Journal of Universal Mathematics
Vol.4 No.1 pp.1-12 (2021)
ISSN-2618-5660
DOI: 10.33773/jum.837840

SOME CLASSES OF q-ANALYTIC FUNCTIONS AND THE q-GREEN'S FORMULA

İLKER GENÇTÜRK, ŞERMIN HÖKELEKLI, AND KERIM KOCA

Abstract. In this paper, we first give two new definitions for q-analytic functions. We also define a new line q-integral. Finally, using these q-integrals we obtain a version of complex q-Green's formula.

1. Introduction and Preliminaries

In complex analysis, q-analogues of classical analytic (holomorphic) functions are defined by several mathematicians in different ways [1, 2, 3, 4]. Moreover, there are many articles where various q-integrals were defined for complex discrete functions on complex discrete sets, and q-Green integrals were obtained using these discrete integrals [2, 5, 6].

In this paper we define a discrete q-line integral for q-analytic functions in the sense of Pashaev-Nalcı, and we present a q-analogue of the Green's formula on the complex plane using this type of an integral.

Now, we will recall some basic definitions in q- calculus:

Let $0 < q < 1$ and $a \in \mathbb{R}$. The q-analogue of a is defined as

$$[a]_q = \frac{1 - q^a}{1 - q}. \tag{1.1}$$

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we use the shorthand notation

$$\begin{aligned} (1 + x)_n &= (1 + x)(1 + qx) \cdots (1 + q^{n-1}x); \quad (1 + x)_0 = 1, \\ (1 + x)_\infty &= \lim_{n\to\infty} (1 + x)_n. \end{aligned} \tag{1.2}$$

For $m, n \in \mathbb{N}$ and $n \geq m$, the q-factorial and the q-analogues of the binomial numbers are defined respectively as

$$[n]_q! = [1]_q[2]_q \ldots [n]_q = \frac{(1 - q)_n}{(1 - q)^n}, \tag{1.3}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q!\ [n - m]_q!} = \frac{(1 - q)_n}{(1 - q)_m\ (1 - q)_{n-m}}. \tag{1.4}$$

Other definitions and concepts will be introduced in the course of the text.

Date: December, 2020.

2000 *Mathematics Subject Classification.* 05A30; 30G25.

Key words and phrases. q-differential, q-integral, q-analytic function.

Let us now consider the discrete set

$$Q = \{(q^m x, q^n y) = q^m x + i q^n y : m, n \in \mathbb{Z};\ x > 0,\ y > 0\}. \tag{1.5}$$

Definition 1.1 ([2]). Given $z_j = x_j + iy_j \in Q$. If z_{j+1} is one of

$$(qx_j, y_j),\ (q^{-1}x_j, y_j),\ (x_j, qy_j),\ (x_j, q^{-1}y_j),$$

then z_j and z_{j+1} are called *adjacent points*.

Definition 1.2. For adjacent points $z_j, z_{j+1} \in Q$, the expression

$$\gamma := \langle z_0, z_1, \ldots, z_n \rangle \tag{1.6}$$

defines a *q-discrete curve* in Q. If $z_i \neq z_j$ for $i \neq j$, the curve is called a *simple discrete curve*. If $z_0 = z_n$, it is called a *simple closed discrete curve*.

Definition 1.3. Let us consider the curve γ as defined in (1.6). The curve

$$\gamma^{-1} := \langle z_n, z_{n-1}, \ldots, z_1, z_0 \rangle \tag{1.7}$$

is called the *opposite-oriented* γ.

Definition 1.4. For $z = x + iy \in Q$, the discrete set

$$S(z) = \{z = x + iy,\ z_1 = x + iq^{-1}y,\ z_2 = qx + iy,\ z_3 = qx + iq^{-1}y\} \tag{1.8}$$

is called a *fundamental set* with respect to z.

Let us denote the elements of Q lying in the discrete closed curve $\gamma := \langle z_0, z_1, \ldots, z_n = z_0 \rangle$ by C, and let $\overline{C} := C \cup \gamma$. Then, every finite subset of Q can be written as the union of fundamental sets

$$\overline{C} = \bigcup_{i=1}^{N} S(z_i). \tag{1.9}$$

Let us also consider the subset

$$T(z) = \{z = x + iy,\ z_1 = x + iq^{-1}y, z_2 = qx + iy\} \subset S(z). \tag{1.10}$$

For $\overline{C}$ as in (1.9), let us define the subset

$$\overline{C}_q := \{z_i : z_i \in S(z_i);\ i = 1, 2, \ldots, N\} \subset Q. \tag{1.11}$$

2. Classes of q-analytic functions

Let $f(z)$ be a discrete function defined on the discrete set Q. We define the discrete partial differential operators

$$D_{q,x} f(z) = \frac{f(z) - f(qx, y)}{(1-q)x}; \quad D_{q,y} f(z) = \frac{f(z) - f(x, qy)}{(1-q)y}. \tag{2.1}$$

We note that in [4], complex q-differential operators $D_{q,z}$ and $D_{q,\overline{z}}$ are defined as

$$D_{q,z} := \frac{1}{2}\left[D_{q,x} - iM^y_{\frac{1}{q}} D_{q,y}\right]; \quad D_{q,\overline{z}} := \frac{1}{2}\left[D_{q,x} + iM^y_{\frac{1}{q}} D_{q,y}\right] \tag{2.2}$$

where $M^y_q f(x, y) = f(x, qy)$ is the dilatation operator.

Definition 2.1 ([4])**.** If a complex valued discrete function $f(x,y)$ satisfies

$$D_{q,\bar{z}}f(x,y)=\frac{1}{2}\left[D_{q,x}f(x,y)+iM^{y}_{\frac{1}{q}}D_{q,y}f(x,y)\right]=0 \tag{2.3}$$

for $z\in T(z)$, then $f(x,y)$ is called a *q-analytic function at point z* in the sense of Pashaev–Nalcı.

Example 2.2. For $n\in\mathbb{N}$, the complex q-binomial expansions

$$\Phi^{(n)}_q(x,y)=(x+iy)(x+iqy)\cdots(x+iq^{n-1}y)\equiv\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\frac{k(k-1)}{2}}x^{n-k}(iy)^k \tag{2.4}$$

are q-analytic in the sense of Pashaev-Nalcı. Moreover, they satisfy

$$D_{q,z}\Phi^{(n)}_q(x,y)=[n]_q\Phi^{(n-1)}_q(x,y).$$

Remark 2.3. In [2], the q-analyticity of $f(z)$ is characterized by the equation

$$D_{q,x}f(x,y)=-iD_{q,y}f(x,y). \tag{2.5}$$

That is, $f(z)$ is called q-analytic in the sense of Harman when the equation

$$\frac{f(z)-f(qx,y)}{(1-q)x}=\frac{f(z)-f(x,qy)}{(1-q)iy} \tag{2.6}$$

holds.

Example 2.4. Let $n\in\mathbb{N}$. The class of function given by

$$\Psi^{(n)}_q(x,y)=\sum_{j=0}^{n}\frac{(iy)^j}{[j]_q!}D_{q,x}(x^j)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q x^j(iy)^{n-j} \tag{2.7}$$

is q-analytic in the sense of Harman, but not in the sense of Pashaev-Nalcı. The functions $\Phi^{(n)}_q(x,y)$ $(n=2,3,\dots)$ defined in (2.4) are not q-analytic in the sense of Harman.

A necessary and sufficient condition for a discrete function $f(z)$ to be q-analytic in the sense of Pashaev-Nalcı is that

$$L_1f(x,y):=(qx+iy)f(x,y)-iyf(qx,y)-qxf(x,q^{-1}y)=0. \tag{2.8}$$

This identity can easily be derived from (2.3).

Similarly, a necessary and sufficient condition for a discrete function $f(z)$ to be q-analytic in the sense of Harman is that

$$L_2f(x,y):=\bar{z}f(z)-xf(x,qy)+iyf(qx,y)=0, \tag{2.9}$$

which be derived from (2.6).

Let $p=q^{-1}$. Consider the differential operator

$$D_{p,x}f(z)=\frac{f(z)-f(px,y)}{(1-p)x};\quad D_{p,y}f(z)=\frac{f(z)-f(x,py)}{(1-p)y} \tag{2.10}$$

for a discrete function $f(z)$.

Definition 2.5. If a complex-valued discrete function $g(x,y)$ satisfies

$$D_{p,x}g(x,y)=-iM^{y}_{\frac{1}{p}}D_{p,y}g(x,y), \tag{2.11}$$

it is called *p-analytic in the sense of Pashaev-Nalcı.*

A necessary and sufficient condition for $g(x,y)$ to be p-analytic on a suitable discrete set Q is that

$$B[g(x,y)] := (px+iy)f(x,y) - iyf(px,y) - pxf(x,p^{-1}y) = 0 \tag{2.12}$$

which can be obtained from (2.11).

Example 2.6. $g(x,y) = x^2 + (1+q^{-1})ixy - q^{-1}y^2$ is p-analytic, but not q-analytic.

In this paper, we present another two classes of q-analytic functions.

Let $D_{q,x}$ and $D_{q,y}$ be the partial q-differential operators as given in (2.1). Using these operators we can define complex differential operators

$$\begin{aligned} D^*_{q,z} &\equiv \frac{1}{2}\left(M^x_{\frac{1}{q}} D_{q,x} - iD_{q,y}\right), \\ D^*_{q,\overline{z}} &\equiv \frac{1}{2}\left(M^x_{\frac{1}{q}} D_{q,x} + iD_{q,y}\right). \end{aligned} \tag{2.13}$$

Definition 2.7. If a discrete function $h(x,y)$ defined on a suitable discrete set Q satisfies

$$D^*_{q,\overline{z}} h(x,y) = 0, \tag{2.14}$$

it is said to be *q-analytic with respect to the operator D^*_q*.

This definition of q-analyticity is different from previous definitions.

Example 2.8. The functions defined by

$$P^{(n)}_q(x,y) = (x+iy)(qx+iy)\cdots(q^{n-1}x+iy); \quad n = 1,2,\ldots \tag{2.15}$$

are all q-analytic with respect to the operator $D^*_{q,\overline{z}}$. In other words, we have

$$D^*_{q,\overline{z}} P^{(n)}_q(x,y) \equiv 0.$$

Moreover, the equality

$$D^*_{q,z} P^{(n)}_q(x,y) = [n]_q P^{(n-1)}_q(x,y)$$

holds.

Remark 2.9. A necessary and sufficient condition for $h(x,y)$ to be q-analytic with respect to D^*_q is that

$$E[h(x,y)] = (x+iqy)h(x,y) - xh(x,qy) - iqyh(q^{-1}x,y) \tag{2.16}$$

which follows from (2.13) and (2.14).

Using equation (2.1) again, we can define complex differential operators

$$\begin{aligned} D^{**}_{q,\overline{z}} &:= \frac{1}{2}\left(M^x_{\frac{1}{q}} D_{q,x} + iM^y_{\frac{1}{q}} D_{q,y}\right), \\ D^{**}_{q,z} &:= \frac{1}{2}\left(M^x_{\frac{1}{q}} D_{q,x} - iM^y_{\frac{1}{q}} D_{q,y}\right). \end{aligned} \tag{2.17}$$

Definition 2.10. If a discrete function $k(x,y)$ defined on a suitable discrete set Q satisfies

$$D^{**}_{q,\overline{z}} k(x,y) = 0,$$

it is called q-analytic with respect to the operator D^{**}_q.

Example 2.11. All functions defined as

$$R_q^{(n)}(x,y)=\sum_{j=0}^{n} q^{-j(n-j)}\begin{bmatrix} n \\ j \end{bmatrix}_q (iy)^j x^{n-j}; \quad n=1,2,\dots \tag{2.18}$$

are q-analytic with respect to D_q^{**}.

In other words, we have

$$D_{q,\bar{z}}^{**}R_q^{(n)}(x,y)\equiv\frac{1}{2}\left[M_{\frac{1}{q}}^{x}D_{q,x}R_q^{(n)}(x,y)+iM_{\frac{1}{q}}^{y}D_{q,y}R_q^{(n)}(x,y)\right]\equiv 0.$$

Moreover, the equality

$$D_{q,z}^{**}R_q^{(n)}(x,y)=[n]_q R_q^{(n-1)}(x,y) \tag{2.19}$$

holds.

3. Complex Line q-Integrals

Let $z_j=x_j+iy_j,\ z_{j+1}=x_{j+1}+iy_{j+1}\in Q$ be two adjacent points. We define the integral of a discrete function $f(z)$ from z_j to z_{j+1} by

$$\int_{z_j}^{z_{j+1}} f(z)\,d_qz=\begin{cases}(z_{j+1}-z_j)f(z_j) & \text{if } z_{j+1}=qx_j+iy_j \text{ or } z_{j+1}=x_j+iq^{-1}y_j\\ (z_{j+1}-z_j)f(z_{j+1}) & \text{if } z_{j+1}=x_j+iqy_j \text{ or } z_{j+1}=q^{-1}x_j+iy_j.\end{cases} \tag{3.1}$$

In this case, on a simple discrete curve

$$\gamma=\langle z_0,z_1,\dots,z_n\rangle$$

lying in the discrete set Q the q-integral of $f(z)$ on γ can be defined as

$$\int_{\gamma} f(z)\,d_qz=\int_{z_0}^{z_n} f(z)\,d_qz=\sum_{j=0}^{n-1}\int_{z_j}^{z_{j+1}} f(z)\,d_qz. \tag{3.2}$$

This integral in (3.2) satisfies the classical properties of line integrals such as additivity, linearity, and orientation-dependence.

Remark 3.1. In [2], the discrete line q-integral is defined (under the same hypotheses) as

$$\int_{z_j}^{z_{j+1}} f(z)\,d_qz=\begin{cases}(z_{j+1}-z_j)f(z_j) & \text{if } z_{j+1}=qx_j+iy_j \text{ or } z_{j+1}=x_j+iqy_j\\ (z_{j+1}-z_j)f(z_{j+1}) & \text{if } z_{j+1}=q^{-1}x_j+iy_j \text{ or } z_{j+1}=x_j+iq^{-1}y_j,\end{cases} \tag{3.3}$$

and

$$\int_{\gamma} f(z)\,d_qz=\int_{z_0}^{z_n} f(z)\,d_qz=\sum_{j=0}^{n-1}\int_{z_j}^{z_{j+1}} f(z)\,d_qz.$$

The integrals in (3.2) and (3.3) are similar but not identical.

Definition 3.2. Let $z_0 \in \overline{C}_q$ be a fixed point, and let $z \in \overline{C}_q$ represent a variable point. The expression

$$F(z) := \int_{z_0}^{z} f(\zeta)\, d_q\zeta \tag{3.4}$$

is called the indefinite q-integral of $f(z)$.

Theorem 3.3. *If a discrete function $f(z)$ is q-analytic on $\overline{C}_q$ in the sense of Pashaev-Nalcı, then the integral in* (3.4) *is path-independent.*

Proof. This can be easily seen from the definition (3.1) and the equality (2.8).

For example, let

$$\gamma_1 = \langle z_1 = x + iy,\ z_2 = x + iq^{-1}y,\ z_3 = qx + iq^{-1}y\rangle,$$
$$\gamma_2 = \langle z_1 = x + iy,\ z_4 = qx + iy,\ z_5 = z_3 = qx + iq^{-1}y\rangle.$$

In this case, we have

$$A = \int_{\gamma_1} f(z)\, d_qz = (q^{-1} - 1)iyf(x, y) + (q - 1)xf(x, q^{-1}y), \tag{3.5}$$

$$B = \int_{\gamma_2} f(z)\, d_qz = (q^{-1} - 1)iyf(qx, y) + (q - 1)xf(x, y), \tag{3.6}$$

and hence, by using (2.8),

$$qxf(x, q^{-1}y) = (qx + iy)f(x, y) - iyf(qx, y). \tag{3.7}$$

If we substitute this in (3.5), we see that $A = B$. □

Theorem 3.4. *If $f(z)$ is q-analytic on $\overline{C}_q$ in the sense of Pashaev-Nalcı, and if $\gamma = \langle z_0, z_1, \ldots, z_n\rangle$ is a simple discrete curve in $\overline{C}_q$, then*

$$\int_{\gamma} f(z)\, d_qz = F(z_n) - F(z_0) \tag{3.8}$$

where $F(z)$ is as given in (3.4)*.*

Proof. This is true since $F(z_0) = 0$ and $F(z_n) = \int_{z_0}^{z_n} f(\zeta)\, d_q\zeta$. □

Theorem 3.5. *If $f(z)$ is q-analytic on $\overline{C}_q$ in the sense of Pashaev-Nalcı, and if $\gamma = \langle z_0, z_1, \ldots, z_n\rangle$ is a simple discrete curve in $\overline{C}_q$, then*

$$\int_{\gamma} D_{q,z}f(z)\, d_qz = f(z_n) - f(z_0). \tag{3.9}$$

Proof. Let us prove the statement for $n = 1$. For $z_0 = x + iy$ and $z_1 = x + iq^{-1}y$ we have

$$\begin{aligned} D_{q,z}f(x, y) &= \frac{1}{2(1 - q)}\left\{\frac{1}{x}\left[f(x, y) - f(qx, y)\right] - \frac{iq}{y}\left[f(x, q^{-1}y) - f(x, y)\right]\right\} \\ &=: \varphi(x, y) \end{aligned}$$

from (2.1) and (2.2).

Thus, using (3.2) we see that

$$\int_{z_0}^{z_1} D_{q,z} f(z)\, d_q z = (z_1 - z_0)\varphi(z_0) = \frac{1-q}{q} iy\varphi(x,y)$$
$$= \frac{1}{2q} iy \left\{ \frac{1}{x}\left[f(x,y) - f(qx,y)\right] - \frac{iq}{y}\left[f(x,q^{-1}y) - f(x,y)\right]\right\}$$
$$= \frac{1}{2}\left[\frac{1}{x} q^{-1} iyf(x,y) - \frac{1}{x} q^{-1} iyf(qx,y) + f(x,q^{-1}y) - f(x,y)\right] \tag{3.10}$$

where $z_0 = x + iy$.

From (2.8) we have

$$q^{-1} y f(qx,y) = xf(x,y) + iq^{-1} yf(x,y) - xf(x,q^{-1}y). \tag{3.11}$$

Using (3.11) in (3.10) we obtain

$$\int_{z_0}^{z_1} D_{q,z} f(z)\, d_q z = f(x,q^{-1}y) - f(x,y) = f(z_1) - f(z_0). \tag{3.12}$$

Considering the property in (3.12) with (3.2), we see that the statement holds true for all $n \in \mathbb{N}$. □

Remark 3.6. We observe that

$$\int_{\gamma} f(z)\, d_q z = - \int_{\gamma^{-1}} f(z)\, d_q z \tag{3.13}$$

from (1.7) and (3.2).

Theorem 3.7. *For $F(z)$ as in* (3.4) *we have*

$$D_{q,z} F(z) = f(z).$$

Proof. Let z_0 be a fixed point. Since

$$F(z) = F(x,y) = \int_{z_0}^{z} f(\zeta)\, d_q\zeta,$$

we have

$$D_{q,z} F(z) = \frac{1}{2(1-q)} \left\{ \frac{1}{x}\left[F(x,y) - F(qx,y)\right] - \frac{iq}{y}\left[F(x,q^{-1}y) - F(x,y)\right]\right\}$$
$$= \frac{1}{2(1-q)} \left\{ \frac{1}{x}\left[\int_{z_0}^{(x,y)} f(\zeta)\, d_q\zeta - \int_{z_0}^{(qx,y)} f(\zeta)\, d_q\zeta\right]\right.$$
$$\left. - \frac{iq}{y}\left[\int_{z_0}^{(x,q^{-1}y)} f(\zeta)\, d_q\zeta - \int_{z_0}^{(x,y)} f(\zeta)\, d_q\zeta\right]\right\} \tag{3.14}$$

using (2.2).

On the other hand, if we let $z_0 = x_0 + iy_0$ and $z = x + iy$, we get from (3.1) that

$$I_1 = \int_{z_0}^{z} f(\zeta)\, d_q\zeta - \int_{z_0}^{(qx,y)} f(\zeta)\, d_q\zeta = \int_{z_0}^{z} f(\zeta)\, d_q\zeta - \int_{z_0}^{z} f(\zeta)\, d_q\zeta - \int_{z}^{(qx,y)} f(\zeta)\, d_q\zeta$$

$$= - \int_{z}^{(qx,y)} f(\zeta)\, d_q\zeta = -(qx + iy - x - iy) f(z) = (1-q) x f(z).$$

Similarly, we have

$$I_2 = \int_{z_0}^{(x,q^{-1}y)} f(\zeta)\, d_q\zeta = \frac{1-q}{q} iy f(z).$$

If we substitute I_1 and I_2 in (3.14), we obtain

$$D_{q,z}F(z) = \frac{1}{2(1-q)} \left[(1-q)f(z) + (1-q)f(z)\right] = f(z).$$

□

Remark 3.8. For $z = x + iy$, $z_1 = qx + iy$, and $z_2 = x + iq^{-1}y$, it is easy to see that

$$\int_{z}^{z_1} f(\zeta)\, d_q\zeta = \int_{z}^{z_2} f(\zeta)\, d_q\zeta + \int_{z_2}^{z_1} f(\zeta)\, d_q\zeta$$

from (3.1).

Theorem 3.9. *Given a simple closed discrete curve $\gamma = \langle z_0, z_1, \ldots, z_{n-1}, z_n = z_0\rangle \subset Q$ and a q-analytic function $f(z)$ in the sense of Pashaev-Nalcı. Then*

$$\int_{\gamma} f(\zeta)\, d_q\zeta = 0. \tag{3.15}$$

Proof. This follows easily from Theorem 3.4. □

Example 3.10. On the discrete set Q, let us consider the discrete curve

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \tag{3.16}$$

where

$$\gamma_1 = \left\langle z_0 = x + iy,\ z_1 = x + iq^{-1}y,\ z_2 = x + iq^{-2}y,\ z_3 = x + iq^{-3}y \right\rangle,$$
$$\gamma_2 = \left\langle z_3 = x + iq^{-3}y,\ z_4 = qx + iq^{-3}y,\ z_5 = q^2x + iq^{-3}y,\ z_6 = q^3x + iq^{-3}y \right\rangle,$$
$$\gamma_3 = \left\langle z_6 = q^3x + iq^{-3}y,\ z_7 = q^3x + iq^{-2}y,\ z_8 = q^3x + iq^{-1}y,\ z_9 = q^3x + iy \right\rangle,$$
$$\gamma_4 = \left\langle z_9 = q^3x + iy,\ z_{10} = q^2x + iy,\ z_{11} = qx + iy,\ z_{12} = x + iy = z_0 \right\rangle.$$

The set that is contained by the simple closed curve γ is

$$C_q = \left\{ z_{13} = qx + iq^{-1}y,\ z_{14} = qx + iq^{-2}y,\ z_{15} = q^2x + iq^{-2}y,\ z_{16} = q^2x + iq^{-1}y \right\}.$$

From (1.11) we have

$$\overline{C}_q = C_q \cup \{z, z_1, z_2, z_{10}, z_{11}\}.$$

Using (3.1) and (3.2) we see that for any discrete function $f(z)$ we have

$$\int_\gamma f(z)\,d_qz = \sum_{j=0}^{11} \int_{z_j}^{z_{j+1}} f(z)\,d_qz$$
$$= \frac{1-q}{q}\left[L_1f(z) + L_1f(z_1) + L_1f(z_2) + L_1f(z_{10}) + L_1f(z_{11}) + L_1f(z_{13}) + L_1f(z_{14}) + L_1f(z_{15}) + L_1f(z_{16})\right].$$

If $f(z)$ is q-analytic in the sense of Pashaev-Nalcı, we have $L_1f(z_k) = 0$, $k = 0, 1, \ldots,$ $z_k \in \overline{C}_q$, and therefore,

$$\int_\gamma f(z)\,d_qz = 0.$$

Remark 3.11. Theorem 3.9 can be thought of as the q-analog of Cauchy's Theorem for analytic functions in classical complex analysis.

Definition 3.12. Given discrete functions $f(x, y)$, $g(x, y)$ and a discrete curve $\gamma = \langle z_0, z_1, \ldots, z_n\rangle$ on the discrete set Q. Let us consider the integral

(3.17)
$$\int_{z_j}^{z_{j+1}} [f(z) * g(z)]\,d_qz = \begin{cases} (z_{j+1} - z_j)f(z_j)g(z_{j+1}); \begin{cases} z_{j+1} = qx_j + iy_j \text{ or} \\ z_{j+1} = x_j + iq^{-1}y_j, \end{cases} \\ (z_{j+1} - z_j)f(z_{j+1})g(z_j); \begin{cases} z_{j+1} = x_j + iqy_j \text{ or} \\ z_{j+1} = q^{-1}x_j + iy_j. \end{cases} \end{cases}$$

The integral given by

$$\int_\gamma [f(z) * g(z)]\,d_qz = \sum_{j=0}^{n-1} \int_{z_j}^{z_{j+1}} [f(z) * g(z)]\,d_qz \tag{3.18}$$

is called the *conjoint integral* of $f(z)$ and $g(z)$ over γ.

Our definition of the conjoint integral is different from the one given in [2].

Theorem 3.13. *On $\overline{C}_q \subset Q$, let $f(x, y)$ be q-analytic in the sense of Pashaev-Nalcı, and $g(x, y)$ in the sense of D_q^*. Then, for any closed discrete simple curve $\gamma = \langle z_0, z_1, \ldots, z_n = z_0\rangle \subset \overline{C}_q$ we have*

$$\int_\gamma [f(z) * g(z)]\,d_qz = 0. \tag{3.19}$$

Proof. Let us prove the statement for the closed simple discrete curve

$$\gamma = \left\langle z_0 = x + iy,\ z_1 = x + iq^{-1}y,\ z_2 = qx + iq^{-1}y,\ z_3 = qx + iy \right\rangle.$$

The proof can be repeated similarly for other closed discrete curves.

Since $f(z)$ is q-analytic in the sense of Pashaev-Nalcı, the equality in (2.8) is satisfied. Thus,

$$(qx + iy)f(x, y) - iyf(qx, y) - qxf(x, q^{-1}y) = 0. \tag{3.20}$$

Since $g(x, y)$ is q-analytic in the sense of D_q^*, we have

$$E\,g(x, y) = (x + iqy)g(x, y) - iqy\,g(q^{-1}x, y) - x\,g(x, qy) = 0 \tag{3.21}$$

by (2.16).

Moreover, from (3.21) we can write

$$E\,g(qx,q^{-1}y)=(qx+iy)g(qx,q^{-1}y)-qx\,g(qx,y)-iy\,g(x,q^{-1}y)=0. \tag{3.22}$$

Using the definition of the integral in (3.17) we have

$$\begin{aligned}\int_\gamma [f(z)*g(z)]\,d_qz &= \sum_{j=0}^{3}\int_{z_j}^{z_{j+1}}[f(z)*g(z)]\,d_qz\\ &=\frac{1-q}{q}\left\{f(x,y)\left[qx\,g(qx,y)+iy\,g(x,q^{-1}y)\right]\right.\\ &\qquad\left.-g(qx,q^{-1}y)\left[qx\,f(x,q^{-1}y)+iy\,f(qx,y)\right]\right\}.\end{aligned} \tag{3.23}$$

By using (3.20) and (3.22) in (3.23) we obtain

$$\begin{aligned}\int_\gamma [f(z)*g(z)]\,d_qz=\frac{1-q}{q}\Big\{&f(x,y)\left[qx\,g(qx,q^{-1}y)+iy\,g(qx,q^{-1}y)\right.\\ &\left.-iy\,g(x,q^{-1}y)+iy\,g(x,q^{-1}y)\right]\\ &-g(qx,q^{-1}y)\left[(qx+iy)f(x,y)\right]\Big\}=0.\end{aligned} \tag{3.24}$$

□

Theorem 3.14. *Let C be and $\overline{C}_q$ be discrete sets as defined in* (1.9) *and* (1.11)*, respectively. Let $f(z)$ and $g(z)$ be two discrete functions on $\overline{C}$. For any closed discrete curve $\gamma=\langle z_0,z_1,\dots,z_n=z_0\rangle$ on $\overline{C}=\overline{C}_q\cup\gamma$ we have*

$$\int_\gamma [f(z)*g(z)]d_qz=\frac{1-q}{q}\sum_{z\in\overline{C}_q}\left[g(qx,q^{-1}y)L_1f(z)-f(z)E\,g(qx,q^{-1}y)\right] \tag{3.25}$$

where the operators L_1 and E are defined in (2.8) *and* (2.16)*.*

Proof. Let us prove the statement for the closed simple discrete curve

$$\begin{aligned}\gamma=\big\langle z_0&=x+iy,\ z_1=x+iq^{-1}y,\ z_2=x+iq^{-2}y,\\ z_3&=qx+iq^{-2}y, z_4=q^2x+iq^{-2}y, z_5=q^2x+iq^{-1}y,\\ z_6&=q^2x+iy,\ z_7=qx+iy,\ z_8=x+iy=z_0\big\rangle.\end{aligned}$$

The proof can be completed by repeating the same argument for other closed discrete curves. From (1.11) we have

$$\begin{aligned}\overline{C}_q=\big\langle z_0=x+iy,\ z_1=x+iq^{-1}y,\ z_3=qx+iq^{-2}y,\ z_4=q^2x+iq^{-2}y,\\ z_5=q^2x+iq^{-1}y,\ z_7=qx+iy,\ z_9=qx+iq^{-1}y\big\rangle.\end{aligned}$$

From (3.17) and (3.18) we obtain

$$\begin{aligned}\int_{\gamma} [f(z)*g(z)]\, d_q z &= \sum_{j=0}^{7} \int_{z_j}^{z_{j+1}} [f(z)*g(z)]\, d_q z \\ &= \frac{1-q}{q}\Big\{ [qx\, g(z_7) + iy\, g(z_1)]\, f(z) + iq^{-1}y\, f(z_1)g(z_2) \\ &\quad - qx\, f(z_2)g(z_3) - q^2 x\, f(z_3)g(z_4) - iq^{-1}y\, f(z_5)g(z_4) \\ &\quad - iy\, f(z_6)g(z_5) + q^2 x\, f(z_7)g(z_6)\Big\}. \end{aligned} \tag{3.26}$$

On the other hand, using (2.16) and the L_1 operator we can compute that

$$\begin{aligned} &[g(z_9)L_1 f(z) - f(z)Eg(z_9)] + [g(z_4)L_1 f(z_9) - f(z_9)Eg(z_4)] \\ &\quad + [g(z_5)L_1 f(z_7) - f(z_7)Eg(z_5)] + [g(z_3)L_1 f(z_1) - f(z_1)Eg(z_3)] \\ &= f(z)\,[qx\, g(z_7) + iy\, g(z_1)] + iq^{-1}y f(z_1)g(z_2) - qx\, f(z_2)g(z_3) - q^2 x\, f(z_3)g(z_4) \\ &\quad - iq^{-1}y\, f(z_5)g(z_4) - iy\, f(z_6)g(z_5) + q^2 x\, f(z_7)g(z_6) \\ &= \frac{q}{1-q}\int_{\gamma} [f(z)*g(z)] d_q z. \end{aligned} \tag{3.27}$$

Comparing (3.26) with (3.27) we see that (3.24) holds true. □

Corollary 3.15. The formula (3.24) is the q-analogue of the classical Green's formula with respect to the integrals (3.17) and (3.18).

4. Conclusion

Various different definitions were given for q-integrals in the literature. The q-Green formula (3.24) takes different forms as the definition of the q-integral changes. For example, in [6], a Green's formula similar to (3.24) was obtained using the well-known *Jackson Integral* in q-analysis.

In this paper, we define a discrete q-line integral for q-analytic functions in the sense of Pashaev-Nalcı. Then using this type of an integral, we present a q-analogue of Green's formula on the complex plane.

Corollary 4.1. If $f(z)$ is q-analytic on a discrete set Q in the sense of Pashaev-Nalcı, then for any discrete function $g(z)$ we have

$$\int_{\gamma} [f(z)*g(z)]\, d_q z = \frac{q}{1-q} \sum_{z\in \overline{C}_q} f(z)Eg(qx.q^{-1}y).$$

References

[1] T. Ernst, A Comprehensive Treatment of q-Calculus, Springer Basel, (2012).
[2] C. J. Harman, Discrete geometric function theory I, Applicable analysis, 7(4)(1978), 315–336.
[3] C. J. Harman, Discrete geometric function theory II, Applicable analysis, 9(3)(1979), 191–203.
[4] O. K. Pashaev and S. Nalci, q-analytic functions, fractals and generalized analytic functions, Journal of Physics A: Mathematical and Theoretical, 47(4)(2014), 045204.
[5] K. Koca K., İ. Gençtürk and M. Aydin, Complex Line q-Integrals and q-Green's Formula on the Plane, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.), LXIV f.1 (2018), 27-45.
[6] K. Koca K., İ. Gençtürk and M. Aydin, q-Green's formula on the complex plane in the sense of Harman, Creat. Math. Inform., 26(3)(2017), 309–320.

(İlker Gençtürk(Corresponding Author)) Kirikkale University,, Department of Mathematics,, 71450 Kirikkale,, Turkey
Email address: ilkergencturk@gmail.com

(Şermin Hökelekli) Kirikkale University,, Department of Mathematics,, 71450 Kirikkale,, Turkey
Email address: serminhokelekli@gmail.com

(Kerim Koca) Kirikkale University,, Department of Mathematics,, 71450 Kirikkale,, Turkey
Email address: kerimkoca@gmail.com