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ON DIGITAL H-GROUPS

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Abstract. In this paper, specific properties of digital H-spaces and digital H-groups are studied. It is shown that there is a contravariant functor from the homotopy category of the pointed digital images to the category of groups and homomorphisms. Then it is proven that a pointed digital image having the same digital homotopy type as a digital H-group is itself a digital H-group.

1. Introduction

The purpose of digital topology is to study the topological properties of discrete objects, such as compactness, connectedness. Digital topology was first studied in [14] by the computer image analysis researcher Rosenfeld. Following years researchers studied the digital versions of many concepts of algebraic topology. Digital homotopy and digital fundamental group are defined in [4] by Boxer. Digital H-space is defined in [7] by Ege and Karaca. H-spaces are an important concept of homotopy theory. An H-space consists of a pointed topological space P with a continuous multiplication $m : X \times X \to X$ and with a constant map $c : X \to X$, such that $m \circ (c, 1_X) \simeq 1_X \simeq m \circ (1_X, c)$. A group structure can be established on H-space, called H-group, by homotopy group operations which are similar to group operations.

2. Preliminaries

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}^n$ the set of all lattice points in Euclidean n-dimensional space. A finite subset X of $\mathbb{Z}^n$ with an adjacency relation κ is called a digital image, denoted by (X, κ).

For a positive integer t with $1 \le t \le n$, two distinct points $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{Z}^n$ are κ_t-adjacent if,

i) there are at most t distinct indices i such that $|x_i - y_i| \neq 1$, and
ii) for all indices j, if $|x_j - y_j| \neq 1$, then $x_j = y_j$.

Consider the following statements for the commonly used adjacency relations:

(1) $p, q \in \mathbb{Z}$ are 2-adjacent if $|p - q| = 1$

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(2) $p, q \in \mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 each coordinate.
(3) $p, q \in \mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ by exactly one coordinate.
(4) $p, q \in \mathbb{Z}^3$ are 26-adjacent if they distinct and differ by at most 1 each coordinate.
(5) $p, q \in \mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ by at most two coordinates.
(6) $p, q \in \mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ by exactly one coordinate.

A digital interval is a set of the form $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$.

The adjacency relation on cartesian product of two digital image is defined as follows.

Definition 2.1. [8] For two digital image (X, κ_1) and (Y, κ_2), the κ^*-adjacency on the product image $X \times Y$ is obtained as follows: $x_1, x_2 \in (X, \kappa_1)$, $y_1, y_2 \in (Y, \kappa_2)$, then (x_1, y_1) and (x_2, y_2) are κ^*-adjacent if and only if one of the following is satisfied:

(1) $x_1 = x_2$ and y_1 and y_2 are κ_2-adjacent,
(2) x_1 and x_2 are κ_1-adjacent and $y_1 = y_2$,
(3) x_1 and x_2 are κ_1-adjacent and y_1 and y_2 are κ_2-adjacent.

Definition 2.2. [6] Let (X, κ_1) and (Y, κ_2) be digital images. Then the function $f : X \to Y$ is (κ_1, κ_2)-continuous if and only if for every $\{x_0, x_1\} \subset X$ such that x_0 and x_1 are κ_1-adjacent, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_2-adjacent.

Example 2.3. Let $f : \mathbb{Z}^2 \to \mathbb{Z}$ be defined as $f(x) = x_1 + x_2$, for all $x = (x_1, x_2) \in \mathbb{Z}^2$. Then it is clear that f is $(4, 2)$-continuous. However it isn't $(8, 2)$-continuous, because $x = (x_1, x_2)$ and $y = (x_1 + 1, x_2 + 1)$ are 8-adjacent but $f(x) = x_1 + x_2$ and $f(y) = x_1 + x_2 + 2$ are not 2-adjacent.

Definition 2.4. [4] Let (X, κ_1) and (Y, κ_2) be two digital image and f and g be (κ_1, κ_2)-continuous functions. If

(1) there exist a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$, such that, for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$,
(2) the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$, $F_x(t) = F(x, t)$ for all $x \in X$ and for all $t \in [0, m]_{\mathbb{Z}}$, is $(2, \kappa_2)$-continuous and
(3) the induced function $F_t : X \to Y$, $F_t(x) = F(x, t)$ for all $x \in X$ and for all $t \in [0, m]_{\mathbb{Z}}$, is (κ_1, κ_2)-continuous,

then f and g are said to be digitally (κ_1, κ_2)-homotopic and F is called a digital (κ_1, κ_2)-homotopy between f and g, written $f \overset{F}{\simeq}_{(\kappa_1, \kappa_2)} g$ (or $f \simeq_{(\kappa_1, \kappa_2)} g$, for short).

The notation $[f]$ is used to denote the digital homotopy class of (κ_1, κ_2)-continuous function $f : X \to Y$, i.e.

$$[f] = \{g : X \to Y \mid \text{g is } (\kappa_1, \kappa_2)\text{-continuous and } f \simeq_{(\kappa_1, \kappa_2)} g\}.$$

The set of all digital homotopy classes of (κ_1, κ_2)-continuous functions is denoted by $[(X, \kappa_1), (Y, \kappa_2)]$, i.e.

$$[(X, \kappa_1), (Y, \kappa_2)] = \{[f] \mid f : (X, \kappa_1) \to (Y, \kappa_2) \text{ is } (\kappa_1, \kappa_2)\text{-continuous}\}.$$

For a digital image (X,κ) and its subset (A,κ), (X,A,κ) is called a digital image pair with κ-adjacency. Also, if A is a singleton set $\{p\}$, then (X,p,κ) is called a pointed digital image.

Definition 2.5. [4] Let (X,κ_1) and (Y,κ_2) be two digital image and f be a (κ_1,κ_2)-continuous function and g be a (κ_2,κ_1)-continuous function, such that

$$f\circ g\simeq_{(\kappa_2,\kappa_2)}1_Y \text{ and } g\circ f\simeq_{(\kappa_1,\kappa_1)}1_X.$$

Then (X,κ_1) and (Y,κ_2) are said to be same (κ_1,κ_2)-homotopy type or (κ_1,κ_2)-homotopy equivalent. Also f and g are called (κ_1,κ_2)-equivalences.

3. Digital H-spaces

In this section some properties of digital H-spaces and digital H-groups are investigated. It is shown that the set of homotopy classes of digitally continuous functions, from a homotopy associative digital H-space to a pointed digital image, is semigroup. Then it is proven that a pointed digital image having the same homotopy type as an abelian digital H-group is itself an abelian digital H-group. Also it is shown that there is a contravariant functor from the homotopy category of the pointed digital images to the category of abelian groups and homomorphisms.

Definition 3.1. [7] Let (X,p,κ) be a pointed digital image. For a digital continuous multiplication $\mu:X\times X\to X$ and the digital constant map $c:X\to X$, defined by $c(x)=p$, if $\mu\circ(c,1_X)\simeq_{(\kappa,\kappa)}\mu\circ(1_X,c)\simeq_{(\kappa,\kappa)}1_X$, then (X,p,κ) is called a digital H-space.

Example 3.2. Let $(\mathbb{Z},0,2)$ be pointed digital image and let $\mu:\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ defined as $\mu(x,y)=x+y$. It is clear that μ is $(4,2)$-continuous.

$$\begin{aligned}\mu\circ(c,1_X)(x) &= \mu(c(x),1_X(x))(x)=\mu(0,x)=0+x=x\\ \mu\circ(1_X,c)(x) &= \mu(1_X(x),c(x))(x)=\mu(x,0)=x+0=x\end{aligned}$$

where $c:\mathbb{Z}\to\mathbb{Z}$ is a constant map, $c(x)=0$. Therefore

$$\mu\circ(c,1_X)\simeq_{(2,2)}1_X\simeq_{(2,2)}\mu\circ(1_X,c).$$

Consequently $(\mathbb{Z},0,2)$ is a digital H-space.

Definition 3.3. [7] Let (X,p,κ) be a digital H-space. If

$$\mu\circ(1_X\times\mu)\simeq_{(\kappa^*,\kappa)}\mu\circ(\mu\times 1_X)$$

then μ is called digital homotopy associative.

If there exists a map $\eta:(X,p,\kappa)\to(X,p,\kappa)$ such that,

$$\mu\circ(\eta,1_X)\simeq_{(\kappa,\kappa)}\mu\circ(1_X,\eta)\simeq_{(\kappa,\kappa)}c$$

then η is called a digital homotopy inverse for μ.

If there exists a map $T:X\times X\to X\times X$, $T(x,y)=(y,x)$, such that

$$\mu\circ T\simeq_{(\kappa^*,\kappa)}\mu$$

then μ is called digital homotopy commutative and (X,p,κ) is called abelian digital H-space.

Theorem 3.4. *Let (X,p,κ_1) be a digital H-space with digital homotopy associative multiplication μ and (Y,q,κ_2) be a pointed digital image. Then $[(Y,q,\kappa_2),(X,p,\kappa_1)]$ is a semigroup with identity.*

Proof. For any $[f], [g] \in [(Y, q, \kappa_2), (X, p, \kappa_1)]$, let define the product

$$[f] \bullet [g] = [\mu \circ (f \times g) \circ \triangle]$$

where $\triangle : Y \to Y \times Y, \triangle(y) = (y, y)$ .Let $[f] = [f']$ and $[g] = [g']$, then there exist (κ_2, κ_1)-homotopies H and G such that $f \overset{G}{\simeq}_{(\kappa_2,\kappa_1)} f'$ and $g \overset{H}{\simeq}_{(\kappa_2,\kappa_1)} g'$. Define a digital homotopy $F : Y \times [0, m]_{\mathbb{Z}} \to X$ as $F = \mu \circ (G, H)$. Then

$$\begin{aligned} F(y, 0) &= \mu \circ (G, H)(y, 0) \\ &= \mu (G(y, 0), H(y, 0)) \\ &= \mu(f(y), g(y)) \\ &= (\mu \circ (f \times g) \circ \triangle)(y) \end{aligned}$$

and similarly $F(y, m) = (\mu \circ (f' \times g') \circ \triangle)(y)$. So

$$(\mu \circ (f \times g) \circ \triangle) \overset{F}{\simeq}_{(\kappa_2,\kappa_1)} (\mu \circ (f' \times g') \circ \triangle).$$

Then

$$\begin{aligned} [f] \bullet [g] &= [\mu \circ (f \times g) \circ \triangle] \\ &= [\mu \circ (f' \times g') \circ \triangle] \\ &= [f'] \bullet [g']. \end{aligned}$$

Consequently " $\bullet$ " is well defined.

Let $c : (X, p, \kappa_1) \to (X, p, \kappa_1)$ be the constant map $c(x) = p, \forall x \in X$. Let define a map $e : (Y, q, \kappa_2) \to (X, p, \kappa_1)$ such that $e(y) = p$, for all $y \in Y$. Then

$$(\mu \circ (e \times f) \circ \triangle)(y) = \mu (p, f(y)) = (\mu \circ (c, 1_X) \circ f)(y)$$

and since $\mu \circ (c, 1_X) \simeq_{(\kappa_1,\kappa_1)} 1_X$, then $[e] \bullet [f] = [\mu \circ (e \times f) \circ \triangle] = [f]$. So $[e]$ is the identity element of $[(Y, q, \kappa_2), (X, p, \kappa_1)]$.

Let $[f], [g], [h] \in [(Y, q, \kappa_2), (X, p, \kappa_1)]$.

$$\begin{aligned} [f] \bullet ([g] \bullet [h]) &= [f] \bullet [\mu \circ (g \times h) \circ \triangle] \\ &= [\mu \circ (f \times (\mu \circ (g \times h) \circ \triangle)) \circ \triangle] \\ &= [\mu \circ (1_X \times \mu) \circ (f \times g \times h) \circ (1_X \times \triangle) \circ \triangle] \\ &= [\mu \circ (\mu \times 1_X) \circ (f \times g \times h) \circ (1_X \times \triangle) \circ \triangle] \\ &= [\mu \circ ((\mu \circ (f \times g) \circ \triangle) \times h) \circ \triangle] \\ &= ([f] \bullet [g]) \bullet [h]. \end{aligned}$$

Therefore " $\bullet$ " is digital homotopy associative. □

Theorem 3.5. [7] *Let (X, p, κ_1) be a digital H-space and (Y, q, κ_2) be a pointed digital image. If (X, p, κ_1) and (Y, q, κ_2) have the same (κ_1, κ_2)-homotopy type, then (Y, q, κ_2) is a digital H-space.*

Definition 3.6. Let (X, p, κ_1) and (Y, q, κ_2) be digital H-spaces. A map $f : X \to Y$ is called a digital H-homomorphism if $f \circ \mu \simeq_{(\kappa_1,\kappa_2)} \eta \circ (f \times f)$, where $\eta : Y \times Y \to Y$.

Theorem 3.7. *Let (X, p, κ_1) be a digital H-space and (Y, q, κ_2) have the same (κ_1, κ_2)-homotopy type with X. Then digital (κ_1, κ_2)-equivalences are digital H-homomorphisms.*

Proof. Let $f : X \to Y$ be a (κ_1, κ_2)-continuous function and let $g : Y \to X$ be a (κ_2, κ_1)-continuous function, such that $f \circ g \simeq_{(\kappa_2,\kappa_2)} 1_Y$ and $g \circ f \simeq_{(\kappa_1,\kappa_1)} 1_X$. Let μ be (κ^*, κ_1)-continuous multiplication of (X, p, κ_1), then (Y, q, κ_2) is a digital H-space with the (κ^*, κ_2)- continuous multiplication $\eta = f \circ \mu \circ (g \times g)$. Then

$$g \circ \eta = g \circ (f \circ \mu \circ (g \times g)) \simeq_{(\kappa^*,\kappa_1)} 1_X \circ \mu \circ (g \times g) = \mu \circ (g \times g).$$

So g is a digital H-homomorphism. Also,

$$\eta \circ (f \times f) = f \circ \mu \circ (g \times g) \circ (f \times f) \simeq_{(\kappa^*,\kappa_2)} f \circ \mu \circ 1_{X\times X} = f \circ \mu.$$

Therefore f is a digital H-homomorphism. □

Definition 3.8. A digital H-group is a digital H-space (X, p, κ) with the digital homotopy associative multiplication μ and digital homotopy inverse η.

It is clear that $(\mathbb{Z}, 0, 2)$ in Example 7 is an abelian digital H-group.

Theorem 3.9. *Let (X, p, κ_1) be a digital H-group. If (X, p, κ_1) and (Y, q, κ_2) have the same (κ_1, κ_2)-homotopy type, then (Y, q, κ_2) is a digital H-group.*

Proof. Let $f : X \to Y$ be a (κ_1, κ_2)-continuous function and let $g : Y \to X$ be a (κ_2, κ_1)-continuous function, such that $f \circ g \simeq_{(\kappa_2,\kappa_2)} 1_Y$ and $g \circ f \simeq_{(\kappa_1,\kappa_1)} 1_X$. Let $\eta = f \circ \mu \circ (g \times g)$ be digital continuous multiplication of (Y, q, κ_2) where $\mu : X \times X \to X$. Then (Y, q, κ_2) is a digital H-space. Since $f \circ g \simeq_{(\kappa_2,\kappa_2)} 1_Y$,

$$\begin{aligned}
\eta \times 1_Y &= (f \circ \mu \circ (g \times g)) \times 1_Y \simeq_{(\kappa^*,\kappa_2)} (f \times f) \circ (\mu \times 1_X) \circ (g \times g \times g) \\
1_Y \times \eta &= 1_Y \times (f \circ \mu \circ (g \times g)) \simeq_{(\kappa^*,\kappa_2)} (f \times f) \circ (1_X \times \mu) \circ (g \times g \times g).
\end{aligned}$$

As (X, p, κ_1) is a digital H-group, μ is digital homotopy associative. Then,

$$\begin{aligned}
\eta \circ (\eta \times 1_Y) &\simeq {}_{(\kappa^*,\kappa_2)} f \circ \mu \circ (g \times g) \circ (f \times f) \circ (\mu \times 1_X) \circ (g \times g \times g) \\
&\simeq {}_{(\kappa^*,\kappa_2)} f \circ \mu \circ 1_{X\times X} \circ (\mu \times 1_X) \circ (g \times g \times g) \\
&= f \circ (\mu \circ (\mu \times 1_X)) \circ (g \times g \times g) \\
&\simeq {}_{(\kappa^*,\kappa_2)} f \circ (\mu \circ (1_X \times \mu)) \circ (g \times g \times g) \\
&= f \circ \mu \circ 1_{X\times X} \circ (1_X \times \mu) \circ (g \times g \times g) \\
&\simeq {}_{(\kappa^*,\kappa_2)} f \circ \mu \circ (g \times g) \circ (f \times f) \circ (1_X \times \mu) \circ (g \times g \times g) \\
&\simeq {}_{(\kappa^*,\kappa_2)} \eta \circ (1_Y \times \eta).
\end{aligned}$$

Therefore η is digital homotopy associative.

Let $\theta : (X, p, \kappa_1) \to (X, p, \kappa_1)$ be digital homotopy inverse for μ and $\theta' = f \circ \theta \circ g$. Then,

$$\begin{aligned}
\eta \circ (\theta', 1_Y) &= (f \circ \mu) \circ (g \times g) \circ (\theta', 1_Y) \\
&= (f \circ \mu) \circ (g \times g) \circ (f \circ \theta \circ g, 1_Y) \\
&= (f \circ \mu) \circ (g \circ f \circ \theta \circ g, g \circ 1_Y) \\
&\simeq {}_{(\kappa_2,\kappa_2)} (f \circ \mu) \circ (1_X \circ \theta \circ g, g) \\
&= f \circ (\mu \circ (\theta, 1_X)) \circ g \\
&\simeq {}_{(\kappa_2,\kappa_2)} f \circ c \circ g \\
&\simeq {}_{(\kappa_2,\kappa_2)} c',
\end{aligned}$$

where $c' : (Y, q, \kappa_2) \to (Y, q, \kappa_2)$, $c'(y) = q$, $\forall y \in Y$. Similarly, $\eta \circ (1_Y \circ \theta') \simeq_{(\kappa_2,\kappa_2)} c'$. Hence θ' is a homotopy inverse for (Y, q, κ_2).

Consequently (Y, q, κ_2) is a digital H-group. □

Theorem 3.10. *Let (X, p, κ_1) be an abelian digital H-group. If (X, p, κ_1) and (Y, q, κ_2) have the same (κ_1, κ_2)-homotopy type, then (Y, q, κ_2) is an abelian digital H-group.*

Proof. Let $f : X \to Y$ be a (κ_1, κ_2)-continuous function and let $g : Y \to X$ be a (κ_2, κ_1)-continuous function, such that $f \circ g \simeq_{(\kappa_2,\kappa_2)} 1_Y$ and $g \circ f \simeq_{(\kappa_1,\kappa_1)} 1_X$. Let μ and $\eta = f \circ \mu \circ (g \times g)$ be digital continuous multiplication of (X, p, κ_1) and (Y, q, κ_2), respectively. As μ is digital homotopy commutative, there exists a function $T : X \times X \to X \times X, T(x_1, x_2) = (x_2, x_1)$ such that $\mu \circ T \simeq_{(\kappa^*,\kappa_1)} \mu$. Now consider the function $T' : Y \times Y \to Y \times Y, T'(y_1, y_2) = (y_2, y_1)$. Then,

$$\eta \circ T' = f \circ \mu \circ (g \times g) \circ T' = f \circ \mu \circ T \circ (g \times g) \simeq_{(\kappa^*,\kappa_2)} f \circ \mu \circ (g \times g) = \eta.$$

So η is digital homotopy commutative. □

Proposition 1. [7] Let (X, p, κ_1) be a digital image, (Y, q, κ_2) be a digital H-group with digital continuous multiplication η. Then $[(X, p, \kappa_1), (Y, q, \kappa_2)]$ is a group under the product $[f] \bullet [g] = [\eta \circ (f \times g) \circ \triangle]$, for all $[f], [g] \in [(X, p, \kappa_1), (Y, q, \kappa_2)]$. Also if (Y, q, κ_2) is an abelian digital H-group, then

$$([(X, p, \kappa_1), (Y, q, \kappa_2)], \bullet)$$

is abelian.

Theorem 3.11. *A homotopy associative digital H-space (X, p, κ) is a digital H-group if and only if the map $\varphi : X \times X \to X \times X$ defined by $\varphi(x, y) = (x, xy)$ is a (κ^*, κ^*)-homotopy equivalence.*

Proof. Let (X, p, κ) be a digital H-group with digital homotopy inverse $\eta : X \to X$. Consider the digital continuous map $j : X \times X \to X \times X$ defined by $j(x, y) = (x, \eta(x)y)$. Now $(\psi \circ j)(x, y) = \psi(x, \eta(x)y) = (x, x\varphi(x)y)$ implies that $\psi \circ j \simeq_{(\kappa^*,\kappa^*)} 1_{X\times X}$, since η is the digital homotopy inverse. Also

$$(j \circ \psi)(x, y) = j(x, xy) = (x, \eta(x)xy)$$

implies $j \circ \psi \simeq_{(\kappa^*,\kappa^*)} 1_{X\times X}$. Hence ψ is (κ^*, κ^*) homotopy equivalence. Conversely, let $\Omega : X \times X \to X \times X$ be the digital homotopy inverse of ψ such that $\psi \circ \Omega \simeq_{(\kappa^*,\kappa^*)} \Omega \circ \psi \simeq_{(\kappa^*,\kappa^*)} 1_{X\times X}$. Now we go ahead to prove that (X, p, κ_1) is a digital H-group. Define $\varphi : X \to X$ by the composite

$$X \overset{i_1}{\to} X \times X \overset{\Omega}{\to} X \times X \overset{p_2}{\to} X.$$

where $i_1 : X \to X \times X$ is defined by $i_1(x) = (x, p)$ and $pi : X \times X \to X$ are the projections . Let $\mu : X \times X \to X$ be the digital continuous multiplication of (X, p, κ), then $p_1 \circ \psi = p_1, p_2 \circ \psi = \mu$ and therefore

$$\begin{aligned} p_1 &\simeq {}_{(\kappa^*,\kappa)} p_1 \circ \psi \circ \Omega = p_1 \circ \Omega \\ p_2 &\simeq {}_{(\kappa^*,\kappa)} p_2 \circ \psi \circ \Omega = \mu \circ \Omega. \end{aligned}$$

In particular, $p_1 \circ \Omega \circ i_1 \simeq_{(\kappa,\kappa)} p_1 \circ i_1 = 1_X$. Hence

$$\begin{aligned} \mu \circ (1_X, \varphi) &= \mu \circ (p_1 \circ \Omega \circ i_1,\, p_2 \circ \Omega \circ i_1) \\ &= \mu \circ (p_1, p_2) \circ (\Omega \circ i_1) \\ &= \mu \circ \Omega \circ i_1 \simeq_{(\kappa,\kappa)} p_2 \circ i_1 = c \end{aligned}$$

where $c : X \to X, \quad c(x) = p$ is the digital constant map. Similarly,

$$\mu \circ (\varphi, 1_X) \simeq_{(\kappa,\kappa)} c.$$

Hence (X, p, κ) is a homotopy associative digital H-space such that φ is the digital homotopy inverse. Consequently, (X, p, κ) is a digital H-group. $\square$

Definition 3.12. [16] Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravariant functor T from $\mathcal{C}$ to $\mathcal{D}$ is a mapping which associates to every object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$ and associates to every morphism $f : X \to Y$ of C a morphism

$$T(f) : T(Y) \to T(X)$$

of D such that, $T(1_X) = 1_{T(X)}$ and $T(gf) = T(f)T(g)$.

Theorem 3.13. [15] *For any category $\mathcal{C}$ and object Y of $\mathcal{C}$, there is a contravariant functor Π^Y from $\mathcal{C}$ to the category of sets and functions which associates to an object X of $\mathcal{C}$ the set $\Pi^Y(X) = \hom(X, Y)$ and to a morphism $f : X \to X'$ the function $\Pi^Y(f) = f^* : \hom(X', Y) \to \hom(X, Y)$ defined by $f^*(g') = g' \circ f$, for $g' : X' \to Y$.*

Definition 3.14. The category whose objects are pointed digital images and the set of morphisms is $\hom((X, p, \kappa_1), (Y, q, \kappa_2)) = [(X, p, \kappa_1), (Y, q, \kappa_2)]$ is called the homotopy category of the pointed digital images.

Theorem 3.15. *Let (X, p, κ) be a digital H-space with the digital continuous multiplication μ and (Y, q, κ_1) and (Z, r, κ_2) be any pointed digital images. Then there exists a homomorphism from $[(Z, r, \kappa_2), (X, p, \kappa)]$ to $[(Y, q, \kappa_1), (X, p, \kappa)]$.*

Proof. Let $h : (Y, q, \kappa_1) \to (Z, r, \kappa_2)$ be a map and $[g], [g'] \in [(Z, r, \kappa_2), (X, p, \kappa)]$. Let define $h^* : [(Z, r, \kappa_2), (X, p, \kappa)] \to [(Y, q, \kappa_1), (X, p, \kappa)]$ as $h^*([g]) = [g \circ h]$. Then

$$\begin{aligned}
h^*([g] \bullet [g']) &= h^*([\mu \circ (g \times g') \circ \triangle]) \\
&= [\mu \circ (g \times g') \circ \triangle \circ h] \\
&= [\mu \circ ((g \circ h) \times (g' \circ h)) \circ \triangle] \\
&= [g \circ h] \bullet [g' \circ h] \\
&= h^*([g]) \bullet h^*([g']).
\end{aligned}$$

Therefore h* is a homomorphism. $\square$

Theorem 3.16. *Let (Y, q, κ) be a digital H-group, then Π^Y is a contravariant functor from the homotopy category of the pointed digital images to the category of groups and homomorphisms.*

Proof. Let (X, p, κ_1) and (Z, r, κ_2) be objects and $[f] \in [(X, p, \kappa_1), (Z, r, \kappa_2)]$ is a morphism of the homotopy category of the pointed digital images.

$\Pi^Y((X, p, \kappa_1)) = \hom((X, p, \kappa_1), (Y, q, \kappa)) = [(X, p, \kappa_1), (Y, q, \kappa)]$.

Therefore $\Pi^Y((X, p, \kappa_1))$ is a group.

$$\begin{aligned}
\Pi^Y([f]) &= f^* : \hom((Z, r, \kappa_2), (Y, q, \kappa)) \to \hom((X, p, \kappa_1), (Y, q, \kappa)) \\
&\Rightarrow f^* : [(Z, r, \kappa_2), (Y, q, \kappa)] \to [(X, p, \kappa_1), (Y, q, \kappa)]
\end{aligned}$$

is a function defined as, $f^*([g]) = [g \circ f]$, for any $[g] \in [(Z, r, \kappa_2), (Y, q, \kappa)]$. So, f^* is a morphism between groups. Also by Theorem 19, f^* is a homomorphism.

Let show that Π^Y is a contravariant functor.

Let $[h] \in [(Z, r, \kappa_2) \to (W, t, \kappa_3)]$. For any morphism $[h'] \in [(W, t, \kappa_3), (Y, q, \kappa)]$,

$$\begin{aligned} \Pi^Y([h])([h']) &= [h' \circ h] \\ \Pi^Y([f])([h' \circ h]) &= [(h' \circ h) \circ f] \\ &= [h' \circ (h \circ f)] \\ &= \Pi^Y([h \circ f])([h']). \end{aligned}$$

Thus,

$$\Pi^Y([f])([h' \circ h]) = \Pi^Y([f])\left(\Pi^Y([h])([h'])\right) = \left(\Pi^Y([f]) \circ \Pi^Y([h])\right)([h']).$$

So, $\Pi^Y([h \circ f]) = \Pi^Y([f]) \circ \Pi^Y([h])$.

Let $[1_X] \in [(X, p, \kappa_1), (X, p, \kappa_1)]$ be the unit morphism of the digital homotopy category of pointed digital images. Then,

$$\Pi^Y([1_X]) = 1_X^* : [(X, p, \kappa_1), (Y, q, \kappa)],\ 1_X^*[(h)] = [h].$$

Consequently Π^Y is a contravariant functor. □

Corollary 3.17. Let (Y, q, κ) be an abelian digital H-group, then Π^Y is a contravariant functor from the homotopy category of the pointed digital images to the category of abelian groups and homomorphisms.

Theorem 3.18. *Let (Y, q, κ_1) and (Z, r, κ_2) be digital H-spaces with the multiplications μ and η, respectiveliy, and $h : (Y, q, \kappa_1) \to (Z, r, \kappa_2)$ be a digital H-homomorphism. Then there exist a homomorphism from $[(X, p, \kappa), (Y, q, \kappa_1)]$ to $[(X, p, \kappa), (Z, r, \kappa_2)]$ for any pointed digital image (X, p, κ).*

Proof. Let $[g], [g'] \in [(X, p, \kappa), (Y, q, \kappa_1)]$ and

$$h_* : [(X, p, \kappa), (Y, q, \kappa_1)] \to [(X, p, \kappa), (Z, r, \kappa_2)]$$

be a map such that $h_*([g]) = [h \circ g]$. Then

$$h_*([g] \bullet [g']) = h_*([\mu \circ (g \times g') \circ \triangle]) = [h \circ \mu \circ (g \times g') \circ \triangle].$$

Since h is a digital homomorphism, then $\eta \circ (h \times h) \simeq_{(\kappa^*, \kappa_2)} h \circ \mu$. So

$$\begin{aligned} [h \circ \mu \circ (g \times g') \circ \triangle] &= [\eta \circ (h \times h) \circ (g \times g') \circ \triangle] \\ &= [h \circ g] \bullet [h \circ g'] \\ &= h_*([g]) \bullet h_*([g']). \end{aligned}$$

Therefore h_* is a homomorphism. □

Corollary 3.19. Let (X, p, κ) be a digital H-space and (Y, q, κ_1) be a digital image. If (X, p, κ) and (Y, q, κ_1) have the same (κ, κ_1)-homotopy type, then there exist homotopy equivalences $f : (X, p, \kappa) \to (Y, q, \kappa_1)$ and $g : (Y, q, \kappa_1) \to (X, p, \kappa)$. For any digital image (Z, r, κ_2)

(1) since f is a digital H-homomorphism, then there exist a homomorphism from $[(Z, r, \kappa_2), (X, p, \kappa)]$ to $[(Z, r, \kappa_2), (Y, q, \kappa_1)]$
(2) since g is a digital H-homomorphism, then there exist a homomorphism from $[(Z, r, \kappa_2), (Y, q, \kappa_1)]$ to $[(Z, r, \kappa_2), (X, p, \kappa)]$.

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