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Rings Whose Pure-Projective Modules Have Maximal or Minimal Projectivity Domain

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Abstract — In this study, we investigate the projectivity domain of pure-projective modules. A pure-projective module is called special-pure-projective (s-pure-projective) module if its projectivity domain contains only regular modules. First, we describe all rings whose pure-projective modules are s-pure-projective, and we show that every ring with an s-pure-projective module. Afterward, we research rings whose pure-projective modules are projective or s-pure-projective. Such rings are said to have $*$-property. We determine the right Noetherian rings have $*$-property.

Keywords *Projectivity domain, pure-projective module, s-pure-projective module, von Neumann regular rings, right Goldie torsion rings*

Mathematics Subject Classification (2020) 16D10, 16D40

1. Introduction

Let R be an associative ring with identity throughout the article, and unless otherwise indicated, any module be a right R-module. Projectivity has been investigated from various angles in the recent studies [1–15]. The class $\{Y \in \mathcal{M}od\text{-}R : X \text{ is } Y\text{-projective}\}$ for a module X is referred to as the projectivity domain of X and is represented by $\mathfrak{Pr}^{-1}(X)$ [16]. It is clear that X is projective if and only if $\mathfrak{Pr}^{-1}(X) = \mathcal{M}od\text{-}R$. Projectively poor (p-poor) modules whose projectivity domains contain only semisimple modules in $\mathcal{M}od\text{-}R$ and rings with no right p-middle class whose modules are projective or p-poor were explored in [1].

We study pure-projective modules in their projectivity domain. Initially, we address the presence of special pure-projective (s-pure-projective) modules, and we prove that an s-pure-projective module exists for every ring. Subsequently, we examine rings, each of whose pure-projective modules is s-pure-projective; these rings are specifically von Neumann regular rings or vNr rings for short. We study rings that have $*$-property, that is, their pure-projective modules are projective or s-pure-projective. For example, semisimple Artinian rings and vNr rings have $*$-property. Additionally, a quasi-Frobenius ring R with a homogeneous right socle and $J(R)^2 = 0$ is also such a ring, for more details, see [5]. We provide the structure of rings that have $*$-property over right Noetherian rings (Theorem 4.10): if R is a ring with $*$-property, then $R \cong \Lambda' \times \Lambda$ where Λ' is semisimple Artinian, and Λ is either zero or an indecomposable ring, which satisfies one of the following cases:

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i. Λ is a right Artinian, and right SI-ring with $J(\Lambda) \neq 0$

ii. Λ is a right Artinian and right Goldie torsion ring with $J(\Lambda) \neq 0$

iii. Λ is a right Artinian ring, and $\mathrm{Soc}(\Lambda_\Lambda) = Z_r(\Lambda) = J(\Lambda) \neq 0$

iv. Λ is a prime ring with $J(\Lambda) = \mathrm{Soc}(\Lambda_\Lambda) = 0$

Finally, we include examples for the cases of Theorem 4.10 as well as a partial answer for the converse of Theorem 4.10.

2. Preliminaries

This section provides some basic notions to be required the following sections. Let X be an R-module. If Y is a submodule, essential submodule, or direct summand of X, we denote $Y \leq X$, $Y \leq_e X$, or $Y \leq_d X$, respectively. For a module X and a ring R, $\mathrm{Rad}(X)$, $J(R)$, $\mathrm{Soc}(X)$, $\mathrm{Soc}(R_R)$, $Z(X)$, $Z_r(R)$, and $Z_2(X)$ stand for the Jacobson radical of X, the Jacobson radical of the ring R, the socle of X, the right socle of the ring R, the singular part of X, the right singular part of the ring R, and the second singular part of X, respectively.

Definition 2.1. A submodule Y of a module X is called pure if there is a monomorphism $i \otimes_R 1_A : Y \otimes_R A \to X \otimes_R A$, for all left R-module A.

Definition 2.2. A module T is called regular if every submodule of T is pure.

The set of all the regular modules is represented by $\mathcal{R}egular$.

Definition 2.3. A short exact sequence

$$\mathbb{E} : 0 \longrightarrow X \xrightarrow{f} Y \longrightarrow Z \longrightarrow 0$$

is called a pure short exact sequence if $\mathrm{Im}(f)$ is a pure submodule of Y.

Definition 2.4. A module T is called pure-projective if it is projective relative to any pure short exact sequence.

$\mathcal{P}$ is our abbreviation for the collection of all pure-projective modules. A module T is pure-projective if and only if T is a summand of a direct sum of finitely presented modules.

Remark 2.5. [1] For a ring R, the following are equivalent.

i. R is semisimple Artinian

ii. Every right R-module is p-poor

iii. There exists a projective p-poor module

iv. $\{0\}$ is p-poor

v. R is p-poor

Proposition 2.6. [1] If R is a (non-semisimple Artinian) quasi-Frobenius ring with homogeneous right socle and $J(R)^2 = 0$, then R has no right p-middle class and $\mathfrak{In}^{-1}(M) = \mathfrak{Pr}^{-1}(M)$ for all right R-module M.

A ring R is called right pure-semisimple if any pure submodule of a module is a direct summand, right SI-ring if every singular right module is injective. A ring R is called semi-primary if $R/J(R)$ is semisimple Artinian and $J(R)$ is a nilpotent ideal, prime if for any two ideals A and B of R, $AB = 0$ implies $A = 0$ or $B = 0$, semiprime if there is no a nonzero nilpotent ideal in R, right Goldie torsion if R is equal to its second singular submodule. For more details, see [16–19].

3. Existence of s-pure-projective Modules

Let X represent an R-module. Then, $\mathfrak{Pr}^{-1}(X)$ is closed under submodules, finite direct sums, and epimorphic images [16]. It is clear from definitions that if T is a regular R-module and X is a pure-projective R-module, then $T \in \mathfrak{Pr}^{-1}(X)$.

Proposition 3.1. $\bigcap_{X\in\mathcal{P}} \mathfrak{Pr}^{-1}(X) = \mathcal{R}egular$.

Proof. Let

$$N \in \bigcap_{X\in\mathcal{P}} \mathfrak{Pr}^{-1}(X)$$

It suffices to show that N is a regular module. Let $K \leq N$. Let F be a finitely presented R-module and $f : F \to N/K$ be any R-module homomorphism. Then, there exists $g : F \to N$, since F is a pure-projective module. The following diagram can be constructed.

$$\begin{tikzcd} & & & F \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ 0 \arrow[r] & K \arrow[r, "i"] & N \arrow[r, "\pi"] & N/K \arrow[r] & 0 \end{tikzcd}$$

where i is a canonical monomorphism and π is a canonical epimorphism. Hence, K is a pure submodule of N. □

Definition 3.2. A pure-projective module X is called s-pure-projective if $\mathfrak{Pr}^{-1}(X) = \mathcal{R}egular$.

The existence problem of s-pure-projective modules, or whether they exist in each ring, is the first question. The following proposition provides a favorable response to this query.

Proposition 3.3. S-pure-projective module is present for any ring R.

Proof. A full set of representations of the isomorphism class of finitely presented R-modules is denoted by $\{X_\gamma \mid \gamma \in \Gamma\}$. Let

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma$$

It is obvious that X is a pure-projective module. Let $T \in \mathfrak{Pr}^{-1}(X)$ and $K \leq T$. Consider the short exact sequence

$$\mathbb{E} : 0 \to K \to T \to T/K \to 0$$

It suffices to show that $\mathbb{E}$ is a pure short exact sequence. Take any R-module homomorphism $f : F \to T/K$, where F is a finitely presented module. Then, F is a T-projective module since X is a T-projective module. Hence, there exists $g : F \to T$ such that $\pi \circ g = f$, that is, we can construct the following commutative diagram

$$\begin{tikzcd} & & & F \arrow[dl, dashed, "g"'] \arrow[d, "f"] & \\ 0 \arrow[r] & K \arrow[r, "i"] & T \arrow[r, "\pi"] & T/K \arrow[r] & 0 \end{tikzcd}$$

where i is a canonical monomorphism and π is a canonical epimorphism. Hence, $\mathbb{E}$ is a pure, short, exact sequence, as desired. □

We can observe that

$$\mathfrak{Pr}^{-1}(X) = \mathfrak{Pr}^{-1}\left(\bigoplus X\right) = \bigcap \mathfrak{Pr}^{-1}(X)$$

In addition, we collect some useful properties of s-pure-projective modules.

Remark 3.4. Let X be a pure-projective R-module and $\{X_i\}_{i\in I}$ be a family of pure-projective R-modules. Then,

i. X is an s-pure-projective module if and only if $\bigoplus X$ is an s-pure-projective module

ii. $\bigoplus\limits_{i\in I}(X_i)^{(J_i)}$ is an s-pure-projective module if and only if $\bigoplus\limits_{i\in I} X_i$ is an s-pure-projective module

Proposition 3.5. Let X be an s-pure-projective R-module. If X is a direct summand of a pure-projective R-module Y, then Y is s-pure-projective.

Proof. Let $Y = X \oplus N$ where N is a module. Let Y be a T-projective. Then, N is pure-projective and X is T-projective, thus T is a regular module by our assumption. Hence,

$$\mathfrak{Pr}^{-1}(X \oplus N) = \mathcal{R}egular$$

that is, Y is s-pure-projective. □

Corollary 3.6. The arbitrary direct sum of s-pure-projective modules is s-pure-projective.

Lemma 3.7. Let R represent a ring. The expressions below are equivalent.

i. Any pure-projective R-module is s-pure-projective

ii. Any finitely presented R-module is s-pure-projective

iii. A projective s-pure-projective R-module exists

iv. $\{0\}$ is an s-pure-projective R-module

v. R is an s-pure-projective R-module

vi. R is a vNr ring

Corollary 3.8. A ring R is an s-pure-projective R-module if and only if $M_n(R)$ is an s-pure-projective $M_n(R)$-module.

Proposition 3.5 is incorrect in its opposite sense, meaning that a direct summand of an s-pure-projective module is not always s-pure-projective.

Example 3.9. A full set of representations of the isomorphism class of finitely presented R-modules is denoted by $\{X_\gamma \mid \gamma \in \Gamma\}$. Let

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma$$

X is an s-pure-projective module by Proposition 3.3. But if R is not vNr, then R is not s-pure-projective see Lemma 3.7. Hence, a copy of R in X as a summand is not s-pure-projective.

In closing this section, we investigate the relationship between p-poor and s-pure-projective modules for pure-projective R-modules. It is clear that if a pure-projective R-module X is p-poor, then it is s-pure-projective. The converse is not true in general; for example, a vNr ring R which is not semisimple Artinian is s-pure-projective but not p-poor, see Remark 2.2 in [1] and Lemma 3.7. As for the converse for a pure semisimple ring R, a pure-projective module X is s-pure-projective if and only if X is p-poor.

4. Rings Whose Pure-Projective Modules are Either s-pure-projective or Projective

This section addresses rings claimed to have $*$-property, meaning that their pure-projective modules are either projective or s-pure-projective. If a ring is not one of these rings, it is considered to has no $*$-property.

Note that a ring R is called right hereditary if every submodule of a projective right module is projective.

Proposition 4.1. Let R be a right hereditary ring. If R has $*$-property, then any pure-projective module that contains an s-pure-projective submodule is s-pure-projective.

Proof. If R is a vNr ring, then any pure-projective R-module is s-pure-projective by Lemma 3.7. Therefore, we may suppose R is not vNr without losing generality. Let X be an s-pure-projective R-module and $X \leq X'$, where X' is a pure-projective R-module. X' is projective or s-pure-projective by our assumption. If X' is projective, then X is projective by our right hereditary assumption. This is impossible by Lemma 3.7 since R is not vNr. Hence, X' must be an s-pure-projective module. □

Lemma 4.2. Let R be a ring with $*$-property and $0 \neq A$ be a finitely generated two-sided ideal of R. Then, $A \leq_d R$ or R/A is a vNr ring.

Proof. R/A is a finitely presented R-module; therefore, it is pure-projective. Then, R/A is projective or s-pure-projective by our assumption, that is, $A \leq_d R$ or $(R/A)_R$ is an s-pure-projective module. If $(R/A)_R$ is an s-pure-projective module, then $(R/A)_R$ is regular since A is fully invariant, and thus R/A is a (R/A)-projective (or quasi-projective) R-module. Hence, $(R/A)_{R/A}$ is regular for any two-sided ideal of R. □

Remark 4.3. Let A be a two-sided ideal of R, a ring with $*$-property. Then,

i. If $X_{R/A}$ is $N_{R/A}$-(R/A)-projective, then X_R is N_R-projective

ii. If $X_{R/A}$ is non-regular, then X_R is non-regular

Factor rings inherit the $*$-property as indicated by the following assertion.

Lemma 4.4. Let A be a two-sided ideal of R, a ring with $*$-property. Therefore, R/A has $*$-property.

Proof. Let X be a pure-projective (R/A)-module, which is not s-pure-projective. It is clear that X is a pure-projective R-module. There exists a non-regular (R/A)-module N such that $X_{R/A}$ is $N_{R/A}$-(R/A)-projective since X is a (R/A)-module, which is not s-pure-projective. This implies that X is N-projective, which means that X_R is not s-pure-projective since N is also a non-regular R-module. Then, X must be a projective R-module by assumption. Hence, X is a projective (R/A)-module. □

Note that a semilocal ring R is a ring, for which $R/J(R)$ is a semisimple Artinian ring. We can easily see that if R is a vNr and right Noetherian ring, then R is a semisimple Artinian ring. Note that a right Noetherian right semiartian ring is right Artinian. Therefore, the next result can be easily obtained with the help of Lemma 4.2.

Corollary 4.5. Let R be a right Noetherian ring with $*$-property, and A be a nonzero two-sided ideal of R. Then, the following are hold:

i. $A \leq_d R$ or R/A is a semisimple Artinian ring

ii. $\mathrm{Soc}(R_R) \leq_d R$ or R is a right Artinian ring

iii. If $J(R)$ is nonzero, then R is a semilocal ring

Lemma 4.6. Let R be a right Noetherian ring with $J(R) \neq J(R)^2$ and has $*$-property. Then, $J(R)^2 = 0$.

Proof. Suppose the contrary that $J(R)^2 \neq 0$. Then, $R/J(R)^2$ is a semisimple Artinian ring by Corollary 4.5. This implies that $J(R) = J(R)^2$, which provides a contradiction. Hence, $J(R) = 0$. □

Lemma 4.7. Let R be a right Noetherian ring with $*$-property. Then, R is either a right Artinian ring with $J(R)^2 = 0$ or a semiprime ring.

PROOF. Suppose that R is not semiprime. Let A be a nilpotent two-sided ideal of R. Then, R/A s-pure-projective by assumption, and nilpotent ideals are small in R. It is clear that R/A is a semisimple Artinian ring by Corollary 4.5. Then, it is clear that $J(R) = A$. Thus, R is a semilocal ring by Corollary 4.5; therefore, a semiprimary ring since $J(R)$ is a nilpotent ideal. Hence, R is a right Artinian ring by Hopkins-Levitzki theorem. Furthermore, $J(R)^2 = 0$ by Lemma 4.6. □

Lemma 4.8. Let R be an indecomposable right Noetherian ring with $*$-property. After that, $\mathrm{Soc}(R_R) \leq_e R$ with either $J(R)^2 = 0$ or $\mathrm{Soc}(R_R) = 0$.

PROOF. $R/\mathrm{Soc}(R_R)$ is a projective or an s-pure-projective R-module by assumption. If $R/\mathrm{Soc}(R_R)$ is a projective R-module, then

$$R = \mathrm{Soc}(R_R) \oplus \Lambda$$

for some right ideal Λ of R. It is obvious that

$$\mathrm{Hom}_R(\mathrm{Soc}(R_R), \Lambda) = 0$$

and

$$\mathrm{Hom}_R(\Lambda, \mathrm{Soc}(R_R)) = 0$$

since Λ is a socle-free R-module and $\mathrm{Soc}(R_R)$ is projective. Hence, Λ is a two-sided ideal of R. Then,

$$\mathrm{Soc}(R_R) = 0 \quad \text{or} \quad \mathrm{Soc}(R_R) = R$$

since R is an indecomposable ring, that is, $\mathrm{Soc}(R_R) \leq_e R$ with $J(R)^2 = 0$ or $\mathrm{Soc}(R_R) = 0$. If $R/\mathrm{Soc}(R_R)$ is an s-pure-projective R-module, then $R/\mathrm{Soc}(R_R)$ is semisimple Artinian, and thus R is a right Artinian by Corollary 4.5. Moreover, this implies that $\mathrm{Soc}(R_R) \leq_e R$ with $J(R)^2 = 0$ or $\mathrm{Soc}(R_R) = 0$ since if $J(R) = 0$, then R is semisimple Artinian, and if $J(R) \neq 0$, then $J(R)$ is nilpotent and $J(R) \neq J(R)^2$, then $J(R)^2 = 0$ by Lemma 4.6. □

Lemma 4.9. Let R be a semiprime right Noetherian ring that is indecomposable and has $*$-property. Then, R is a semisimple Artinian ring if it is not prime.

PROOF. Let A be a nonzero two-sided ideal of R. R/A is a projective R-module or R/A is an s-pure-projective R-module by our assumption. If R/A is a projective R-module, then a right ideal B exists, such as $R = A \oplus B$. $BA = 0$ because of the direct sum property and $AB = 0$ since $(AB)^2 = 0$ and R is semiprime. Then, $A = R$ since R is an indecomposable ring. If R/A is an s-pure-projective R-module, then R/A is semisimple Artinian by Corollary 4.5. We observe that R/A is a semisimple Artinian ring for any nonzero two-sided ideal A of R. Suppose that R is not prime. Let I_1 and I_2 be nonzero two-sided ideals of R such that $I_1 I_2 = 0$ and $I_1 \cap I_2 = 0$ since R is a semiprime ring. Then, there is an R-monomorphism $R_R \to R/I_1 \oplus R/I_2$. Hence, R is a semisimple Artinian ring since R/I_1 and R/I_2 are semisimple Artinian rings by our first observation. □

Theorem 4.10. Let R be a right Noetherian ring. If R has $*$-property, then $R \cong \Lambda' \times \Lambda$ where Λ' is semisimple Artinian ring and Λ is either zero or an indecomposable ring, which satisfies one of the following cases.

i. Λ is a right Artinian, and right SI-ring with $J(\Lambda) \neq 0$

ii. Λ is a right Artinian and right Goldie torsion ring with $J(\Lambda) \neq 0$

iii. Λ is a right Artinian ring, and $\mathrm{Soc}(\Lambda_\Lambda) = Z_r(\Lambda) = J(\Lambda) \neq 0$

iv. Λ is a prime ring with $J(\Lambda) = \mathrm{Soc}(\Lambda_\Lambda) = 0$

PROOF. We can write

$$R = R_1 \oplus R_2 \oplus ... \oplus R_n$$

where $n \in \mathbb{Z}^+$ and R_i is an indecomposable ring, for all $i \in \{1, 2, ..., n\}$. Either R is a semisimple Artinian, or there exists an $i \in \{1, 2, ..., n\}$ such that R_i is not a semisimple Artinian ring. Let A be a right ideal of R_j, for $i \neq j$. By our assumption, R_j/A is either s-pure-projective or projective since R_j/A is a pure-projective R-module. It is clear that

$$\mathrm{Hom}_R(R_j/A, R_i/B) = 0$$

for any submodule B of R_i. Then,

$$R_i \in \mathfrak{Pr}^{-1}(R_j/A) \neq \mathcal{R}egular$$

since R_i is not regular, that is, R_j/A is not an s-pure-projective R-module. Then, R_j/A must be projective; therefore, R_j is a semisimple Artinian ring. Thus, we have

$$R \cong \Lambda' \times \Lambda$$

where Λ' is semisimple Artinian, and Λ is either zero or an indecomposable ring. Λ is a ring with $*$-property, and Λ is either right Artinian with $J(R)^2 = 0$ or semiprime by Lemma 4.4 and Lemma 4.7.

We divide the proof into three cases.

First case: Suppose that Λ is a right Artinian with $J(R)^2 = 0$ and $Z_2(R) = 0$. Then, Λ is a right SI-ring by Proposition 3.5 [19]. This case provides us (i) of Theorem 4.10.

Second case: Suppose that Λ is a right Artinian with $J(R)^2 = 0$ and $Z_2(R) \neq 0$. Then, $R/Z_2(R)$ is projective or s-pure-projective. If $R/Z_2(R)$ is projective, then $Z_2(R) = R$. Hence, R is a right Goldie torsion ring. This case provides (ii) of Theorem 4.10.

If $R/Z_2(R)$ is not projective, then $R/Z_2(R)$ is s-pure-projective and thus semisimple Artininan by Corollary 4.5. Hence, $J(\Lambda) \leq Z_r(\Lambda)$, since $J(\Lambda)$ is a semisimple R-module and $Z_r(\Lambda) \leq_e Z_2(R)$. Λ has a decomposition

$$\Lambda = \bigoplus_{i=1}^{n} \Lambda_i$$

where each Λ_i is a local module since Λ is Artinian. It is clear that

$$J(\Lambda_i) \subseteq Z_r(\Lambda_i) \neq \Lambda_i$$

where $i \in \{1, 2, ..., n\}$. This implies $J(\Lambda) \subseteq Z_r(\Lambda)$, that is, $J(\Lambda) = Z_r(\Lambda)$. Suppose the contrary that

$$J(\Lambda) \neq \mathrm{Soc}(\Lambda_\Lambda)$$

Then, for some Λ_i is simple. We can suppose that the simple components of the decomposition are Λ_1, Λ_2, $\cdots$, Λ_k where $1 \leq k \leq n$. Let

$$I_1 = \{i \in \{1, 2, ..., n\} \mid \Lambda_i \cong \Lambda_t\}$$

and

$$I_2 = \{1, 2, ..., n\} - I_1$$

Let

$$D = \bigoplus_{i \in I_1} \Lambda_i$$

Let $i \in I_1$ and $j \in I_2$. Clearly, $\mathrm{Hom}_R(\Lambda_i, \Lambda_j) = 0$. Moreover, $\mathrm{Hom}_R(\Lambda_j, \Lambda_i) = 0$ by the singularity of $J(\Lambda)$. This is a contradiction since Λ indecomposable; therefore, $J(\Lambda) = \mathrm{Soc}(\Lambda_\Lambda)$, that is, we get *(iii)* of Theorem 4.10.

Third case: Let Λ be a semiprime ring. It is clear that Λ is a prime ring by Lemma 4.9. Assume that $\mathrm{Soc}(\Lambda_\Lambda) = 0$. If $J(\Lambda) \neq 0$, then $J(\Lambda)^2 = J(\Lambda)$ by Lemma 4.6. By Nakayama's Lemma, $J(\Lambda)_\Lambda$ is infinitely generated, which is impossible by the right Noetherianity; therefore, $J(\Lambda) = 0$. This provides the last case of Theorem 4.10. But what about the case; where Λ is a semiprime ring and $\mathrm{Soc}(\Lambda_\Lambda) \neq 0$. $\mathrm{Soc}(\Lambda_\Lambda) \leq_e \Lambda$ and $J(\Lambda)^2 = 0$ by Lemma 4.8, and $J(\Lambda) = 0$ by assumption. We must have $Z_r(\Lambda) = 0$ since Λ is prime, and $\mathrm{Soc}(\Lambda_\Lambda) \neq 0$. Besides, we have that

$$Z_r(\Lambda)\,\mathrm{Soc}(\Lambda_\Lambda) = 0$$

Then, Λ is a right SI-ring by Corollary 3.7 [19]. According to Proposition 10.15 [16], if Λ has a finitely generated socle, it becomes a semisimple Artinian ring. This is impossible since Λ is not a semisimple Artinian ring. Further, $\mathrm{Soc}(\Lambda_\Lambda)$ can not be infinitely generated since Λ is a right Noetherian ring. Hence, we have no extra cases for Theorem 4.10. This completes the proof of Theorem 4.10. □

In the next example, we illustrate for each case in Theorem 4.10.

Example 4.11. *i.* Let F be a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. It is well-known that R is an Artinian SI-ring and $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$

ii. Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_4 \end{bmatrix}$. R is an Artinian ring and we can easily see that $Z_r(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 2\mathbb{Z}_4 \end{bmatrix}$ and $Z_2(R_R) = R$. Thus, R is a right Goldie torsion ring

iii. Let p be a prime number and $R = \mathbb{Z}/p^2\mathbb{Z}$. Then, R is an Artinian ring with

$$\mathrm{Soc}(R_R) = Z_r(R) = J(R) = p\mathbb{Z}/p^2\mathbb{Z}$$

iv. Let $R = \mathbb{Z}$. Then, it is well known that R is a prime ring with $J(R) = \mathrm{Soc}(R) = 0$

The following proposition provides a partial answer to the converse of Theorem 4.10. The answer follows from any pure-projective p-poor module is s-pure-projective and Proposition 3.14 [1].

Proposition 4.12. If R is a (non-semisimple Artinian) quasi-Frobenius ring with a homogeneous right socle and $J(R)^2 = 0$, then R has $*$-property.

5. Conclusion

We study pure-projective modules in their projectivity domains. After showing that each ring with an s-pure-projective module, we characterize rings all of whose modules are s-pure-projective as vNr rings. Afterwards, we investigate rings whose pure-projective modules are projective or s-pure-projective, called rings has $*$-property. Semisimple Artinian rings and vNr rings are examples of these rings. We provide the structure of right Noetherian rings that have $*$-property (Theorem 4.10). Furthermore, we present a partial answer for the converse of Proposition 4.12. Consequently, the results can be generalized to non-Noetherian rings, and the full characterization of Theorem 4.10 can be studied. Since it is common and important to study certain classes of modules in module theory; in addition, one can continue to look at the projectivity domains of some other special classes of modules.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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