcal{T}_g$ -sets, each of which belongs to a  $\mathcal{T}_g$ -space. Diagrams establish the various relationships amongst the classes presented here and in the literature, and an ad hoc application supports the overall theory.

**Keywords** — Generalized topology, generalized topological space, generalized operations, generalized open sets, generalized closed sets

**Mathematics Subject Classification (2020)** — 54A05, 54B05

## 1. Introduction

Just as the notion of  $\mathcal{T}$ -set (open or closed set relative to ordinary topology) is fundamental and indispensable in the study of  $\mathcal{T}$ -sets in  $\mathcal{T}$ -spaces (arbitrary sets in ordinary topological spaces) and in the formulation of the concept of  $\mathbf{g}\mathcal{T}$ -set (generalized  $\mathcal{T}$ -open or  $\mathcal{T}$ -closed set relative to ordinary topology) in the study of  $\mathbf{g}\mathcal{T}$ -sets in  $\mathcal{T}$ -spaces (generalized sets in ordinary topological spaces) [1–6], so is the notion of  $\mathcal{T}_g$ -set (open or closed set relative to generalized topology) in the study of  $\mathcal{T}_g$ -sets in  $\mathcal{T}_g$ -spaces (arbitrary sets in generalized topological spaces) and in the formulation of the concept of  $\mathbf{g}\mathcal{T}_g$ -set (generalized  $\mathcal{T}_g$ -open or  $\mathcal{T}_g$ -closed set relative to generalized topology) in the study of  $\mathbf{g}\mathcal{T}_g$ -sets in  $\mathcal{T}_g$ -spaces (generalized sets in generalized topological spaces) [7]. Thus, the  $\mathbf{g}$ -topology maps  $\mathcal{T}_g : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  from the power set  $\mathcal{P}(\Omega)$  of  $\Omega$  into itself, thereby inducing  $\mathbf{g}$ -topologies on the underlying set  $\Omega$ , are classes of distinguished open subsets of a  $\mathcal{T}$ -space which are not  $\mathcal{T}$ -open sets but are  $\mathcal{T}_g$ -open sets which are related to the families of  $\mathbf{g}\mathcal{T}$ -open sets [8, 9]. Examples of  $\mathbf{g}\mathcal{T}$ -sets in  $\mathcal{T}$ -spaces are  $\alpha$ -open and  $\alpha$ -closed sets [10],  $\beta$ -open sets [11], and  $\gamma$ -open sets [12]. Examples of  $\mathbf{g}\mathcal{T}_g$ -sets in  $\mathcal{T}_g$ -spaces are  $\Delta_\mu$ -sets and  $\nabla_\mu$ -sets [13],  $\omega$ -open sets [2], and  $\theta$ -sets [14]. From these  $\alpha$ ,  $\beta$ ,  $\gamma$ -sets and  $\Delta_\mu$ ,  $\nabla_\mu$ ,  $\omega$ ,  $\theta$ -sets, the theories of  $\mathbf{g}\mathcal{T}$ -sets and  $\mathbf{g}\mathcal{T}_g$ -sets then appear to be subjects of primary interest.

To the best of our knowledge, the theory of  $\mathbf{g}\mathcal{T}$ -sets is well-known and that of  $\mathbf{g}\mathcal{T}_g$ -sets less-known. The earliest works on the theory of  $\mathbf{g}\mathcal{T}$ -sets are those of Levine [15, 16], Njåstad [10], and

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Császár [14, 17–20], and the latest works on the theory of  $\mathbf{g}\mathfrak{T}$ -sets are those of Rajeshwari et al. [21], Jeyanthi et al. [3, 13], Ghour et al. [2], and Tyagi et al. [6], among others. Levine [16] introduced and investigated the weaker forms of open sets, Njåstad [10] introduced and investigated the structures of some classes of more or less nearly open sets, and Császár [20] introduced the notion of  $\mathbf{g}$ -topologies; [21] introduced the weaker forms of closed sets and studied some of their characterizations, Jeyanthi et al. [3] gave a unified framework for the study of several types of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets, Ghour et al. [2] extended the notion of a type of  $\mathbf{g}\mathfrak{T}$ -sets in a  $\mathcal{T}$ -space to its analogue in a  $\mathcal{T}_{\mathbf{g}}$ -space, and Tyagi et al. [6] introduced and investigated several types of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets in  $\mathcal{T}_{\mathbf{g}}$ -spaces.

Several other specific classes of  $\mathbf{g}\mathfrak{T}$ ,  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets have been defined and investigated by other authors for various purposes from time to time in the literature of  $\mathcal{T}$ ,  $\mathcal{T}_{\mathbf{g}}$ -spaces [9, 22–38]. The fruitfulness of all these references have made significant contributions to the theory of  $\mathcal{T}$ ,  $\mathcal{T}_{\mathbf{g}}$ -spaces, among others.

In this paper, we will show how further contributions can be added to the field in a unified way. The rest of this paper is structured in this manner: In Section 2, preliminary notions are described in Subsection 2.1 and the main results of the theory of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets in  $\mathcal{T}_{\mathbf{g}}$ -spaces are reported in Section 3. In Section 4, the establishment of the various relationships between the classes of  $\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathfrak{T}_{\mathbf{g}}$ -closed sets and the classes of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -closed sets in the  $\mathcal{T}_{\mathbf{g}}$ -space  $\mathfrak{T}_{\mathbf{g}}$  are discussed and illustrated through diagrams in Subsection 4.1. To support the work, a nice application, concentrating on fundamental concepts from the standpoint of the theory of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets is presented in Subsection 4.2. Finally, Subsection 5 provides concluding remarks and future directions of the theory of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets in  $\mathcal{T}_{\mathbf{g}}$ -spaces.

## 2. Theory

### 2.1. Preliminaries

Our discussion starts by recalling a carefully chosen set of terms used in this study [39]. Throughout this manuscript, the structures  $\mathfrak{T} = (\Omega, \mathcal{T})$  and  $\mathfrak{T}_{\mathbf{g}} = (\Omega, \mathcal{T}_{\mathbf{g}})$ , respectively, are called ordinary and generalized topological spaces (briefly,  $\mathcal{T}$ -space and  $\mathcal{T}_{\mathbf{g}}$ -space). The symbols  $\mathcal{T}$  and  $\mathcal{T}_{\mathbf{g}}$ , respectively, are called ordinary topology and generalized topology (briefly, topology and  $\mathbf{g}$ -topology). Subsets of  $\mathfrak{T}$  and  $\mathfrak{T}_{\mathbf{g}}$ , respectively, are called  $\mathfrak{T}$ -sets and  $\mathfrak{T}_{\mathbf{g}}$ -sets; subsets of  $\mathcal{T}$  and  $\mathcal{T}_{\mathbf{g}}$ , respectively, are called  $\mathcal{T}$ -open and  $\mathcal{T}_{\mathbf{g}}$ -open sets, and their complements are called  $\mathcal{T}$ -closed and  $\mathcal{T}_{\mathbf{g}}$ -closed sets. Generalizations of  $\mathfrak{T}$ -sets,  $\mathcal{T}$ -open and  $\mathcal{T}$ -closed sets in  $\mathcal{T}$ , respectively, are called  $\mathbf{g}\mathfrak{T}$ -sets,  $\mathbf{g}\mathcal{T}$ -open and  $\mathbf{g}\mathcal{T}$ -closed sets; generalizations of  $\mathfrak{T}_{\mathbf{g}}$ -sets,  $\mathcal{T}_{\mathbf{g}}$ -open and  $\mathcal{T}_{\mathbf{g}}$ -closed sets in  $\mathcal{T}_{\mathbf{g}}$ , respectively, are called  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets,  $\mathbf{g}\mathcal{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\mathcal{T}_{\mathbf{g}}$ -closed sets;  $\mathfrak{U}$  stands for the universe of discourse, fixed within the framework of the theory of  $\mathbf{g}\mathfrak{T}_{\mathbf{g}}$ -sets and containing as elements all sets  $(\Omega, \Gamma\text{-sets}; \mathcal{T}, \mathbf{g}\mathcal{T}, \mathfrak{T}, \mathbf{g}\mathfrak{T}\text{-sets}; \mathcal{T}_{\mathbf{g}}, \mathbf{g}\mathcal{T}_{\mathbf{g}}, \mathfrak{T}_{\mathbf{g}}, \mathbf{g}\mathfrak{T}_{\mathbf{g}}\text{-sets})$  considered in this theory, and  $I_n^0 := \{\nu \in \mathbb{N}^0 : \nu \leq n\}$ ; index sets  $I_{\infty}^0, I_n^*, I_{\infty}^*$  are defined similarly. A set  $\Gamma \subset \mathfrak{U}$  is a subset of the set  $\Omega \subset \mathfrak{U}$  and, for some  $\mathcal{T}_{\mathbf{g}}$ -open set  $\mathcal{O}_{\mathbf{g}} \in \mathcal{T} \cup \mathbf{g}\mathcal{T} \cup \mathcal{T}_{\mathbf{g}} \cup \mathbf{g}\mathcal{T}_{\mathbf{g}}$ , these implications hold:

$$\mathcal{O}_{\mathbf{g}} \in \mathcal{T} \Rightarrow \mathcal{O}_{\mathbf{g}} \in \mathbf{g}\mathcal{T} \Rightarrow \mathcal{O}_{\mathbf{g}} \in \mathcal{T}_{\mathbf{g}} \Rightarrow \mathcal{O}_{\mathbf{g}} \in \mathbf{g}\mathcal{T}_{\mathbf{g}} \Rightarrow \mathcal{O}_{\mathbf{g}} \subset \Omega \subset \mathfrak{U} \quad (1)$$

In a natural way, a monotonic map  $\mathcal{T}_{\mathbf{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$  from the power set  $\mathcal{P}(\Omega)$  of  $\Omega$  into itself can be associated to a given mapping  $\pi_{\mathbf{g}} : \Omega \longrightarrow \Omega$ , thereby inducing a  $\mathbf{g}$ -topology  $\mathcal{T}_{\mathbf{g}} \subset \mathcal{P}(\Omega)$  on the underlying set  $\Omega$  [9]. Therefore, the definition of a  $\mathcal{T}_{\mathbf{g}}$ -space can be presented in a nice way. Thus, retaining the axioms to be satisfied by its  $\mathbf{g}$ -topology [33], and assuming no separation axioms, unless otherwise stated, the following definition is suggestive:

**Definition 2.1** ( $\mathcal{T}_{\mathbf{g}}$ -Space [39]). Let  $\Omega \subset \mathfrak{U}$  be a given set and let  $\mathcal{P}(\Omega) := \{\mathcal{O}_{\mathbf{g},\nu} : \mathcal{O}_{\mathbf{g},\nu} \subseteq \Omega\}$  be the family of all subsets  $\mathcal{O}_{\mathbf{g},1}, \mathcal{O}_{\mathbf{g},2}, \dots$ , of  $\Omega$ . Then, every one-valued map of the type  $\mathcal{T}_{\mathbf{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$  satisfying the following axioms:

$$i. \mathcal{T}_{\mathbf{g}}(\emptyset) = \emptyset$$

$$ii. \mathcal{T}_{\mathbf{g}}(\mathcal{O}_{\mathbf{g}}) \subseteq \mathcal{O}_{\mathbf{g}}$$

$$iii. \mathcal{T}_{\mathfrak{g}}(\bigcup_{\nu \in I_{\infty}^*} \mathcal{O}_{\mathfrak{g},\nu}) = \bigcup_{\nu \in I_{\infty}^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$$

is called a “ $\mathfrak{g}$ -topology on  $\Omega$ ,” and the structure  $\mathfrak{T}_{\mathfrak{g}} := (\Omega, \mathcal{T}_{\mathfrak{g}})$  is called a “ $\mathcal{T}_{\mathfrak{g}}$ -space.”

In Definition 2.1, by Ax. *i.*, Ax. *ii.*, and Ax. *iii.*, respectively, are meant that the unary operation  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  preserves nullary union, is contracting and preserves binary union. Any element  $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Omega)$  of the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$  is called a  $\mathcal{T}_{\mathfrak{g}}$ -open set and its complement element  $\mathcal{C}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \mathcal{K}_{\mathfrak{g}} \notin \mathcal{T}_{\mathfrak{g}}(\Omega)$  is called a  $\mathcal{T}_{\mathfrak{g}}$ -closed set. If there exists a  $\nu \in I_{\infty}^*$  such that  $\mathcal{O}_{\mathfrak{g},\nu} = \Omega$ , then  $\mathfrak{T}_{\mathfrak{g}}$  is called a strong  $\mathcal{T}_{\mathfrak{g}}$ -space [9,19]. Moreover, if the relation  $\mathcal{T}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}) = \bigcap_{\nu \in I_n^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$  holds for any index set  $I_n^* \subset I_{\infty}^*$  such that  $n < \infty$ , then  $\mathfrak{T}_{\mathfrak{g}}$  is called a quasi  $\mathcal{T}_{\mathfrak{g}}$ -space [17].

**Definition 2.2** ( $\mathfrak{g}$ -Closure,  $\mathfrak{g}$ -Interior Operators [39]). Let  $\mathfrak{T}_{\mathfrak{g}}$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space on the set  $\Omega \subset \mathfrak{U}$  with a  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ . Then,

- i.* The operator  $\text{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  carrying each  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  into its closure  $\text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$  is called a “ $\mathfrak{g}$ -closure operator.”
- ii.* The operator  $\text{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  carrying each  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  into its interior  $\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} \setminus \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$  is called a “ $\mathfrak{g}$ -interior operator.”

By convention, we let  $\mathcal{T}_{\mathfrak{g}}(\Omega)$  and  $\neg\mathcal{T}_{\mathfrak{g}}(\Omega)$ , respectively, stand for the classes of all  $\mathcal{T}_{\mathfrak{g}}$ -open and  $\mathcal{T}_{\mathfrak{g}}$ -closed sets relative to the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}$ . Their proper definitions are contained below.

**Definition 2.3** (Classes:  $\mathcal{T}_{\mathfrak{g}}$ -Open,  $\mathcal{T}_{\mathfrak{g}}$ -Closed Sets [39]). Let  $\mathfrak{T}_{\mathfrak{g}}$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space, let  $\mathcal{C} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  denotes the absolute complement with respect to the underlying set  $\Omega \subset \mathfrak{U}$ , and let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be any  $\mathfrak{T}_{\mathfrak{g}}$ -set. The classes

$$\mathcal{T}_{\mathfrak{g}}(\Omega) := \{\mathcal{O}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}\} \quad \text{and} \quad \neg\mathcal{T}_{\mathfrak{g}}(\Omega) := \{\mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : \mathcal{C}(\mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}}\} \quad (2)$$

respectively, denote the classes of all  $\mathcal{T}_{\mathfrak{g}}$ -open and  $\mathcal{T}_{\mathfrak{g}}$ -closed sets relative to the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}$ , and the classes

$$\mathcal{C}_{\mathcal{T}_{\mathfrak{g}}}^{\text{sub}}[\mathcal{S}_{\mathfrak{g}}] := \{\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}} : \mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}\} \quad \text{and} \quad \mathcal{C}_{\neg\mathcal{T}_{\mathfrak{g}}}^{\text{sup}}[\mathcal{S}_{\mathfrak{g}}] := \{\mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g}} : \mathcal{K}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}}\} \quad (3)$$

respectively, denote the classes of  $\mathcal{T}_{\mathfrak{g}}$ -open subsets and  $\mathcal{T}_{\mathfrak{g}}$ -closed supersets (complements of the  $\mathcal{T}_{\mathfrak{g}}$ -open subsets) of the  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  relative to the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}$ .

That  $\mathcal{C}_{\mathcal{T}_{\mathfrak{g}}}^{\text{sub}}[\mathcal{S}_{\mathfrak{g}}] \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega)$  and  $\neg\mathcal{T}_{\mathfrak{g}}(\Omega) \supseteq \mathcal{C}_{\neg\mathcal{T}_{\mathfrak{g}}}^{\text{sup}}[\mathcal{S}_{\mathfrak{g}}]$  are true for the  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  in question are clear from the context. To this end, the  $\mathfrak{g}$ -closure and the  $\mathfrak{g}$ -interior of a  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space define themselves as

$$\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) := \bigcup_{\mathcal{O}_{\mathfrak{g}} \in \mathcal{C}_{\mathcal{T}_{\mathfrak{g}}}^{\text{sub}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}} \quad \text{and} \quad \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) := \bigcap_{\mathcal{K}_{\mathfrak{g}} \in \mathcal{C}_{\neg\mathcal{T}_{\mathfrak{g}}}^{\text{sup}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{K}_{\mathfrak{g}} \quad (4)$$

We note in passing that,  $\text{cl}_{\mathfrak{g}}(\cdot) \neq \text{cl}(\cdot)$  and  $\text{int}_{\mathfrak{g}}(\cdot) \neq \text{int}(\cdot)$ , because the resulting sets obtained from the intersection of all  $\mathcal{T}_{\mathfrak{g}}$ -closed supersets and the union of all  $\mathcal{T}_{\mathfrak{g}}$ -open subsets, respectively, relative to the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}$  are not necessarily equal to those which would be obtained from the intersection of all  $\mathcal{T}$ -closed supersets and the union of all  $\mathcal{T}$ -open subsets relative to the topology  $\mathcal{T}$  [23]. Throughout this work, by  $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\cdot)$ ,  $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot)$ , and  $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot)$ , respectively, are meant  $\text{cl}_{\mathfrak{g}}(\text{int}_{\mathfrak{g}}(\cdot))$ ,  $\text{int}_{\mathfrak{g}}(\text{cl}_{\mathfrak{g}}(\cdot))$ , and  $\text{cl}_{\mathfrak{g}}(\text{int}_{\mathfrak{g}}(\text{cl}_{\mathfrak{g}}(\cdot)))$ ; other composition operators are defined in a similar way. Also, the backslash  $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}$  refers to the set-theoretic relative complement of  $\mathcal{S}_{\mathfrak{g}}$  in  $\mathfrak{T}_{\mathfrak{g}}$ . Finally, for convenience of notation, let  $\mathcal{P}^*(\Omega) = \mathcal{P}(\Omega) \setminus \{\emptyset\}$ ,  $\mathcal{T}_{\mathfrak{g}}^* = \mathcal{T}_{\mathfrak{g}} \setminus \{\emptyset\}$ , and  $\neg\mathcal{T}_{\mathfrak{g}}^* = \neg\mathcal{T}_{\mathfrak{g}} \setminus \{\emptyset\}$ .

**Definition 2.4** ( $\mathfrak{g}$ -Operation [39]). Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space. Then, a mapping  $\text{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  on  $\mathcal{P}(\Omega)$  ranging in  $\mathcal{P}(\Omega)$  is called a “ $\mathfrak{g}$ -operation” if and only if the following statements hold:

$$(\forall \mathcal{S}_{\mathfrak{g}} \in \mathcal{P}^*(\Omega)) (\exists (\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g}}^* \times \neg\mathcal{T}_{\mathfrak{g}}^*) [(\text{op}_{\mathfrak{g}}(\emptyset) = \emptyset) \vee (\neg\text{op}_{\mathfrak{g}}(\emptyset) = \emptyset) \vee (\mathcal{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))] \quad (5)$$

where  $\neg\text{op}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  is called the “complementary  $\mathfrak{g}$ -operation” on  $\mathcal{P}(\Omega)$  ranging in  $\mathcal{P}(\Omega)$  and, for all  $(\mathcal{S}_{\mathfrak{g}}, \mathcal{U}_{\mathfrak{g},\mu}, \mathcal{V}_{\mathfrak{g},\nu}) \in \bigotimes_{\alpha \in I_3^*} \mathcal{P}^*(\Omega)$  such that  $\mathcal{W}_{\mathfrak{g}} = \mathcal{U}_{\mathfrak{g},\mu} \cup \mathcal{V}_{\mathfrak{g},\nu}$  and  $(\hat{\mathcal{W}}_{\mathfrak{g}}, \neg\hat{\mathcal{W}}_{\mathfrak{g}}) = (\text{op}_{\mathfrak{g}}(\mathcal{W}_{\mathfrak{g}}), \neg\text{op}_{\mathfrak{g}}(\mathcal{W}_{\mathfrak{g}}))$ , the following axioms are satisfied:

- i.  $(\mathcal{S}_g \subseteq \text{op}_g(\mathcal{O}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g(\mathcal{K}_g))$
  - ii.  $(\text{op}_g(\mathcal{S}_g) \subseteq \text{op}_g \circ \text{op}_g(\mathcal{O}_g)) \vee (\neg \text{op}_g(\mathcal{S}_g) \supseteq \neg \text{op}_g \circ \neg \text{op}_g(\mathcal{K}_g))$
  - iii.  $\left( \hat{\mathcal{W}}_g \subseteq \bigcup_{\sigma=\mu,\nu} \text{op}_g(\mathcal{O}_{g,\sigma}) \right) \vee \left( \neg \hat{\mathcal{W}}_g \supseteq \bigcup_{\sigma=\mu,\nu} \neg \text{op}_g(\mathcal{K}_{g,\sigma}) \right)$
  - iv.  $(\mathcal{U}_{g,\mu} \subseteq \mathcal{V}_{g,\nu} \longrightarrow \text{op}_g(\mathcal{O}_{g,\mu}) \subseteq \text{op}_g(\mathcal{O}_{g,\nu})) \vee (\mathcal{U}_{g,\mu} \supseteq \mathcal{V}_{g,\nu} \longleftarrow \neg \text{op}_g(\mathcal{K}_{g,\mu}) \supseteq \neg \text{op}_g(\mathcal{K}_{g,\nu}))$
- for some  $(\mathcal{O}_g, \mathcal{O}_{g,\mu}, \mathcal{O}_{g,\nu}) \in \bigotimes_{\alpha \in I_3^*} \mathcal{T}_g^*$  and  $(\mathcal{K}_g, \mathcal{K}_{g,\mu}, \mathcal{K}_{g,\nu}) \in \bigotimes_{\alpha \in I_3^*} \neg \mathcal{T}_g^*$ .

The formulation of Definition 2.5 is based on the axioms of the Čech closure operator [25] and the various axioms used by many mathematicians to define closure operators [36]. The class  $\mathcal{L}_g[\Omega]$  stands for the class of all possible  $g$ -operators and their complementary  $g$ -operators in the  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ .

**Definition 2.5** ( $\text{op}_g(\cdot)$ -Elements [39]). Let  $\mathfrak{T}_g$  be a  $\mathcal{T}_g$ -space. The elements of the class  $\mathcal{L}_g[\Omega] = \mathcal{L}_g^\omega[\Omega] \times \mathcal{L}_g^\kappa[\Omega]$ , where

$$\mathcal{L}_g[\Omega] := \{ \mathbf{op}_{g,\nu\mu}(\cdot) = (\text{op}_{g,\nu}(\cdot), \neg \text{op}_{g,\mu}(\cdot)) : (\nu, \mu) \in I_3^0 \times I_3^0 \} \quad (6)$$

in the  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$  are defined as:

$$\begin{aligned} \text{op}_g(\cdot) &\in \mathcal{L}_g^\omega[\Omega] := \{ \text{op}_{g,0}(\cdot), \text{op}_{g,1}(\cdot), \text{op}_{g,2}(\cdot), \text{op}_{g,3}(\cdot) \} \\ &= \{ \text{int}_g(\cdot), \text{cl}_g \circ \text{int}_g(\cdot), \text{int}_g \circ \text{cl}_g(\cdot), \text{cl}_g \circ \text{int}_g \circ \text{cl}_g(\cdot) \} \\ \neg \text{op}_g(\cdot) &\in \mathcal{L}_g^\kappa[\Omega] := \{ \neg \text{op}_{g,0}(\cdot), \neg \text{op}_{g,1}(\cdot), \neg \text{op}_{g,2}(\cdot), \neg \text{op}_{g,3}(\cdot) \} \\ &= \{ \text{cl}_g(\cdot), \text{int}_g \circ \text{cl}_g(\cdot), \text{cl}_g \circ \text{int}_g(\cdot), \text{int}_g \circ \text{cl}_g \circ \text{int}_g(\cdot) \} \end{aligned} \quad (7)$$

We remark in passing that,  $\mathbf{op}_{g,11}(\cdot) = \neg \mathbf{op}_{g,22}(\cdot)$ , and the use of  $\mathbf{op}_g(\cdot) = (\text{op}_g(\cdot), \neg \text{op}_g(\cdot)) \in \mathcal{L}_g[\Omega]$  on a class of  $\mathfrak{T}_g$ -sets will construct a new class of  $g$ - $\mathfrak{T}_g$ -sets, just as the use of  $\mathcal{L}[\Omega] := \{ \mathbf{op}_\nu(\cdot) = (\text{op}_\nu(\cdot), \neg \text{op}_\nu(\cdot)) : \nu \in I_3^0 \}$  on the class of  $\mathfrak{T}$ -sets have constructed the new class of  $g$ - $\mathfrak{T}$ -sets. But since  $\text{cl}_g(\cdot) \neq \text{cl}(\cdot)$  and  $\text{int}_g(\cdot) \neq \text{int}(\cdot)$ , in general, it follows that  $\mathbf{op}_g(\cdot) \neq \mathbf{op}(\cdot)$  and, therefore, the new class of  $g$ - $\mathfrak{T}_g$ -sets that will be obtained from the first construction will, in general, differ from the new class of  $g$ - $\mathfrak{T}$ -sets that had been obtained from the second construction.

**Definition 2.6** ( $g$ - $\nu$ - $\mathfrak{T}_g$ -Set [39]). A  $\mathfrak{T}_g$ -set  $\mathcal{S}_g \subset \mathfrak{T}_g$  in a  $\mathcal{T}_g$ -space is called a “ $g$ - $\mathfrak{T}_g$ -set” if and only if there exist a pair  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathcal{T}_g \times \neg \mathcal{T}_g$  of  $\mathcal{T}_g$ -open and  $\mathcal{T}_g$ -closed sets, and a  $g$ -operator  $\mathbf{op}_g(\cdot) \in \mathcal{L}_g[\Omega]$  such that the following statement holds:

$$(\exists \xi) [(\xi \in \mathcal{S}_g) \wedge ((\mathcal{S}_g \subseteq \text{op}_g(\mathcal{O}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g(\mathcal{K}_g)))] \quad (8)$$

The  $g$ - $\mathfrak{T}_g$ -set  $\mathcal{S}_g \subset \mathfrak{T}_g$  is said to be of category  $\nu$  if and only if it belongs to the following class of  $g$ - $\nu$ - $\mathfrak{T}_g$ -sets:

$$g\text{-}\nu\text{-S}[\mathfrak{T}_g] := \{ \mathcal{S}_g \subset \mathfrak{T}_g : (\exists \mathcal{O}_g, \mathcal{K}_g, \mathbf{op}_{g,\nu}(\cdot)) [(\mathcal{S}_g \subseteq \text{op}_{g,\nu}(\mathcal{O}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_{g,\nu}(\mathcal{K}_g))] \} \quad (9)$$

It is called a  $g$ - $\nu$ - $\mathfrak{T}_g$ -open set if it satisfies the first property in  $g\text{-}\nu\text{-S}[\mathfrak{T}_g]$  and a  $g$ - $\nu$ - $\mathfrak{T}_g$ -closed set if it satisfies the second property in  $g\text{-}\nu\text{-S}[\mathfrak{T}_g]$ . The classes of  $g$ - $\nu$ - $\mathfrak{T}_g$ -open and  $g$ - $\nu$ - $\mathfrak{T}_g$ -closed sets, respectively, are defined by

$$\begin{aligned} g\text{-}\nu\text{-O}[\mathfrak{T}_g] &:= \{ \mathcal{S}_g \subset \mathfrak{T}_g : (\exists \mathcal{O}_g, \mathbf{op}_{g,\nu}(\cdot)) [\mathcal{S}_g \subseteq \text{op}_{g,\nu}(\mathcal{O}_g)] \} \\ g\text{-}\nu\text{-K}[\mathfrak{T}_g] &:= \{ \mathcal{S}_g \subset \mathfrak{T}_g : (\exists \mathcal{K}_g, \mathbf{op}_{g,\nu}(\cdot)) [\mathcal{S}_g \supseteq \neg \text{op}_{g,\nu}(\mathcal{K}_g)] \} \end{aligned} \quad (10)$$

From the class  $g\text{-}\nu\text{-S}[\mathfrak{T}_g]$ , consisting of the classes  $g\text{-}\nu\text{-O}[\mathfrak{T}_g]$  and  $g\text{-}\nu\text{-K}[\mathfrak{T}_g]$ , respectively, of  $g$ - $\nu$ - $\mathfrak{T}_g$ -open and  $g$ - $\nu$ - $\mathfrak{T}_g$ -closed sets of category  $\nu$ , where  $\nu \in I_3^0$ , there results in the following definition.

**Definition 2.7** ( $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\text{-Set}$  [39]). Let  $\mathfrak{T}_{\mathfrak{g}}$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space. If, for each  $\nu \in I_3^0$ ,  $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , respectively, denote the classes of  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets of category  $\nu$  then,

$$\begin{aligned}\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] &= \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} (\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]) \\ &= (\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]) \cup (\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]) \\ &= \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\end{aligned}\quad (11)$$

In the sequel, it is interesting to view the concepts of open, semi-open, pre-open, semi-pre-open sets as  $\mathfrak{g}\text{-}\mathfrak{T}$ -open sets of categories 0, 1, 2, and 3; likewise, to view the concepts of closed, semi-closed, pre-closed, semi-pre-closed sets as  $\mathfrak{g}\text{-}\mathfrak{T}$ -closed sets of categories 0, 1, 2, and 3. These can be realised by omitting the subscript “ $\mathfrak{g}$ ” in all symbols of the above definitions.

**Definition 2.8** ( $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}\text{-Set}$  [39]). A  $\mathfrak{T}$ -set  $\mathcal{S} \subset \mathfrak{T}$  in a  $\mathcal{T}$ -space is called a “ $\mathfrak{g}\text{-}\mathfrak{T}$ -set” if and only if there exists a pair  $(\mathcal{O}, \mathcal{K}) \in \mathcal{T} \times \neg\mathcal{T}$  of  $\mathcal{T}$ -open and  $\mathcal{T}$ -closed sets, and an operator  $\mathbf{op}(\cdot) \in \mathcal{L}[\Omega]$  such that the following statement holds:

$$(\exists \xi) [(\xi \in \mathcal{S}) \wedge ((\mathcal{S} \subseteq \mathbf{op}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg \mathbf{op}(\mathcal{K})))] \quad (12)$$

The  $\mathfrak{g}\text{-}\mathfrak{T}$ -set  $\mathcal{S} \subset \mathfrak{T}$  is said to be of category  $\nu$  if and only if it belongs to the following class of  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}$ -sets:

$$\mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}] := \{\mathcal{S} \subset \mathfrak{T} : (\exists \mathcal{O}, \mathcal{K}, \mathbf{op}_{\nu}(\cdot)) [(\mathcal{S} \subseteq \mathbf{op}_{\nu}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg \mathbf{op}_{\nu}(\mathcal{K}))]\} \quad (13)$$

It is called a  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -open set if it satisfies the first property in  $\mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}]$  and a  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -closed set if it satisfies the second property in  $\mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}]$ . The classes of  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -open and  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -closed sets, respectively, are defined by

$$\begin{aligned}\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] &:= \{\mathcal{S} \subset \mathfrak{T} : (\exists \mathcal{O}, \mathbf{op}_{\nu}(\cdot)) [\mathcal{S} \subseteq \mathbf{op}_{\nu}(\mathcal{O})]\} \\ \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] &:= \{\mathcal{S} \subset \mathfrak{T} : (\exists \mathcal{K}, \mathbf{op}_{\nu}(\cdot)) [\mathcal{S} \supseteq \neg \mathbf{op}_{\nu}(\mathcal{K})]\}\end{aligned}\quad (14)$$

As in the previous definitions, from the class  $\mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}]$ , consisting of the classes  $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}]$  and  $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]$ , respectively, of  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -open and  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -closed sets of category  $\nu$ , where  $\nu \in I_3^0$ , there results in the following definition.

**Definition 2.9** (Class:  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\text{-Sets}$  [39]). Let  $\mathfrak{T}$  be a  $\mathcal{T}$ -space. If, for each  $\nu \in I_3^0$ ,  $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}]$  and  $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]$ , respectively, denote the classes of  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -open and  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}$ -closed sets of category  $\nu$  then,

$$\begin{aligned}\mathfrak{g}\text{-S}[\mathfrak{T}] &= \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} (\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}] \cup \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]) \\ &= (\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}]) \cup (\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]) \\ &= \mathfrak{g}\text{-O}[\mathfrak{T}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}]\end{aligned}\quad (15)$$

The classes of  $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$  as well as the classes of  $\mathfrak{T}$ -open and  $\mathfrak{T}$ -closed sets in a  $\mathcal{T}$ -space  $\mathfrak{T}$  are defined as thus:

**Definition 2.10** (Families:  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\text{-Open Sets}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\text{-Closed Sets}$  [39]). Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space and let  $\mathfrak{T} = (\Omega, \mathcal{T})$  be a  $\mathcal{T}$ -space.

- i. The classes  $\mathbf{O}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathbf{K}[\mathfrak{T}_{\mathfrak{g}}]$  denote the families of  $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets, respectively, in  $\mathfrak{T}_{\mathfrak{g}}$ , with  $\mathbf{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathbf{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathbf{K}[\mathfrak{T}_{\mathfrak{g}}]$ .
- ii. The classes  $\mathbf{O}[\mathfrak{T}]$  and  $\mathbf{K}[\mathfrak{T}]$  denote the families of  $\mathfrak{T}$ -open and  $\mathfrak{T}$ -closed sets, respectively, in  $\mathfrak{T}$ , with  $\mathbf{S}[\mathfrak{T}] = \mathbf{O}[\mathfrak{T}] \cup \mathbf{K}[\mathfrak{T}]$ .

In the following sections, the main results of the theory of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets are presented.

### 3. Main Results

**Theorem 3.1.** Let  $\text{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$  and  $\text{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ , respectively, be  $\mathfrak{g}$ -closure and  $\mathfrak{g}$ -interior operators in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ . Then,

- i.  $\text{cl}_{\mathfrak{g}}(\cdot)$  and  $\text{int}_{\mathfrak{g}}(\cdot)$  are enhancing and contracting, respectively.
- ii.  $\text{cl}_{\mathfrak{g}}(\cdot)$  and  $\text{int}_{\mathfrak{g}}(\cdot)$  are idempotent.
- iii.  $\text{cl}_{\mathfrak{g}}(\cdot)$  and  $\text{int}_{\mathfrak{g}}(\cdot)$  are monotone.

PROOF.

- i. Since the following logical statement

$$\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : (\forall \xi) [(\xi \in \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \longleftarrow \xi \in \mathcal{S}_{\mathfrak{g}}) \vee (\xi \in \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \longrightarrow \xi \in \mathcal{S}_{\mathfrak{g}})]$$

holds, it follows that  $\mathcal{S}_{\mathfrak{g}} \subseteq \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  or  $\mathcal{S}_{\mathfrak{g}} \supseteq \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ .

- ii. If  $\mathcal{S}_{\mathfrak{g}}$  is open, then  $\mathcal{S}_{\mathfrak{g}} = \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ ; if it is closed,  $\mathcal{S}_{\mathfrak{g}} = \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ . Consequently, the substitutions  $\mathcal{S}_{\mathfrak{g}} \longmapsto \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  and  $\mathcal{S}_{\mathfrak{g}} \longmapsto \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ , respectively, give  $\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \text{int}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  and  $\text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \text{cl}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ .
- iii. Let  $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  such that  $\mathcal{R}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$ . Then,  $\mathcal{R}_{\mathfrak{g}} \subseteq \text{cl}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}})$ ,  $\mathcal{R}_{\mathfrak{g}} \supseteq \text{int}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}})$ ,  $\mathcal{S}_{\mathfrak{g}} \subseteq \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ , and  $\mathcal{S}_{\mathfrak{g}} \supseteq \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  by i. Consequently,  $\text{int}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}}) \subseteq \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  and  $\text{cl}_{\mathfrak{g}}(\mathcal{R}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ .

□

**Lemma 3.2.** Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set of a  $\mathcal{T}_{\mathfrak{g}}$ -space. Then,

- i.  $(\mathcal{S}_{\mathfrak{g}} = \emptyset) \wedge (\Omega \in \mathcal{T}_{\mathfrak{g}}) \Rightarrow (\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \emptyset) \wedge (\text{cl}_{\mathfrak{g}}(\emptyset) = \emptyset)$
- ii.  $(\mathcal{S}_{\mathfrak{g}} = \emptyset) \wedge (\Omega \notin \mathcal{T}_{\mathfrak{g}}) \Rightarrow (\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) = \emptyset) \wedge (\text{cl}_{\mathfrak{g}}(\emptyset) \neq \emptyset)$

PROOF.

- i. If  $\mathcal{S}_{\mathfrak{g}} = \emptyset$  and  $\Omega \in \mathcal{T}_{\mathfrak{g}}$ , then  $(\emptyset \in C_{\mathcal{T}_{\mathfrak{g}}}^{\text{sub}}[\emptyset]) \wedge (\emptyset \in C_{\mathcal{T}_{\mathfrak{g}}}^{\text{sup}}[\emptyset])$ . Consequently,  $\text{int}_{\mathfrak{g}}(\emptyset) = \emptyset$  and  $\text{cl}_{\mathfrak{g}}(\emptyset) = \emptyset$ .
- ii. If  $\mathcal{S}_{\mathfrak{g}} = \emptyset$  and  $\Omega \notin \mathcal{T}_{\mathfrak{g}}$ , then  $(\emptyset \in C_{\mathcal{T}_{\mathfrak{g}}}^{\text{sub}}[\emptyset]) \wedge (\emptyset \notin C_{\mathcal{T}_{\mathfrak{g}}}^{\text{sup}}[\emptyset])$ . Consequently,  $\text{int}_{\mathfrak{g}}(\emptyset) = \emptyset$  and  $\text{int}_{\mathfrak{g}}(\emptyset) \neq \emptyset$ .

□

According to Sarsak [40] and Noiri [41], the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$  may be called a  $\mu$ -space when  $\text{cl}_{\mathfrak{g}}(\emptyset) = \emptyset$ .

**Theorem 3.3.** If  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$  are  $n \geq 1$   $\mathfrak{T}_{\mathfrak{g}}$ -sets of a  $\mathcal{T}_{\mathfrak{g}}$ -space, then,

- i.  $\text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) = \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$
- ii.  $\text{int}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) = \bigcap_{\nu \in I_n^*} \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$

PROOF. Expressed in set-builder notation, the  $\mathfrak{g}$ -closure and the  $\mathfrak{g}$ -interior of a  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space can also be defined as thus:

$$\begin{aligned} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) &:= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g}} \cap \text{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) &:= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g}} \cap \text{int}(\mathcal{O}_{\mathfrak{g}}) = \text{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \end{aligned}$$

respectively, from which it is easily seen that,

$$\begin{aligned}
 \text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &= \bigcup_{\nu \in I_n^*} \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g},\nu} \cap \text{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\
 &= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : ((\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \cap \text{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\
 &= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : (\bigcup_{\nu \in I_n^*} (\mathcal{S}_{\mathfrak{g},\nu} \cap \text{cl}(\mathcal{O}_{\mathfrak{g}})) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\
 &= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : \bigvee_{\nu \in I_n^*} ((\mathcal{S}_{\mathfrak{g},\nu} \cap \text{cl}(\mathcal{O}_{\mathfrak{g}}) \neq \emptyset) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})) \} \\
 &= \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})
 \end{aligned}$$

Likewise, it is also easily seen that,

$$\begin{aligned}
 \text{int}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &= \bigcap_{\nu \in I_n^*} \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g},\nu} \cap \text{int}(\mathcal{O}_{\mathfrak{g}}) = \text{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\
 &= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : ((\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \cap \text{int}(\mathcal{O}_{\mathfrak{g}}) = \text{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\
 &= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : (\bigcap_{\nu \in I_n^*} (\mathcal{S}_{\mathfrak{g},\nu} \cap \text{int}(\mathcal{O}_{\mathfrak{g}})) = \text{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}) \} \\
 &= \{ \xi \in \mathfrak{T}_{\mathfrak{g}} : \bigwedge_{\nu \in I_n^*} ((\mathcal{S}_{\mathfrak{g},\nu} \cap \text{int}(\mathcal{O}_{\mathfrak{g}}) = \text{int}(\mathcal{O}_{\mathfrak{g}})) \wedge (\xi \in \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}})) \} \\
 &= \bigcap_{\nu \in I_n^*} \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})
 \end{aligned}$$

□

Clearly,  $\mathcal{S}_{\mathfrak{g},\mu} \subseteq \bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}$  and  $\mathcal{S}_{\mathfrak{g},\mu} \supseteq \bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}$  hold true for any  $\mu \in I_n^*$ . The following corollary, then, is an immediate consequence of the above theorem.

**Corollary 3.4.** If  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$  are  $n \geq 1$   $\mathfrak{T}_{\mathfrak{g}}$ -sets of a  $\mathcal{T}_{\mathfrak{g}}$ -space, then,

- i.  $\text{cl}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \subseteq \bigcap_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$
- ii.  $\text{int}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \supseteq \bigcup_{\nu \in I_n^*} \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$

**Proposition 3.5.** For any  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , the following statement holds:

$$\mathfrak{T}_{\mathfrak{g}} \setminus (\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cup \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})) = \emptyset \quad (16)$$

PROOF. Let  $\xi \in \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})$ . Then,  $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}$  since,  $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}} \subseteq \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})$ . But,  $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})$  and, consequently,  $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ . Hence, there follows that,  $\text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subseteq \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ . Conversely, let  $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ . Then,  $\xi \in \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$ , since  $\mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$ . But, since  $\mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})$  and  $\text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$ , and, consequently,  $\xi \in \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$ . Hence,  $\mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}})$ . Since  $\text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  is equivalent to

$$(\text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \subseteq \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})) \wedge (\text{cl}_{\mathfrak{g}}(\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}) \supseteq \mathfrak{T}_{\mathfrak{g}} \setminus \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}))$$

the proof of the proposition at once follows. □

**Proposition 3.6.** Let  $\text{cl}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$  and  $\text{int}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ , respectively, be  $\mathfrak{g}$ -closure and  $\mathfrak{g}$ -interior operators in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ . If  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$  are  $n \geq 1$   $\mathfrak{T}_{\mathfrak{g}}$ -sets of the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , then,

- i.  $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \supseteq \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$
- ii.  $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \supseteq \bigcup_{\nu \in I_n^*} \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$
- iii.  $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \supseteq \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$



PROOF. Since the relations

$$\text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) = \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}), \quad \text{int}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \supseteq \bigcup_{\nu \in I_n^*} \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$$

hold, it follows that

$$\begin{aligned} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &\supseteq \text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})) \\ &= \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}) \\ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &\supseteq \text{int}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})) \\ &= \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}) \\ \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &= \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})) \\ &\supseteq \bigcup_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}) \end{aligned}$$

□

From the above proposition, it is obvious that their duals are

$$\begin{aligned} \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &\subseteq \bigcap_{\nu \in I_n^*} \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}) \\ \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &\subseteq \bigcap_{\nu \in I_n^*} \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}) \\ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) &\subseteq \bigcap_{\nu \in I_n^*} \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu}) \end{aligned} \quad (17)$$

respectively. On this basis, we have the following corollary:

**Corollary 3.7.** Let  $\text{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$  be a  $\mathfrak{g}$ -operator in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ . If  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \subset \mathfrak{T}_{\mathfrak{g}}$  are  $n \geq 1$   $\mathfrak{T}_{\mathfrak{g}}$ -sets of the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , then,

$$i. \text{ op}_{\mathfrak{g}} \circ \neg \text{op}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \supseteq \bigcup_{\nu \in I_n^*} \text{op}_{\mathfrak{g}} \circ \neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$$

$$ii. \neg \text{op}_{\mathfrak{g}} \circ \text{op}_{\mathfrak{g}}(\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}) \subseteq \bigcap_{\nu \in I_n^*} \neg \text{op}_{\mathfrak{g}} \circ \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\nu})$$

**Theorem 3.8.** If  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$  are  $n \geq 1$   $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets of a class  $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , then  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ .

PROOF. The statement  $\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$  for every  $\nu \in I_n^*$  is identical to the logical statement:

$$\exists (\mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathcal{T}_{\mathfrak{g}} \times \neg \mathcal{T}_{\mathfrak{g}} : (\mathcal{S}_{\mathfrak{g},\nu} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))$$

On the other hand, if  $\text{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$  is a  $\mathfrak{g}$ -operator in the  $\mathcal{T}_{\mathfrak{g}}$ -space, then

$$\begin{aligned} \text{op}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}) &= \bigcup_{\nu \in I_n^*} \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \\ \neg \text{op}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}) &= \bigcup_{\nu \in I_n^*} \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}) \end{aligned}$$

Consequently,

$$\begin{aligned} &\bigvee_{\nu \in I_n^*} ((\mathcal{S}_{\mathfrak{g},\nu} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))) \\ \Rightarrow &((\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \bigcup_{\nu \in I_n^*} \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \bigcup_{\nu \in I_n^*} \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))) \\ \Rightarrow &((\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \text{op}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu})) \vee (\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \text{op}_{\mathfrak{g}}(\bigcup_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}))) \end{aligned}$$

But,  $\bigcup_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu} \in \mathcal{T}_{\mathfrak{g}}$  and  $\bigcup_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu} \in \neg \mathcal{T}_{\mathfrak{g}}$ . Hence,  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$ .

□

**Theorem 3.9.** If  $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \dots, \mathcal{S}_{g,n} \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  are  $n \geq 1$   $\mathbf{g}\text{-}\mathfrak{T}_g$ -sets of a class  $\mathbf{g}\text{-S}[\mathfrak{T}_g]$  in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ , then

$$\left(\bigcap_{\nu \in I_n^*} \mathcal{S}_{g,\nu} \in \mathbf{g}\text{-S}[\mathfrak{T}_g]\right) \vee \left(\bigcap_{\nu \in I_n^*} \mathcal{S}_{g,\nu} \notin \mathbf{g}\text{-S}[\mathfrak{T}_g]\right) \quad (18)$$

PROOF. Because,  $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \dots, \mathcal{S}_{g,n} \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  by hypothesis, the trueness of  $\bigcap_{\nu \in I_n^*} \mathcal{S}_{g,\nu} \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  and  $\bigcap_{\nu \in I_n^*} \mathcal{S}_{g,\nu} \notin \mathbf{g}\text{-S}[\mathfrak{T}_g]$  evidently depend on the following property:

$$\bigwedge_{\nu \in I_n^*} ((\mathcal{S}_{g,\nu} \subseteq \text{op}_g(\mathcal{O}_{g,\nu})) \vee (\mathcal{S}_{g,\nu} \supseteq \neg \text{op}_g(\mathcal{K}_{g,\nu})))$$

where  $(\mathcal{O}_{g,\nu}, \mathcal{K}_{g,\nu}) \in \mathcal{T}_g \times \neg \mathcal{T}_g$  for every  $\nu \in I_n^*$ . Furthermore, because the  $\mathbf{g}\text{-}\mathfrak{T}_g$ -set-theoretic operations concern finite intersections, it suffices to prove the theorem for  $n = 2$ . Set the first property preceding  $\vee$  to  $P(\nu)$  and that following  $\vee$  to  $Q(\nu)$ . Then, its decomposition gives

$$\begin{aligned} \bigwedge_{\nu \in I_2^*} (P(\nu) \vee Q(\nu)) &= (\bigwedge_{\nu \in I_2^*} P(\nu)) \vee (\bigwedge_{\nu \in I_2^*} Q(\nu)) \\ &= (P(1) \wedge Q(2)) \vee (P(2) \wedge Q(1)) \end{aligned}$$

If  $\mathcal{S}_{g,1}, \mathcal{S}_{g,2} \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  are both  $\mathbf{g}\text{-}\mathfrak{T}_g$ -open sets then  $\bigwedge_{\nu \in I_2^*} P(\nu)$  is true, and if they are both  $\mathbf{g}\text{-}\mathcal{T}_g$ -closed sets then  $\bigwedge_{\nu \in I_2^*} Q(\nu)$  is true. In these two cases,  $\bigcap_{\nu \in I_2^*} \mathcal{S}_{g,\nu} \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ . Because, in general, there does not necessarily exists  $\mathbf{g}\text{-}\mathfrak{T}_g$ -set which is simultaneously  $\mathbf{g}\text{-}\mathcal{T}_g$ -open and  $\mathbf{g}\text{-}\mathcal{T}_g$ -closed, both  $P(1) \wedge Q(2)$  and  $P(2) \wedge Q(1)$  are untrue; thus,  $\bigcap_{\nu \in I_2^*} \mathcal{S}_{g,\nu} \notin \mathbf{g}\text{-S}[\mathfrak{T}_g]$ .  $\square$

**Theorem 3.10.** Let  $\mathcal{S}_g \subset \mathfrak{T}_g$  be a  $\mathfrak{T}_g$ -set and let  $\text{op}_g(\cdot) \in \mathcal{L}_g[\Omega]$  be a  $\mathbf{g}$ -operator in a  $\mathcal{T}_g$ -space. If  $\mathcal{S}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ , then

$$(\text{op}_g(\mathcal{S}_g) \in \mathbf{g}\text{-S}[\mathfrak{T}_g]) \vee (\neg \text{op}_g(\mathcal{S}_g) \in \mathbf{g}\text{-S}[\mathfrak{T}_g]) \quad (19)$$

PROOF. Let  $\mathcal{S}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ . Then,  $(\mathcal{S}_g \subseteq \text{op}_g(\mathcal{O}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g(\mathcal{K}_g))$  for some pair  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathcal{T}_g \times \neg \mathcal{T}_g$  of  $\mathcal{T}_g$ -open and  $\mathcal{T}_g$ -closed sets relative to  $\mathcal{T}_g$ . Consequently,  $\text{op}_g(\mathcal{S}_g) \subseteq \text{op}_g \circ \text{op}_g(\mathcal{O}_g)$  or  $\neg \text{op}_g(\mathcal{S}_g) \supseteq \neg \text{op}_g \circ \neg \text{op}_g(\mathcal{K}_g)$ . But,  $\text{op}_g \circ \text{op}_g(\mathcal{O}_g) \subseteq \text{op}_g(\mathcal{O}_g)$  and  $\neg \text{op}_g \circ \neg \text{op}_g(\mathcal{K}_g) \supseteq \neg \text{op}_g(\mathcal{K}_g)$ . Thus, there follows that  $\text{op}_g(\mathcal{S}_g) \subseteq \text{op}_g(\mathcal{O}_g)$  or  $\neg \text{op}_g(\mathcal{S}_g) \supseteq \neg \text{op}_g(\mathcal{K}_g)$ . Hence,  $\text{op}_g(\mathcal{S}_g) \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  or  $\neg \text{op}_g(\mathcal{S}_g) \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ .  $\square$

**Proposition 3.11.** Let  $\mathcal{S}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$  and suppose the logical statement

$$(\exists \mathcal{R}_g \subset \mathfrak{T}_g) [(\mathcal{R}_g \subseteq \text{op}_g(\mathcal{S}_g)) \vee (\mathcal{R}_g \supseteq \neg \text{op}_g(\mathcal{S}_g))] \quad (20)$$

holds, then  $\mathcal{R}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ .

PROOF. Let there exists a  $\mathfrak{T}_g$ -set  $\mathcal{R}_g \subset \mathfrak{T}_g$  such that  $\mathcal{R}_g \subseteq \text{op}_g(\mathcal{S}_g)$  or  $\mathcal{R}_g \supseteq \neg \text{op}_g(\mathcal{S}_g)$ . But  $\mathcal{S}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  implies  $\text{op}_g(\mathcal{S}_g) \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$  or  $\neg \text{op}_g(\mathcal{S}_g) \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ . Thus,  $\mathcal{R}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ .  $\square$

**Corollary 3.12.** Let  $\mathfrak{T}_g$  be a  $\mathcal{T}_g$ -space. If  $\mathbf{g}\text{-S}[\mathfrak{T}_g] = \mathbf{g}\text{-O}[\mathfrak{T}_g] \cup \mathbf{g}\text{-K}[\mathfrak{T}_g]$  denotes a class of  $\mathbf{g}\text{-}\mathfrak{T}_g$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_g$ -closed sets, and  $\mathbf{S}[\mathfrak{T}_g] = \mathbf{O}[\mathfrak{T}_g] \cup \mathbf{K}[\mathfrak{T}_g]$  denotes a class of  $\mathfrak{T}_g$ -open and  $\mathfrak{T}_g$ -closed sets, then

$$\mathbf{g}\text{-S}[\mathfrak{T}_g] \supseteq \mathbf{g}\text{-O}[\mathfrak{T}_g] \cup \mathbf{g}\text{-K}[\mathfrak{T}_g] \supseteq \mathbf{O}[\mathfrak{T}_g] \cup \mathbf{K}[\mathfrak{T}_g] \supseteq \mathbf{S}[\mathfrak{T}_g] \quad (21)$$

An important remark should be pointed out at this stage.

**Remark 3.13.** The converse of the statement “if  $\mathcal{S}_g \in \mathbf{S}[\mathfrak{T}_g]$  then  $\mathcal{S}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]$ ” is obviously untrue. Because, the negation of this statement gives

$$(\mathcal{S}_g \in \mathbf{S}[\mathfrak{T}_g]) \wedge (\neg (\mathcal{S}_g \in \mathbf{g}\text{-S}[\mathfrak{T}_g]))$$

which is an untrue statements.

**Theorem 3.14.** Let  $\mathfrak{T}_g$  be a  $\mathcal{T}_g$ -space. If  $\mathcal{S}_g \subset \mathfrak{T}_g$ , then

$$\mathcal{S}_g \in \mathfrak{g}\text{-S}[\mathfrak{T}_g] \Leftrightarrow (\mathcal{S}_g \subseteq \text{op}_g \circ \neg \text{op}_g(\mathcal{S}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g \circ \text{op}_g(\mathcal{S}_g)) \quad (22)$$

PROOF.

( $\Leftarrow$ ) : Let

$$(\mathcal{S}_g \subseteq \text{op}_g \circ \neg \text{op}_g(\mathcal{S}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g \circ \text{op}_g(\mathcal{S}_g))$$

Then, the substitution of  $\neg \text{op}_g(\mathcal{S}_g) = \mathcal{O}_g$  in the logical statement preceding  $\vee$  and  $\text{op}_g(\mathcal{S}_g) = \mathcal{K}_g$  in that following  $\vee$  gives  $(\mathcal{S}_g \subseteq \text{op}_g(\mathcal{O}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g(\mathcal{K}_g))$ .

( $\Rightarrow$ ) : Let  $\mathcal{S}_g \in \mathfrak{g}\text{-S}[\mathfrak{T}_g]$ . Then,  $(\mathcal{S}_g \subseteq \text{op}_g(\mathcal{O}_g)) \vee (\mathcal{S}_g \supseteq \neg \text{op}_g(\mathcal{K}_g))$ . Consequently, substituting  $\mathcal{O}_g = \neg \text{op}_g(\mathcal{S}_g)$  in the logical statement preceding  $\vee$  and  $\mathcal{K}_g = \text{op}_g(\mathcal{S}_g)$  in that following  $\vee$ , the required logical statement at once follows, which proves the theorem.  $\square$

The class  $\mathfrak{g}\text{-S}[\mathfrak{T}_g]$  forms a  $\mathfrak{g}$ -topology on  $\Omega$ , which will be denoted by  $\mathcal{T}_{g\text{-S}}$ .

**Theorem 3.15.** Let  $\mathfrak{g}\text{-S}[\mathfrak{T}_g]$  be a given  $\mathfrak{g}$ -class in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ . Then, the one-valued map  $\mathcal{T}_{g\text{-S}} : \mathfrak{g}\text{-S}[\mathfrak{T}_g] \longrightarrow \mathfrak{g}\text{-S}[\mathfrak{T}_g]$  forms a  $\mathfrak{g}$ -topology on  $\Omega$  in the  $\mathcal{T}_g$ -space.

PROOF. By definition,  $(\emptyset = \text{op}_g(\emptyset)) \vee (\emptyset = \neg \text{op}_g(\emptyset))$ . Since, either  $\text{op}_g(\emptyset) \subseteq \text{op}_g(\mathcal{O}_g)$  or  $\neg \text{op}_g(\emptyset) \supseteq \neg \text{op}_g(\mathcal{K}_g)$  holds, where  $\mathcal{O}_g, \mathcal{K}_g \subset \mathfrak{T}_g$ , respectively, are some  $\mathcal{T}_g$ -open and  $\mathcal{T}_g$ -closed sets in  $\mathfrak{T}_g$ , it follows that  $\emptyset \in \mathfrak{g}\text{-S}[\mathfrak{T}_g]$  and, hence,  $\mathcal{T}_{g\text{-S}}(\emptyset) = \emptyset$ . Let  $\mathcal{S}_g \in \mathfrak{g}\text{-S}[\mathfrak{T}_g]$ . Then, since  $\mathfrak{g}\text{-S}[\mathfrak{T}_g] \subseteq \mathfrak{g}\text{-S}[\mathfrak{T}_g]$ , it follows that  $\mathcal{S}_g$  is a superset of  $\mathcal{T}_{g\text{-S}}(\mathcal{S}_g)$ . Hence,  $\mathcal{T}_{g\text{-S}}(\mathcal{S}_g) \subseteq \mathcal{S}_g$ . Let  $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \dots$  be  $\mathfrak{T}_g$ -sets satisfying, for every  $\nu \in I_\infty^*$ ,  $\mathcal{S}_{g,\nu}$ . Then, there exist classes  $\{\mathcal{O}_{g,\nu} \in \mathcal{T}_g : \nu \in I_\infty^*\}$  and  $\{\mathcal{K}_{g,\nu} \in \neg \mathcal{T}_g : \nu \in I_\infty^*\}$ , respectively, of  $\mathcal{T}_g$ -open and  $\mathcal{T}_g$ -closed sets such that

$$(\bigcup_{\nu \in I_\infty^*} \mathcal{S}_{g,\nu} \subseteq \text{op}_g(\bigcup_{\nu \in I_\infty^*} \mathcal{O}_{g,\nu})) \vee (\bigcup_{\nu \in I_\infty^*} \mathcal{S}_{g,\nu} \supseteq \neg \text{op}_g(\bigcup_{\nu \in I_\infty^*} \mathcal{K}_{g,\nu}))$$

a relation established on the following expressions:

$$\begin{aligned} \bigcup_{\nu \in I_\infty^*} \text{op}_g(\mathcal{O}_{g,\nu}) &= \text{op}_g(\bigcup_{\nu \in I_\infty^*} \mathcal{O}_{g,\nu}) \\ \bigcup_{\nu \in I_\infty^*} \neg \text{op}_g(\mathcal{K}_{g,\nu}) &= \neg \text{op}_g(\bigcup_{\nu \in I_\infty^*} \mathcal{K}_{g,\nu}) \end{aligned}$$

Consequently,  $\bigcup_{\nu \in I_\infty^*} \mathcal{S}_{g,\nu} \in \mathfrak{g}\text{-S}[\mathfrak{T}_g]$ , since  $\bigcup_{\nu \in I_\infty^*} \mathcal{O}_{g,\nu} \in \mathcal{T}_g$  is a  $\mathcal{T}_g$ -open set and  $\bigcup_{\nu \in I_\infty^*} \mathcal{O}_{g,\nu} \in \neg \mathcal{T}_g$  is a  $\mathcal{T}_g$ -closed set. Hence,

$$\mathcal{T}_{g\text{-S}}(\bigcup_{\nu \in I_\infty^*} \mathcal{S}_{g,\nu}) = \bigcup_{\nu \in I_\infty^*} \mathcal{T}_{g\text{-S}}(\mathcal{S}_{g,\nu})$$

$\square$

An immediate consequence of the above theorem is the following corollary.

**Corollary 3.16.** Let a  $\mathfrak{T}_g$  be a  $\mathcal{T}_g$ -space. Then, the structure  $(\Omega, \mathcal{T}_{g\text{-S}})$ , where  $\mathcal{T}_{g\text{-S}} : \mathfrak{g}\text{-S}[\mathfrak{T}_g] \longrightarrow \mathfrak{g}\text{-S}[\mathfrak{T}_g]$ , is a  $\mathcal{T}_g$ -space.

To condense the set-builder notation describing the classes  $\mathfrak{g}\text{-S}[\mathfrak{T}_g]$  and then classify it into subclasses, predicates must be introduced, and the choice made is to consider the so-called *Boolean-valued functions* on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg \mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ , the definition of which are given below.

**Definition 3.17.** Let  $(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g) \in \mathfrak{T}_g \times \mathcal{T}_g \times \neg \mathcal{T}_g$  and let  $\mathbf{op}_g(\cdot) \in \mathcal{L}_g[\Omega]$  be a  $\mathfrak{g}$ -operator in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ . The first two predicates

$$\begin{aligned} P_g(\mathcal{S}_g, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq) &:= (\exists \mathcal{O}_g, \text{op}_g(\cdot)) (\mathcal{S}_g \subseteq \text{op}_g(\mathcal{O}_g)) \\ P_g(\mathcal{S}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq) &:= (\exists \mathcal{K}_g, \neg \text{op}_g(\cdot)) (\mathcal{S}_g \supseteq \neg \text{op}_g(\mathcal{K}_g)) \\ P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq) &:= P_g(\mathcal{S}_g, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq) \\ &\quad \vee P_g(\mathcal{S}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq) \end{aligned} \quad (23)$$

are called a Boolean-valued functions on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg \mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ .

In this respect,  $\mathbf{g}\text{-S}[\mathfrak{T}_{\mathbf{g}}] := \{\mathcal{S}_{\mathbf{g}} \subset \mathfrak{T}_{\mathbf{g}} : P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{O}_{\mathbf{g}}, \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq, \supseteq)\}$ . Moreover, employing the set-builder notations, the class of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets, denoted by  $\mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}]$  and  $\mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}]$ , respectively, may then be defined as thus:

**Definition 3.18.** Let  $\mathfrak{T}_{\mathbf{g}}$  be a  $\mathcal{T}_{\mathbf{g}}$ -space. The classes

$$\begin{aligned}\mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] &:= \{\mathcal{S}_{\mathbf{g}} \subset \mathfrak{T}_{\mathbf{g}} : P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{O}_{\mathbf{g}}, \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq)\} \\ \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}] &:= \{\mathcal{S}_{\mathbf{g}} \subset \mathfrak{T}_{\mathbf{g}} : P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{K}_{\mathbf{g}}, \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq)\}\end{aligned}\quad (24)$$

respectively, such that  $\mathbf{g}\text{-S}[\mathfrak{T}_{\mathbf{g}}] = \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] \cup \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}]$ , denote the families of all  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets in  $\mathfrak{T}_{\mathbf{g}}$ .

It is interesting to demonstrate their usefulness. In this direction, let us prove in a different way that  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -set-theoretic operations is closed under arbitrary unions.

$$\begin{aligned}P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{O}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq) &:= (\exists \mathcal{O}_{\mathbf{g}}, \mathbf{op}_{\mathbf{g}}(\cdot)) (\mathcal{S}_{\mathbf{g}} \subseteq \mathbf{op}_{\mathbf{g}}(\mathcal{O}_{\mathbf{g}})) \\ P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{K}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq) &:= (\exists \mathcal{K}_{\mathbf{g}}, \neg \mathbf{op}_{\mathbf{g}}(\cdot)) (\mathcal{S}_{\mathbf{g}} \supseteq \neg \mathbf{op}_{\mathbf{g}}(\mathcal{K}_{\mathbf{g}})) \\ P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{O}_{\mathbf{g}}, \mathcal{K}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq, \supseteq) &:= P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{O}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq) \\ &\quad \vee P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g}}, \mathcal{K}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq)\end{aligned}\quad (25)$$

**Theorem 3.19.** If  $\{\mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] : \nu \in I_n^*\}$  and  $\{\mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}] : \nu \in I_n^*\}$ , respectively, are finite collections of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets in a  $\mathcal{T}_{\mathbf{g}}$ -space  $\mathfrak{T}_{\mathbf{g}}$ , then

$$\begin{aligned}\bigcup_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_{\mathbf{g}} : (\exists \nu \in I_{\mu}^*) (\xi \in \mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}])\} &\subseteq \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] \\ \bigcap_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_{\mathbf{g}} : (\forall \nu \in I_{\mu}^*) (\xi \in \mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}])\} &\subseteq \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}]\end{aligned}\quad (26)$$

PROOF. Let  $\{\mathcal{R}_{\mathbf{g},\nu} \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] : \nu \in I_n^*\}$  and  $\{\mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}] : \nu \in I_n^*\}$ , respectively, be finite collections of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets in a  $\mathcal{T}_{\mathbf{g}}$ -space  $\mathfrak{T}_{\mathbf{g}}$ . Then, since  $(\mathcal{R}_{\mathbf{g},\nu}, \mathcal{S}_{\mathbf{g},\nu}) \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] \times \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}]$ , there exists  $(\mathcal{O}_{\mathbf{g},\nu}, \mathcal{K}_{\mathbf{g},\nu}) \in \mathcal{T}_{\mathbf{g}} \times \neg \mathcal{T}_{\mathbf{g}}$  such that the propositional formulas  $P_{\mathbf{g}}(\mathcal{R}_{\mathbf{g},\nu}, \mathcal{O}_{\mathbf{g},\nu}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq)$  and  $P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g},\nu}, \mathcal{K}_{\mathbf{g},\nu}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq)$  hold true for every index  $\nu \in I_n^*$ . Consequently, the propositional formulas  $\bigvee_{\nu \in I_n^*} P_{\mathbf{g}}(\mathcal{R}_{\mathbf{g},\nu}, \mathcal{O}_{\mathbf{g},\nu}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq)$  and  $\bigwedge_{\nu \in I_n^*} P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g},\nu}, \mathcal{K}_{\mathbf{g},\nu}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq)$  also hold true. Since

$$\begin{aligned}\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu} &\longleftrightarrow \bigcup_{\mu \in I_n^*} \{\mathcal{R}_{\mathbf{g},\nu} \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}] : \nu \in I_{\mu}^*\} \\ &\longleftrightarrow \bigcup_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_{\mathbf{g}} : (\exists \nu \in I_{\mu}^*) (\xi \in \mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}])\} \subseteq \bigcup_{\nu \in I_n^*} \mathbf{op}_{\mathbf{g}}(\mathcal{O}_{\mathbf{g},\nu}) \\ \bigcup_{\nu \in I_n^*} \mathbf{op}_{\mathbf{g}}(\mathcal{O}_{\mathbf{g},\nu}) &\longleftrightarrow \bigcap_{\mu \in I_n^*} \{\mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}] : \nu \in I_{\mu}^*\} \\ &\longleftrightarrow \bigcap_{\mu \in I_n^*} \{\xi \in \mathfrak{T}_{\mathbf{g}} : (\forall \nu \in I_{\mu}^*) (\xi \in \mathcal{S}_{\mathbf{g},\nu} \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}])\} \supseteq \bigcap_{\nu \in I_n^*} \neg \mathbf{op}_{\mathbf{g}}(\mathcal{K}_{\mathbf{g},\nu})\end{aligned}$$

it results that  $\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu} \subseteq \bigcup_{\nu \in I_n^*} \mathbf{op}_{\mathbf{g}}(\mathcal{O}_{\mathbf{g},\nu})$  and  $\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathbf{g},\nu} \supseteq \bigcap_{\nu \in I_n^*} \neg \mathbf{op}_{\mathbf{g}}(\mathcal{K}_{\mathbf{g},\nu})$ . But it holds that  $\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu} \subseteq \mathbf{op}_{\mathbf{g}}(\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu}) \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}]$  and  $\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathbf{g},\nu} \supseteq \neg \mathbf{op}_{\mathbf{g}}(\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathbf{g},\nu}) \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}]$ . Consequently, there exists  $(\mathcal{O}_{\mathbf{g}}, \mathcal{K}_{\mathbf{g}}) \in \mathcal{T}_{\mathbf{g}} \times \neg \mathcal{T}_{\mathbf{g}}$  such that  $\bigcup_{\mu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu} \subseteq \mathbf{op}_{\mathbf{g}}(\mathcal{O}_{\mathbf{g}}) \in \mathbf{g}\text{-O}[\mathfrak{T}_{\mathbf{g}}]$  and  $\bigcap_{\mu \in I_n^*} \mathcal{S}_{\mathbf{g},\nu} \supseteq \neg \mathbf{op}_{\mathbf{g}}(\mathcal{K}_{\mathbf{g}}) \in \mathbf{g}\text{-K}[\mathfrak{T}_{\mathbf{g}}]$ , implying that both  $P_{\mathbf{g}}(\bigcup_{\nu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu}, \bigcup_{\nu \in I_n^*} \mathcal{O}_{\mathbf{g},\nu}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq)$  and  $P_{\mathbf{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathbf{g},\nu}, \bigcup_{\nu \in I_n^*} \mathcal{K}_{\mathbf{g},\nu}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq)$ , respectively, hold true. But,

$$\begin{aligned}P_{\mathbf{g}}(\bigcup_{\nu \in I_n^*} \mathcal{R}_{\mathbf{g},\nu}, \mathcal{O}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq) &= \bigvee_{\nu \in I_n^*} P_{\mathbf{g}}(\mathcal{R}_{\mathbf{g},\nu}, \mathcal{O}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \subseteq) \\ P_{\mathbf{g}}(\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathbf{g},\nu}, \mathcal{K}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq) &= \bigwedge_{\nu \in I_n^*} P_{\mathbf{g}}(\mathcal{S}_{\mathbf{g},\nu}, \mathcal{K}_{\mathbf{g}}; \mathbf{op}_{\mathbf{g}}(\cdot); \supseteq)\end{aligned}$$

Hence, it suffices to set

$$\begin{aligned} P_g(\mathcal{R}_g, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq) &= \bigvee_{\nu \in I_n^*} P_g(\mathcal{R}_{g,\nu}, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq) \\ P_g(\mathcal{S}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq) &= \bigvee_{\nu \in I_n^*} P_g(\mathcal{S}_{g,\nu}, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq) \end{aligned}$$

and the theorem is proved.  $\square$

If in  $P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq)$  it be assumed that  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathbf{g-S}[\mathfrak{T}_g] \times \mathbf{g-S}[\mathfrak{T}_g]$ , we have the following theorem:

**Theorem 3.20.** Let  $(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g) \in \mathfrak{T}_g \times \mathcal{T}_g \times \neg\mathcal{T}_g$  in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ . If  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathbf{g-O}[\mathfrak{T}_g] \times \mathbf{g-K}[\mathfrak{T}_g]$ , then

$$\{\mathcal{S}_g \subset \mathfrak{T}_g : P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq)\} \subseteq \mathbf{g-S}[\mathfrak{T}_g] \quad (27)$$

PROOF. It is clear that

$$P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq) = P_g(\mathcal{S}_g, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq) \vee P_g(\mathcal{S}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq)$$

and the Boolean-valued functions surrounding  $\vee$  hold on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg\mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ . Consequently, the following two cases must be considered in proving the theorem:

CASE I. Let  $P_g(\mathcal{S}_g, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq)$  hold on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg\mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ . Then,  $\mathcal{S}_g \subseteq \mathbf{op}_g(\mathcal{O}_g)$ . But,  $\mathcal{O}_g \in \mathbf{g-O}[\mathfrak{T}_g]$ , and consequently, it follows that  $\mathcal{O}_g \subseteq \mathbf{op}_g(\mathcal{O}_{g,\nu})$  and  $\mathbf{op}_g(\mathcal{O}_g) \subseteq \mathbf{op}_g \circ \mathbf{op}_g(\mathcal{O}_{g,\nu}) \subseteq \mathbf{op}_g(\mathcal{O}_{g,\nu})$  for some  $\mathcal{O}_{g,\nu} \in \mathcal{T}_g$ , by the properties of the  $\mathbf{g}$ -operator. Hence,  $P_g(\mathcal{S}_g, \mathcal{O}_{g,\nu}; \mathbf{op}_g(\cdot); \subseteq)$  holds on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg\mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ .

CASE II. Let  $P_g(\mathcal{S}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq)$  hold on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg\mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ . Then,  $\mathcal{S}_g \supseteq \neg\mathbf{op}_g(\mathcal{K}_g)$ . But,  $\mathcal{K}_g \in \mathbf{g-K}[\mathfrak{T}_g]$ , and consequently, it follows that  $\mathcal{K}_g \supseteq \neg\mathbf{op}_g(\mathcal{K}_{g,\nu})$  and  $\mathbf{op}_g(\mathcal{K}_g) \supseteq \neg\mathbf{op}_g \circ \neg\mathbf{op}_g(\mathcal{K}_{g,\nu}) \supseteq \neg\mathbf{op}_g(\mathcal{K}_{g,\nu})$  for some  $\mathcal{K}_{g,\nu} \in \neg\mathcal{T}_g$ , by the properties of the  $\mathbf{g}$ -operator. Hence,  $P_g(\mathcal{S}_g, \mathcal{K}_{g,\nu}; \mathbf{op}_g(\cdot); \supseteq)$  holds on  $\mathfrak{T}_g \times \mathcal{T}_g \cup \neg\mathcal{T}_g \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ .

From CASE I. and CASE II., it follows that

$$\begin{aligned} \{\mathcal{S}_g \subset \mathfrak{T}_g : P_g(\mathcal{S}_g, \mathcal{O}_g; \mathbf{op}_g(\cdot); \subseteq)\} &\subseteq \mathbf{g-O}[\mathfrak{T}_g] \\ \{\mathcal{S}_g \subset \mathfrak{T}_g : P_g(\mathcal{S}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \supseteq)\} &\subseteq \mathbf{g-K}[\mathfrak{T}_g] \end{aligned}$$

But, since  $\mathbf{g-S}[\mathfrak{T}_g] = \mathbf{g-O}[\mathfrak{T}_g] \cup \mathbf{g-K}[\mathfrak{T}_g]$ , the proof of the theorem at once follows.  $\square$

The following theorem shows that the class  $\mathbf{g-S}[\mathfrak{T}_g]$ , upon satisfaction of two conditions, is the smallest class of  $\mathbf{g-T}_g$ -sets in the  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ .

**Theorem 3.21.** Let  $\mathbf{g-S}_0[\mathfrak{T}_g] = \mathbf{g-O}_0[\mathfrak{T}_g] \cup \mathbf{g-K}_0[\mathfrak{T}_g]$  be a class of  $\mathbf{g-T}_g$ -sets in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$  such that the following two conditions are satisfied:

- i. If  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathbf{g-O}_0[\mathfrak{T}_g] \times \mathbf{g-K}_0[\mathfrak{T}_g]$  and  $P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq)$  holds on  $\mathfrak{T}_g \times \mathbf{g-O}_0[\mathfrak{T}_g] \times \mathbf{g-K}_0[\mathfrak{T}_g] \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$ , then  $\mathcal{S}_g \in \mathbf{g-S}_0[\mathfrak{T}_g]$ .
- ii. The relation  $\mathcal{S}_g \in \mathbf{S}[\mathfrak{T}_g]$  implies  $\mathcal{S}_g \in \mathbf{g-S}_0[\mathfrak{T}_g]$ .

Then,  $\mathbf{g-S}[\mathfrak{T}_g] \subseteq \mathbf{g-S}_0[\mathfrak{T}_g]$ .

PROOF. Let  $\mathcal{S}_g \in \mathbf{g-S}[\mathfrak{T}_g]$ . Then,  $P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq)$  holds on  $\mathfrak{T}_g \times \mathbf{O}[\mathfrak{T}_g] \times \mathbf{K}[\mathfrak{T}_g] \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$  for some pair  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathbf{O}[\mathfrak{T}_g] \times \mathbf{K}[\mathfrak{T}_g]$ . But,  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathbf{O}[\mathfrak{T}_g] \times \mathbf{K}[\mathfrak{T}_g]$  implies  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathbf{g-O}_0[\mathfrak{T}_g] \times \mathbf{g-K}_0[\mathfrak{T}_g]$  by (i), and the latter together with the trueness of  $P_g(\mathcal{S}_g, \mathcal{O}_g, \mathcal{K}_g; \mathbf{op}_g(\cdot); \subseteq, \supseteq)$  on  $\mathfrak{T}_g \times \mathbf{g-O}_0[\mathfrak{T}_g] \times \mathbf{g-K}_0[\mathfrak{T}_g] \times \mathcal{L}_g[\Omega] \times \{\subseteq, \supseteq\}$  implies  $\mathcal{S}_g \in \mathbf{g-S}_0[\mathfrak{T}_g]$  by (ii). Thus,  $\mathbf{g-S}[\mathfrak{T}_g] \subseteq \mathbf{g-S}_0[\mathfrak{T}_g]$ , which completes the proof.  $\square$

In the earlier discussion, the set  $\Omega \subset \mathfrak{U}$  carried the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}(\Omega)$ . A  $\mathfrak{g}$ -topology of this kind will be termed an *absolute  $\mathfrak{g}$ -topology*. To this end, if  $\Gamma \subseteq \Omega$  is any subset of  $\Omega$  then, obviously, we would expect  $\Gamma$  to carry the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}(\Gamma)$ . But, since  $\overline{\mathcal{T}_{\mathfrak{g}}}(\Gamma) \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega)$ , as a consequence of the fact that  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Gamma) \longrightarrow \mathcal{P}(\Gamma)$  is the one-valued restriction map of  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ , which follows from the statement,  $\Gamma \subseteq \Omega$  implies  $\mathcal{P}(\Gamma) \subseteq \mathcal{P}(\Omega)$ , it does make sense to term  $\mathcal{T}_{\mathfrak{g}}(\Gamma)$  a *relative  $\mathfrak{g}$ -topology*. In order to determine what any  $\mathfrak{g}$ -set-theoretic concepts for the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$  becomes when discussion is restricted to  $\Gamma \subseteq \Omega$ , it merely suffices to regard  $\Gamma$  as the set which carries the relative  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}}(\Gamma)$  and carry over the discussion verbatim.

**Definition 3.22** ( $\mathcal{T}_{\mathfrak{g}}$ -Subspace). Let  $\mathfrak{T}_{\mathfrak{g}}(\Omega) := (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , where  $\Omega \subset \mathfrak{U}$  carries the absolute  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \longrightarrow \mathcal{P}(\Omega)$ , and let  $\mathcal{P}(\Gamma) := \{\mathcal{O}_{\mathfrak{g},\nu} : \mathcal{O}_{\mathfrak{g},\nu} \subset \Gamma\}$  be the family of all subsets  $\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \dots$ , of any subset  $\Gamma \subseteq \Omega$  of  $\Omega$ , then every one-valued restriction map of the type

$$\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Gamma) \longmapsto \mathcal{T}_{\mathfrak{g}}(\Gamma) := \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Omega)\} \quad (28)$$

defines a “relative  $\mathfrak{g}$ -topology on  $\Gamma$ ,” and the structure  $\mathfrak{T}_{\mathfrak{g}}(\Gamma) := (\Gamma, \mathcal{T}_{\mathfrak{g}}(\Gamma))$  is called a “ $\mathcal{T}_{\mathfrak{g}}$ -subspace.”

**Theorem 3.23.** Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}(\Gamma) \subseteq \mathfrak{T}_{\mathfrak{g}}(\Omega)$ , where  $\mathfrak{T}_{\mathfrak{g}}(\Gamma) = (\Gamma, \mathcal{T}_{\mathfrak{g}}(\Gamma))$  is the  $\mathcal{T}_{\mathfrak{g}}$ -subspace of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}(\Omega) = (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$ . If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$ , then  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$ .

PROOF. If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$ , then  $P_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$  holds on  $\mathfrak{T}_{\mathfrak{g}}(\Omega) \times \mathcal{T}_{\mathfrak{g}}(\Omega) \cup \neg \mathcal{T}_{\mathfrak{g}}(\Omega) \times \mathcal{L}_{\mathfrak{g}}[\Omega] \times \{\subseteq, \supseteq\}$ . Therefore, if  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$ , then  $P_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}} \cap \Gamma, \mathcal{O}_{\mathfrak{g}} \cap \Gamma, \mathcal{K}_{\mathfrak{g}} \cap \Gamma; \mathbf{op}_{\mathfrak{g}}(\cdot); \subseteq, \supseteq)$  holds on  $\mathfrak{T}_{\mathfrak{g}}(\Gamma) \times \mathcal{T}_{\mathfrak{g}}(\Gamma) \cup \neg \mathcal{T}_{\mathfrak{g}}(\Gamma) \times \mathcal{L}_{\mathfrak{g}}[\Gamma] \times \{\subseteq, \supseteq\}$ . But, since  $\mathcal{S}_{\mathfrak{g}} \cap \Gamma = \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$ ,  $\mathcal{O}_{\mathfrak{g}} \cap \Gamma = \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Gamma)$ , and  $\mathcal{K}_{\mathfrak{g}} \cap \Gamma = \mathcal{K}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}(\Gamma)$ , it follows that  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Gamma)]$  whenever  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}(\Omega)]$ , and the theorem is proved.  $\square$

**Definition 3.24** (Cartesian Product). The Cartesian product of an arbitrary family  $\{\Omega_{\nu} \subset \mathfrak{U} : \nu \in I_n^*\}$  of sets is the set of functions  $\phi : I_n^* \longrightarrow \bigcup_{\nu \in I_n^*} \Omega_{\nu}$  such that  $\phi : \nu \longmapsto \Omega_{\nu}$  for every  $\nu \in I_n^*$ . It is denoted by  $\bigotimes_{\nu \in I_n^*} \Omega_{\nu}$  and satisfies the following properties:

$$i. \quad \bigotimes_{\nu=\mu} \Omega_{\nu} = \Omega_{\mu} \quad \forall \mu \in I_n^*$$

$$ii. \quad \bigotimes_{\nu \in I_{\mu+1}^*} \Omega_{\nu} = \left( \bigotimes_{\nu \in I_{\mu}^*} \Omega_{\nu} \right) \times \Omega_{\mu+1} \quad \forall \mu \in I_{n-1}^*$$

The projection map which gives the projection of the Cartesian product set  $\bigotimes_{\nu \in I_n^*} \Omega_{\nu}$  onto the  $\mu^{\text{th}}$  factor of  $\bigotimes_{\nu \in I_n^*} \Omega_{\nu}$  is defined as thus.

**Definition 3.25** (Projection). Let  $\{\Omega_{\nu} \subset \mathfrak{U} : \nu \in I_n^*\}$  be any class of sets and let  $\bigotimes_{\nu \in I_n^*} \Omega_{\nu}$  denotes the Cartesian product of these sets. The map

$$\text{proj}_{\mu} : \bigotimes_{\nu \in I_n^*} \Omega_{\nu} \longrightarrow \Omega_{\mu} \quad (\text{proj}_{\mu}(\bigotimes_{\nu \in I_n^*} \Omega_{\nu}) = \Omega_{\mu}) \quad (29)$$

is called the projection of the Cartesian product set  $\bigotimes_{\nu \in I_n^*} \Omega_{\nu}$  onto the  $\mu^{\text{th}}$  factor of  $\bigotimes_{\nu \in I_n^*} \Omega_{\nu}$ .

To generate all  $\mathcal{T}_{\mathfrak{g}}$ -open sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , a basis  $\mathcal{B}[\mathcal{T}_{\mathfrak{g}}]$  for  $\mathfrak{T}_{\mathfrak{g}}$  must be supplied, and the following definition is worth considering.

**Definition 3.26** ( $\mathcal{T}_{\mathfrak{g}}$ -Basis). A subclass  $\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})] \subseteq \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$  consisting of  $\mathcal{T}_{\mathfrak{g}}$ -open sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}(\Omega_{\mu}) := (\Omega_{\mu}, \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu}))$ , defined by

$$\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})] := \{\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} : (\nu, \mu, \sigma(\nu, \mu)) \in I_{\infty}^* \times \{\mu\} \times I_{\infty}^*\} \quad (30)$$

is said to be a base for  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega_{\mu}) \longrightarrow \mathcal{P}(\Omega_{\mu})$  if and only if

$$\forall (\mu, \sigma(\mu), \mathcal{O}_{\mathfrak{g},\sigma(\mu)}) \in \{\mu\} \times I_{\infty}^* \times \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu}), \exists I_{\sigma(\mu)} \subseteq I_{\infty}^* : \mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \quad (31)$$

With regards to the terminology employed,  $\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})]$  is called a  $\mathcal{T}_{\mathfrak{g}}$ -basis and its elements,  $\mathcal{B}_{\mathcal{T}_{\mathfrak{g}}}$ -open sets, because they are  $\mathcal{T}_{\mathfrak{g}}$ -open sets of  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega_{\mu}) \rightarrow \mathcal{P}(\Omega_{\mu})$ . With regards to the definition itself, an immediate consequence follows. By the relation  $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}$ , is meant, for every  $(\nu, \mu, \sigma(\mu), \sigma(\nu, \mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_{\infty}^* \times I_{\infty}^*$ , that  $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})]$  and  $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$  in the relation  $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}$ , where  $\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})]$  and  $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$  are given by

$$\begin{aligned} \text{proj}_{\alpha} : \bigotimes_{\mu \in I_n^*} \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})] &\longrightarrow \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\alpha})] \\ \text{proj}_{\alpha} : \bigotimes_{\mu \in I_n^*} \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu}) &\longrightarrow \mathcal{T}_{\mathfrak{g}}(\Omega_{\alpha}) \quad \forall \alpha \in I_n^* \end{aligned} \quad (32)$$

respectively. To this end, a Cartesian product topology (Cartesian  $\mathcal{T}_{\mathfrak{g}}$ -product) is one that having for  $\mathcal{T}_{\mathfrak{g}}$ -basis all  $\mathcal{B}_{\mathcal{T}_{\mathfrak{g}}}$ -open sets of the form  $\text{proj}_{\mu}^{-1}(\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)})$ , where  $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})]$  for every  $(\nu, \mu, \sigma(\nu, \mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_{\infty}^*$ . Therefore, in order to define a Cartesian product  $\mathcal{T}_{\mathfrak{g}}$ -space, it suffices to take the above descriptions into account and postulate a proper definition on this ground. The following definition presents itself.

**Definition 3.27.** Let  $\{\mathfrak{T}_{\mathfrak{g}}(\Omega_{\mu}) := (\Omega_{\mu}, \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})) : \mu \in I_n^*\}$  be a class of  $n \geq 1$   $\mathcal{T}_{\mathfrak{g}}$ -spaces and, for every  $\mu \in I_n^*$ , let  $\mathcal{T}_{\mathfrak{g},\Omega_{\mu}} : \mathcal{P}(\Omega_{\mu}) \rightarrow \mathcal{P}(\Omega_{\mu})$  be the  $\mathfrak{g}$ -topology for  $\mathfrak{T}_{\mathfrak{g}}(\Omega_{\mu})$ . The Cartesian  $\mathcal{T}_{\mathfrak{g}}$ -product  $:= \bigotimes_{\mu \in I_n^*} \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$  on the Cartesian product set  $\Omega := \bigotimes_{\mu \in I_n^*} \Omega_{\mu}$  is that having for  $\mathcal{T}_{\mathfrak{g}}$ -basis all  $\mathcal{B}_{\mathcal{T}_{\mathfrak{g}}}$ -open sets belonging to the following class:

$$\mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega)] := \{\text{proj}_{\mu}^{-1}(\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}) : \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})] \forall (\nu, \mu, \sigma(\nu, \mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_{\infty}^*\} \quad (33)$$

The structure  $\mathfrak{T}_{\mathfrak{g}}(\Omega) := (\Omega, \mathcal{T}_{\mathfrak{g}}(\Omega))$  is called a “Cartesian product  $\mathcal{T}_{\mathfrak{g}}$ -space.”

The fact that  $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})]$  and  $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$  hold for every  $(\nu, \mu, \sigma(\mu), \sigma(\nu, \mu)) \in I_{\sigma(\mu)}^* \times I_n^* \times I_{\infty}^* \times I_{\infty}^*$  makes it reasonable to write

$$\begin{aligned} \bigotimes_{\mu \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\mu)} &\in \bigotimes_{\mu \in I_n^*} \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu}) \\ \bigotimes_{\mu \in I_n^*} (\bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}) &= \bigcup_{\vec{\nu} \in \bigotimes_{\alpha \in I_n^*} I_{\sigma(\alpha)}^*} (\bigotimes_{\alpha \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}) \\ &\in \bigotimes_{\mu \in I_n^*} \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})] \end{aligned} \quad (34)$$

where  $\vec{\nu} := (\nu_1, \nu_2, \dots, \nu_n)$  and, for every  $\alpha \in I_n^*$ ,  $\nu_{\alpha} \in I_{\sigma(\alpha)}^*$ . An immediate consequence of such relation is contained in the following lemma.

**Lemma 3.28.** If  $\mathcal{T}_{\mathfrak{g}} : \mathcal{Q}(\Omega) \rightarrow \mathcal{Q}(\Omega)$  is a one-valued map on the Cartesian product set  $\Omega = \bigotimes_{\mu \in I_n^*} \Omega_{\mu}$ , where

$$\mathcal{Q}(\Omega) := \left\{ \mathcal{O}_{\mathfrak{g},\sigma} = \bigcup_{\vec{\nu} \in \bigotimes_{\alpha \in I_n^*} I_{\sigma(\alpha)}^*} (\bigotimes_{\alpha \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)}) : \mathcal{O}_{\mathfrak{g},\sigma} \in \bigotimes_{\mu \in I_n^*} \mathcal{B}[\mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})] \right\} \quad (35)$$

then  $\mathcal{T}_{\mathfrak{g}} : \mathcal{Q}(\Omega) \rightarrow \mathcal{Q}(\Omega)$  is a  $\mathfrak{g}$ -topology on the Cartesian product set  $\bigotimes_{\mu \in I_n^*} \Omega_{\mu}$ .

PROOF. Let  $\mathcal{O}_{\mathfrak{g},\sigma} = \bigcup_{\vec{\nu} \in \bigotimes_{\alpha \in I_n^*} I_{\sigma(\alpha)}^*} (\bigotimes_{\alpha \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)})$ . Since  $\mathcal{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)} \in \mathcal{T}_{\mathfrak{g}}(\Omega_{\mu})$  holds true for every  $(\nu_{\alpha}, \alpha, \sigma(\nu_{\alpha}, \alpha)) \in I_{\sigma(\alpha)}^* \times I_n^* \times I_{\infty}^*$ , it is evident that  $\mathcal{O}_{\mathfrak{g},\sigma} = \emptyset$  only if, for every  $(\nu_{\alpha}, \alpha, \sigma(\nu_{\alpha}, \alpha)) \in I_{\sigma(\alpha)}^* \times I_n^* \times I_{\infty}^*$ ,  $\mathcal{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)} = \emptyset$ . Thus,  $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$ .

Let  $\mathcal{O}_{\mathfrak{g},\sigma} = \bigcup_{\vec{\nu} \in \bigotimes_{\alpha \in I_n^*} I_{\sigma(\alpha)}^*} (\bigotimes_{\alpha \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_{\alpha},\alpha)})$ . Then, since  $\mathcal{Q}(\Omega) \subseteq \mathcal{Q}(\Omega)$ , it follows that  $\mathcal{O}_{\mathfrak{g},\sigma}$  is a superset of  $\mathcal{T}_{\mathfrak{g}}(\Omega)$ . Thus,  $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma}) \subseteq \mathcal{O}_{\mathfrak{g},\sigma}$ .

Let  $\vec{\nu} = (\nu_1, \dots, \nu_n)$  and  $\vec{\kappa} = (\kappa_1, \dots, \kappa_n)$ , and consider

$$\begin{aligned}\mathcal{O}_{\mathfrak{g},\sigma} &= \bigcup_{\vec{\nu} \in \bigotimes_{\alpha \in I_n^*} I_{\sigma(\alpha)}^*} \left( \bigotimes_{\alpha \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_\alpha, \alpha)} \right) \\ \mathcal{O}_{\mathfrak{g},\tau} &= \bigcup_{\vec{\kappa} \in \bigotimes_{\beta \in I_n^*} I_{\tau(\beta)}^*} \left( \bigotimes_{\beta \in I_n^*} \mathcal{O}_{\mathfrak{g},\tau(\kappa_\beta, \beta)} \right)\end{aligned}$$

Further, let us assume that  $\vec{\eta} = (\nu_1, \dots, \nu_n, \kappa_1, \dots, \kappa_n)$ ,  $\mathbb{I}_{\sigma(\alpha)}^* := \bigotimes_{\alpha \in I_n^*} I_{\sigma(\alpha)}^*$ , and  $\mathbb{I}_{\sigma(\beta)}^* := \bigotimes_{\beta \in I_n^*} I_{\tau(\beta)}^*$ . Then,

$$\mathcal{O}_{\mathfrak{g},\sigma} \cup \mathcal{O}_{\mathfrak{g},\tau} = \bigcup_{\vec{\eta} \in \mathbb{I}_{\sigma(\alpha)}^* \times \mathbb{I}_{\sigma(\beta)}^*} \left( \bigotimes_{\mu \in I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu_\mu, \mu)} \right) \cup \left( \bigotimes_{\beta \in I_n^*} \mathcal{O}_{\mathfrak{g},\tau(\kappa_\beta, \beta)} \right)$$

Thus,  $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma} \cup \mathcal{O}_{\mathfrak{g},\tau}) \subseteq \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma}) \cup \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\tau})$ .  $\square$

**Theorem 3.29.** Let  $\mathfrak{T}_{\mathfrak{g},1}(\Omega), \mathfrak{T}_{\mathfrak{g},2}(\Omega), \dots, \mathfrak{T}_{\mathfrak{g},n}(\Omega)$  be  $n \geq 1$   $\mathcal{T}_{\mathfrak{g}}$ -spaces and let  $\mathfrak{T}_{\mathfrak{g}}(\Omega) := \bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)$  be the  $\mathcal{T}_{\mathfrak{g}}$ -space product. If the relation  $(\mathcal{S}_{\mathfrak{g},1}, \dots, \mathcal{S}_{\mathfrak{g},n}) \in \bigotimes_{\nu \in I_n^*} \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\nu}]$  holds, then  $\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-S}[\bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)]$ .

PROOF. For every  $\sigma \in I_n^*$ , let

$$\mathbf{op}_{\mathfrak{g},12\dots\sigma}(\cdot) = (\mathbf{op}_{\mathfrak{g},12\dots\sigma}(\cdot), \neg \mathbf{op}_{\mathfrak{g},12\dots\sigma}(\cdot)) \in \mathcal{L}_{\mathfrak{g},12\dots\sigma}[\Omega]$$

denotes the  $\mathfrak{g}$ -operator in  $\bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)$  and, for every  $\nu \in I_n^*$ , let  $(\mathcal{S}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\nu}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\nu}] \times \mathcal{T}_{\mathfrak{g},\nu} \times \neg \mathcal{T}_{\mathfrak{g},\nu}$ . Then,

$$\begin{aligned}\mathbf{op}_{\mathfrak{g},12\dots n}(\bigotimes_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}) &= \bigotimes_{\nu \in I_n^*} \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu}) \\ \neg \mathbf{op}_{\mathfrak{g},12\dots n}(\bigotimes_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}) &= \bigotimes_{\nu \in I_n^*} \neg \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu})\end{aligned}$$

On the other hand, for every  $\nu \in I_n^*$ , the logical statement

$$(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu}))$$

holds in  $\mathfrak{T}_{\mathfrak{g},\nu}$ . Consequently,

$$\begin{aligned}& \bigotimes_{\nu \in I_n^*} ((\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu}))) \\ \Rightarrow & ((\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \bigotimes_{\nu \in I_n^*} \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\nu})) \vee (\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \bigotimes_{\nu \in I_n^*} \neg \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\nu}))) \\ \Rightarrow & ((\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathbf{op}_{\mathfrak{g},12\dots n}(\bigotimes_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu})) \vee (\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \supseteq \neg \mathbf{op}_{\mathfrak{g},12\dots n}(\bigotimes_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu})))\end{aligned}$$

Therefore, the Boolean-valued functions

$$P_{\mathfrak{g}}(\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu}, \bigotimes_{\nu \in I_n^*} \mathcal{O}_{\mathfrak{g},\nu}, \bigotimes_{\nu \in I_n^*} \mathcal{K}_{\mathfrak{g},\nu}; \mathbf{op}_{\mathfrak{g},12\dots n}(\cdot); \subseteq, \supseteq)$$

holds on  $\mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathcal{T}_{\mathfrak{g}} \times \neg \mathcal{T}_{\mathfrak{g}} \times \mathcal{L}_{\mathfrak{g},12\dots n}[\Omega] \times \{\subseteq, \supseteq\}$  and, hence, it follows that

$$\bigotimes_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-G}[\bigotimes_{\nu \in I_n^*} \mathfrak{T}_{\mathfrak{g},\nu}(\Omega)]$$

$\square$

The categorical classifications of  $\mathfrak{T}$ -sets and  $\mathfrak{g}\text{-}\mathfrak{T}$ -sets in the  $\mathcal{T}$ -space  $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$  and,  $\mathfrak{T}_{\mathfrak{g}}$ -sets and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$  are discussed and diagrammed on this ground in the next sections.



## 4. Discussion

### 4.1. Categorical Classifications

Having adopted a categorical approach in the classifications of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , the twofold purposes here are to establish the various relationships between the classes of  $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets and the classes of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , and to illustrate them through diagrams.

We have seen that,  $S[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}S[\mathfrak{T}_{\mathfrak{g}}]$ . But,  $S[\mathfrak{T}_{\mathfrak{g}}] = O[\mathfrak{T}_{\mathfrak{g}}] \cup K[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}S[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}K[\mathfrak{T}_{\mathfrak{g}}]$ . Consequently,  $O[\mathfrak{T}_{\mathfrak{g}}], K[\mathfrak{T}_{\mathfrak{g}}] \subseteq S[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}O[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-}K[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}S[\mathfrak{T}_{\mathfrak{g}}]$ ;  $O[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}S[\mathfrak{T}_{\mathfrak{g}}]$  and  $K[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}K[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}S[\mathfrak{T}_{\mathfrak{g}}]$ . In Figure 1, we present the relationships between the class  $S[\mathfrak{T}_{\mathfrak{g}}] = O[\mathfrak{T}_{\mathfrak{g}}] \cup K[\mathfrak{T}_{\mathfrak{g}}]$  of  $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets and the class  $\mathfrak{g}\text{-}S[\mathfrak{T}_{\mathfrak{g}}] = \mathfrak{g}\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \cup \mathfrak{g}\text{-}K[\mathfrak{T}_{\mathfrak{g}}]$  of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ .

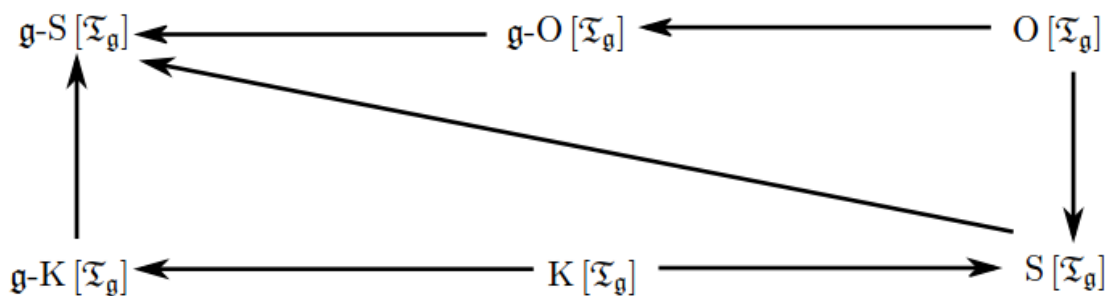


Fig. 1. Relationships: classes of  $\mathfrak{T}_{\mathfrak{g}}$ -sets and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets

It is plain that  $\mathfrak{g}\text{-}\nu\text{-}O[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}O[\mathfrak{T}]$  and  $\mathfrak{g}\text{-}\nu\text{-}O[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}\nu\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}O[\mathfrak{T}_{\mathfrak{g}}]$  for every  $\nu \in I_3^0$ . Moreover, it is also clear that,  $\mathfrak{g}\text{-}2\text{-}O[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}3\text{-}O[\mathfrak{T}]$  and  $\mathfrak{g}\text{-}0\text{-}O[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}1\text{-}O[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}3\text{-}O[\mathfrak{T}]$ , and  $\mathfrak{g}\text{-}2\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}3\text{-}O[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}0\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}1\text{-}O[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}3\text{-}O[\mathfrak{T}_{\mathfrak{g}}]$ . In fact, for every  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ , the relation  $\text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  holds. Consequently,

$$\text{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \supseteq \text{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \quad \forall \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} \quad (36)$$

In Figure 2, we present the relationships between the class  $\mathfrak{g}\text{-}O[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}O[\mathfrak{T}_{\mathfrak{g}}]$  of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open sets of categories 0, 1, 2 and 3 in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , and the class  $\mathfrak{g}\text{-}O[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}O[\mathfrak{T}]$  of  $\mathfrak{g}\text{-}\mathfrak{T}$ -open sets of categories 0, 1, 2 and 3 in the  $\mathcal{T}$ -space  $\mathfrak{T} \subset \mathfrak{T}_{\mathfrak{g}}$ .

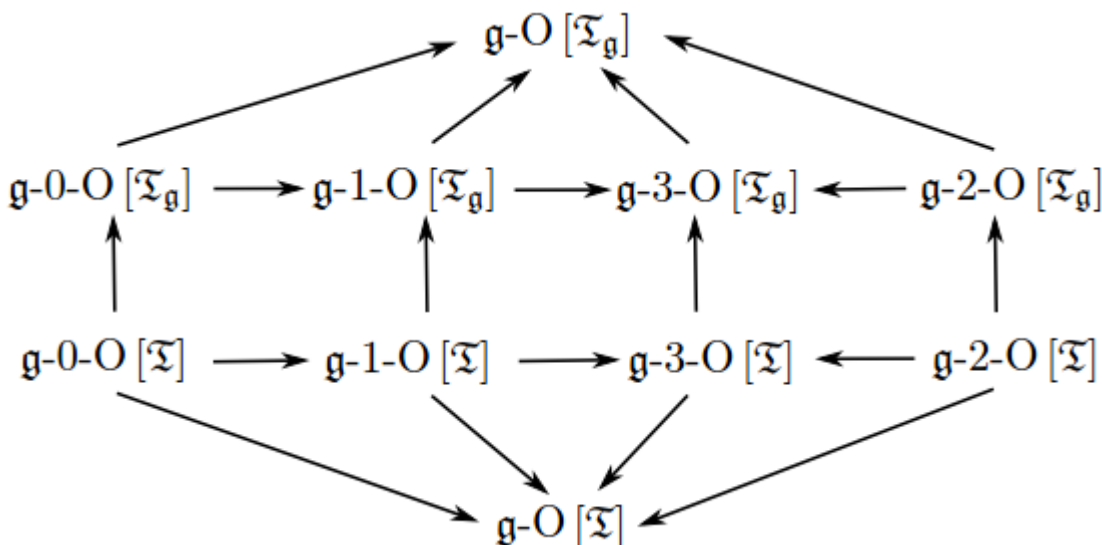
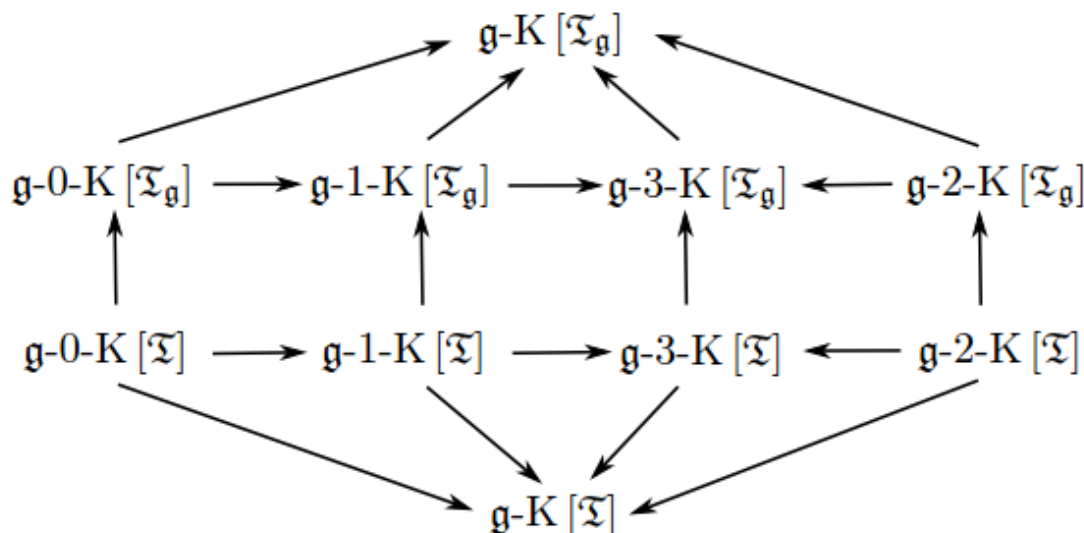


Fig. 2. Relationships: classes of  $\mathfrak{g}\text{-}\mathfrak{T}$ -open sets and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open sets

It is plain that,  $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}]$  and  $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  for every  $\nu \in I_3^0$ . Moreover, it is also clear that,  $\mathfrak{g}\text{-}2\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}3\text{-K}[\mathfrak{T}]$  and  $\mathfrak{g}\text{-}0\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}1\text{-K}[\mathfrak{T}] \subseteq \mathfrak{g}\text{-}3\text{-K}[\mathfrak{T}]$ , and  $\mathfrak{g}\text{-}2\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}3\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}0\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}1\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}3\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ . Because, for every  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$ , the relations  $\text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})$  holds. Consequently,

$$\neg \text{op}_{\mathfrak{g},0}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg \text{op}_{\mathfrak{g},1}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg \text{op}_{\mathfrak{g},3}(\mathcal{S}_{\mathfrak{g}}) \subseteq \neg \text{op}_{\mathfrak{g},2}(\mathcal{S}_{\mathfrak{g}}) \quad \forall \mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}} \quad (37)$$

In Figure 3, we present the relations between the class  $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets of categories 0, 1, 2 and 3 in the  $\mathfrak{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , and the class  $\mathfrak{g}\text{-K}[\mathfrak{T}] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}]$  of  $\mathfrak{g}\text{-}\mathfrak{T}$ -closed sets of categories 0, 1, 2 and 3 in the  $\mathfrak{T}$ -space  $\mathfrak{T} \subseteq \mathfrak{T}_{\mathfrak{g}}$ .



**Fig. 3.** Relationships: classes of  $\mathfrak{g}\text{-}\mathfrak{T}$ -closed sets and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets

As in the papers of Caldas et al. [42], Dontchev [43], Jun et al. [8], and Tyagi et al. [6], among others, the manner we have positioned the arrows is solely to stress that, in general, none of the implications in FIGS 1, 2 and 3 is reversible.

At this stage, a nice application is worth considering, and is presented in the following section.

## 4.2. A Nice Application

Concentrating on fundamental concepts from the standpoint of the theory of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets, we shall now present a nice application. Let  $\Omega = \{\xi_{\nu} : \nu \in I_5^*\}$  denotes the underlying set and consider the  $\mathfrak{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ , where

$$\begin{aligned} \mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\emptyset, \{\xi_1\}, \{\xi_3, \xi_4\}, \{\xi_1, \xi_3, \xi_4\}\} \\ &= \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \mathcal{O}_{\mathfrak{g},3}, \mathcal{O}_{\mathfrak{g},4}\} \end{aligned} \quad (38)$$

$$\begin{aligned} \neg \mathcal{T}_{\mathfrak{g}}(\Omega) &= \{\Omega, \{\xi_2, \xi_3, \xi_4, \xi_5\}, \{\xi_1, \xi_2, \xi_5\}, \{\xi_2, \xi_5\}\} \\ &= \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},3}, \mathcal{K}_{\mathfrak{g},4}\} \end{aligned} \quad (39)$$

respectively, stand for the classes of  $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets. Since conditions  $\mathcal{T}_{\mathfrak{g}}(\emptyset) = \emptyset$ ,  $\mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \mathcal{O}_{\mathfrak{g},\nu}$  for every  $\nu \in I_4^*$ , and  $\mathcal{T}_{\mathfrak{g}}(\bigcup_{\nu \in I_4^*} \mathcal{O}_{\mathfrak{g},\nu}) = \bigcup_{\nu \in I_4^*} \mathcal{T}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu})$  are satisfied, it is clear that the one-valued map  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\{\xi_{\nu} : \nu \in I_5^*\})$  is a  $\mathfrak{g}$ -topology. Furthermore, it is easily checked that,  $\mathcal{O}_{\mathfrak{g},\mu} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}]$  for every  $(\nu, \mu) \in I_3^0 \times I_4^*$ . Hence, the  $\mathfrak{T}_{\mathfrak{g}}$ -open sets forming the  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g}} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  of the  $\mathfrak{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  are  $\mathfrak{g}\text{-}\mathfrak{T}$ -open sets relative to the  $\mathfrak{T}$ -space  $\mathfrak{T} = (\Omega, \mathcal{T})$ .

After calculations, the classes  $\mathbf{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathbf{g}}]$  and  $\mathbf{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathbf{g}}]$ , respectively, of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets of categories  $\nu \in \{0, 2\}$  then take the following forms:

$$\begin{aligned}\mathbf{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathbf{g}}] &= \mathcal{T}_{\mathbf{g}} \cup \{\{\xi_3\}, \{\xi_4\}, \{\xi_1, \xi_3\}, \{\xi_1, \xi_4\}\} \\ \mathbf{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathbf{g}}] &= \neg\mathcal{T}_{\mathbf{g}} \cup \{\{\xi_2, \xi_4, \xi_5\}, \{\xi_1, \xi_2, \xi_3, \xi_5\} \\ &\quad \{\xi_1, \xi_2, \xi_4, \xi_5\}, \{\xi_2, \xi_3, \xi_5\}\}, \quad \forall \nu \in \{0, 2\}\end{aligned}\quad (40)$$

On the other hand, those of categories  $\nu \in \{1, 3\}$  take the following forms:

$$\begin{aligned}\mathbf{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathbf{g}}] &= \mathcal{T}_{\mathbf{g}} \cup \{\mathcal{O}_{\mathbf{g}} : \mathcal{O}_{\mathbf{g}} \in \mathcal{P}(\Omega) \setminus \mathcal{T}_{\mathbf{g}}\} \\ \mathbf{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathbf{g}}] &= \neg\mathcal{T}_{\mathbf{g}} \cup \{\mathcal{K}_{\mathbf{g}} : \mathcal{K}_{\mathbf{g}} \in \mathcal{P}(\Omega) \setminus \neg\mathcal{T}_{\mathbf{g}}\}, \quad \forall \nu \in \{1, 3\}\end{aligned}\quad (41)$$

The discussions carried out in the preceding sections can be easily verified from this nice application. The next section provides concluding remarks and future directions of the theory of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -sets discussed in the preceding sections.

## 5. Conclusion

In this paper, we developed a new theory, called *Theory of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -Sets*. In its own rights, the proposed theory has several advantages. The very first advantage is that the theory holds equally well when  $(\Omega, \mathcal{T}_{\mathbf{g}}) = (\Omega, \mathcal{T})$  and other features adapted on this basis, in which case it might be called *Theory of  $\mathbf{g}\text{-}\mathfrak{T}$ -Sets*. Hence, in a  $\mathcal{T}_{\mathbf{g}}$ -space the theoretical framework categorises such pairs of concepts as  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets,  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -semi-open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -semi-closed sets,  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -pre-open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -pre-closed sets, and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -semi-pre-open and  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -semi-pre-closed sets as  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a  $\mathcal{T}$ -space it categorises such pairs of concepts as  $\mathbf{g}\text{-}\mathfrak{T}$ -open and  $\mathbf{g}\text{-}\mathfrak{T}$ -closed sets,  $\mathbf{g}\text{-}\mathfrak{T}$ -semi-open and  $\mathbf{g}\text{-}\mathfrak{T}$ -semi-closed sets,  $\mathbf{g}\text{-}\mathfrak{T}$ -pre-open and  $\mathbf{g}\text{-}\mathfrak{T}$ -pre-closed sets, and  $\mathbf{g}\text{-}\mathfrak{T}$ -semi-pre-open and  $\mathbf{g}\text{-}\mathfrak{T}$ -semi-pre-closed sets as  $\mathbf{g}\text{-}\mathfrak{T}$ -sets of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -sets of mixed categories. More precisely, for some pair  $(\nu, \mu) \in I_3^0 \times I_3^0$  such that  $\nu \neq \mu$ , to develop the theory of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -open sets belonging to the class  $\{\mathcal{O}_{\mathbf{g}} = \mathcal{O}_{\mathbf{g},\nu} \cup \mathcal{O}_{\mathbf{g},\mu} : (\mathcal{O}_{\mathbf{g},\nu}, \mathcal{O}_{\mathbf{g},\mu}) \in \mathbf{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathbf{g}}] \times \mathbf{g}\text{-}\mu\text{-O}[\mathfrak{T}_{\mathbf{g}}]\}$  and the theory of  $\mathbf{g}\text{-}\mathfrak{T}_{\mathbf{g}}$ -closed sets belonging to the class  $\{\mathcal{K}_{\mathbf{g}} = \mathcal{K}_{\mathbf{g},\nu} \cup \mathcal{K}_{\mathbf{g},\mu} : (\mathcal{K}_{\mathbf{g},\nu}, \mathcal{K}_{\mathbf{g},\mu}) \in \mathbf{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathbf{g}}] \times \mathbf{g}\text{-}\mu\text{-K}[\mathfrak{T}_{\mathbf{g}}]\}$  in a  $\mathcal{T}_{\mathbf{g}}$ -space  $\mathfrak{T}_{\mathbf{g}}$ , as Andrijević [22] and Caldas et al. [44] developed the theory of  $b$ -open and  $b$ -closed sets in a  $\mathcal{T}$ -space  $\mathfrak{T}$ . Such two theories are what we thought would certainly be worth considering, and the discussion of this work ends here.

## Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the manuscript.

## Conflicts of Interest

The authors declare no conflict of interest.

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