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ON NEW INTUITIONISTIC FUZZY (ε, η)-NEGATION AND (ε, η)-IMPLICATION

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Abstract. Continuing an older author's research over some special types of negations and implications over intuitionistic fuzzy sets, in the paper new (ε, η)-negation and (ε, η)-implication are introduced. Some of their basic properties are discussed.

1. Introduction

The concept of an Intuitionistic Fuzzy Set (IFS) was introduced in 1983 in [1] and again there in it, the first definition of operation intuitionistic fuzzy negation over IFS was introduced. Later, in [2], two forms of the operation intuitionistic fuzzy implication were introduced. As of the 2000s, more than 200 different implications and more than 50 different negations arise and in [5 – 21, 23, 24], their basic properties were discussed. In [3, 4], two negations, namely ε-negation and (ε, η)-negation, were introduced and over their bases ε- and (ε, η)-implications were defined, respectively. It was shown that the (ε, η)-negation and (ε, η)-implication are extensions of the ε-negation and ε-implication, respectively.

In the present paper another pair of a negation and an implication, in some sense dual to the older (ε, η)-negation and (ε, η)-implication, will be introduced and some of their basic properties will be described.

2. Preliminaries

Let a set E be fixed. An IFS A in E is an object of the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

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Let for every $x \in E$:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Therefore, function π determines the degree of uncertainty.

Obviously, for every ordinary fuzzy set, $\pi_A(x) = 0$ for each $x \in E$ and the fuzzy set has the form:

$$\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle | x \in E\}.$$

Let everywhere below, the universe E be given. One of the geometrical interpretations of the IFSs uses the intuitionistic fuzzy interpretational triangle F on Fig. 1.

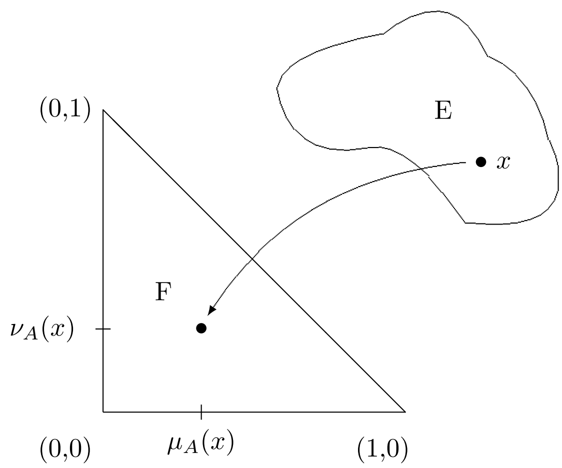

Figure 1. The intuitionistic fuzzy interpretational triangle

For every two IFSs A and B a lot of relations and operations have been defined (see, e.g. [1, 5]). The most important of them are following[1]:

$$\begin{array}{lll}
A \subset B & \text{iff} & (\forall x \in E)(\mu_A(x) \leq \mu_B(x) \;\&\; \nu_A(x) \geq \nu_B(x)); \\
A \supset B & \text{iff} & B \subset A; \\
A = B & \text{iff} & (\forall x \in E)(\mu_A(x) = \mu_B(x) \;\&\; \nu_A(x) = \nu_B(x)); \\
\neg_1 A & = & \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in E\}; \\
A \cap B & = & \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\}; \\
A \cup B & = & \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle | x \in E\}.
\end{array}$$

The IFS A is a tautological set iff for each $x \in E$: $\mu_A(x) = 1$ and $\nu_A(x) = 0$, and it is an Intuitionistic Fuzzy Tautological Set (IFTS) iff each $x \in E$: $\mu_A(x) \geq \nu_A(x)$.

In [3, 4] we introduced two intuitionistic fuzzy negations and two intuitionistic fuzzy implications, but the first pair of a negation and an implication are partial cases of the second pair, respectively, which have the forms:

$$\neg^{\varepsilon,\eta} A = \{\langle x, \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta), \min(1, \mu_A(x) + \eta)\rangle | x \in E\},$$

$$A \to^{\varepsilon,\eta} B = \{\langle x, \min(1, \max(\mu_B(x), \nu_A(x) + \varepsilon)) \max(0, \min(\nu_B(x), mu_A(x) - \eta)\rangle | x \in E\}.$$

[1] "iff" is an abbreviation of "if and only if"

3. Definitions of the new (ε,η)-negation and (ε,η)-implication

Here, by analogy with the ideas from [3, 4], we introduce a pair of a new negation and a new implication over IFSs. They generalize the classical negation and implication over IFSs, but on the other hand, they have some non-classical properties.

We construct a set of IF-negations that has the form

$$N = \{\neg_{\varepsilon,\eta} \mid 0 \le \varepsilon < 1 \ \&\ 0 \le \eta < 1\},$$

where for each IFS A,

$$\neg_{\varepsilon,\eta} A = \{\langle x, \max(0, \nu_A(x) - \varepsilon), \min(1, \mu_A(x) + \eta)\rangle | x \in E\}.$$

We will remark that the existing (ε,η)-negation and (ε,η)-implication are denoted with the index (ε,η) in superscript while the new ones are denoted with this index being subscript. The logic behind the denotation is that the first negation increase the result than the classical negation $\neg_1$), while the new negation decrease this result.

Below, we study some basic properties of an arbitrary element of N.

For ε and η there are two cases.

- $\eta > \varepsilon$, but this case is impossible, because, for example, if $\mu_A(x) = 0.6, \nu_A(x) = 0.3, \varepsilon = 0.2, \eta = 0.8$, then

$$\max(0, \nu_A(x) - \varepsilon) + \min(1, \mu_A(x) + \eta) = \max(0, 0.1) + \min(1, 1.4) = 1.1 > 1.$$

- $\eta \le \varepsilon$.

Let everywhere below $0 \le \eta \le \varepsilon < 1$ be fixed.

First, we see that set $\neg_{\varepsilon,\eta} A$ is an IFS, because for each $x \in E$

$$\max(0, \nu_A(x) - \varepsilon), \min(1, \mu_A(x) + \eta) \in [0, 1],$$

and if $\nu_A(x) \le \varepsilon$, then,

$$\max(0, \nu_A(x) - \varepsilon) + \min(1, \mu_A(x) + \eta) = \min(1, \mu_A(x) + \eta) \le 1;$$

if $\nu_A(x) \ge \varepsilon$, then,

$$\begin{aligned} &\max(0, \nu_A(x) - \varepsilon) + \min(1, \mu_A(x) + \eta) \\ &= \nu_A(x) - \varepsilon + \min(1, \mu_A(x) + \eta) \\ &\le \nu_A(x) - \varepsilon + \mu_A(x) + \eta \le 1. \end{aligned}$$

Therefore, the more exact form of N is

$$N = \{\neg_{\varepsilon,\eta} \mid 0 \le \eta \le \varepsilon < 1\}.$$

Second, let

$$\begin{aligned} O^* &= \{\langle x, 0, 1\rangle | x \in E\}, \\ U^* &= \{\langle x, 0, 0\rangle | x \in E\}, \\ E^* &= \{\langle x, 1, 0\rangle | x \in E\}. \end{aligned}$$

Then

$$\begin{aligned} \neg_{\varepsilon,\eta} O^* &= \{\langle x, \max(0, 1-\varepsilon), \min(1, 0+\eta)\rangle | x \in E\} = \{\langle x, 1-\varepsilon, \eta\rangle | x \in E\}, \\ \neg_{\varepsilon,\eta} U^* &= \{\langle x, \max(0, 0-\varepsilon), \min(1, 0+\eta)\rangle | x \in E\} = \{\langle x, 0, \eta\rangle | x \in E\}, \\ \neg_{\varepsilon,\eta} E^* &= \{\langle x, \max(0, 0-\varepsilon), \min(1, 1+\eta)\rangle | x \in E\} = \{\langle x, 0, 1\rangle | x \in E\} = O^*. \end{aligned}$$

Third, in Fig. 2, x and $\neg_1 x$ are shown, where if $\langle x, \mu_A(x), \nu_A(x)\rangle \in A$, then $\langle \neg_1 x,\ \nu_A(x),\ \mu_A(x)\rangle \in \neg_1 A$.

In Fig. 3 y and $\neg_{\varepsilon,\eta}y$ are shown, where if $\langle y, \mu_A(y), \nu_A(y)\rangle \in A$, then $\langle \neg_{\varepsilon,\eta}w, \nu_A(w) - \eta, \mu_A(w) + \varepsilon\rangle \in \neg_{\varepsilon,\eta}A$.

In Fig. 4, z and $\neg_{\varepsilon,\eta}z$ are shown, where if $\langle z, \mu_A(z), \nu_A(z)\rangle \in A$, then $\langle \neg_{\varepsilon,\eta}z, \nu_A(z) - \eta, \mu_A(z) + \varepsilon\rangle \in \neg_{\varepsilon,\eta}A$.

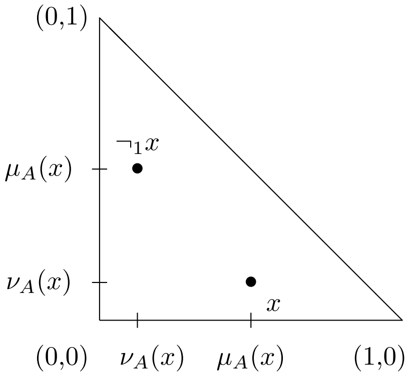

Figure 2. Geometrical interpretation of negation $\neg_1$

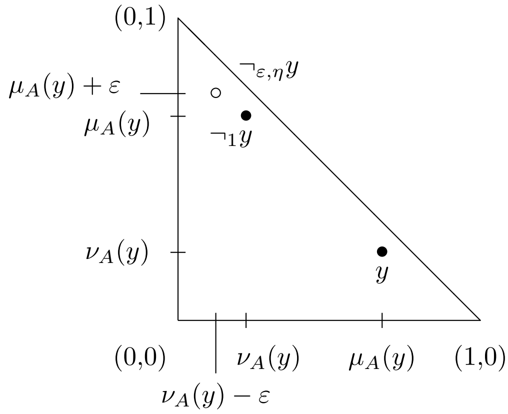

Figure 3. Geometrical interpretation of negation $\neg_{\varepsilon,\eta}$ (first case)

Fourth, we construct a set of IF-implications that has the form

$$I = \{\to_{\varepsilon,\eta} \mid 0 \leq \eta \leq \varepsilon < 1\},$$

where for every two IFSs A and B,

$$A \to_{\varepsilon,\eta} B = \{\langle x, \max(\mu_B(x), \nu_A(x) - \varepsilon), \min(\nu_B(x), \mu_A(x) + \eta)\rangle | x \in E\}.$$

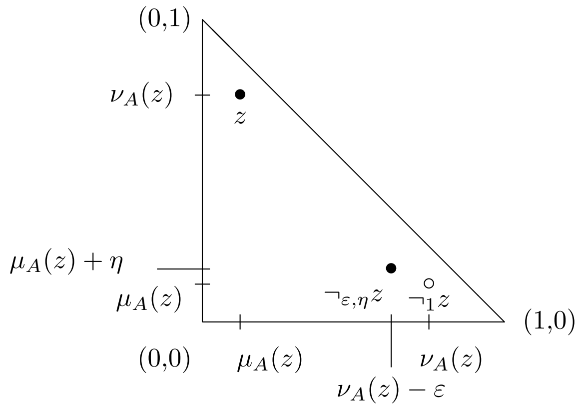

Figure 4. Geometrical interpretation of negation $\neg_{\varepsilon,\eta}$ (second case)

The set $A \rightarrow_{\varepsilon,\eta} B$ is an IFS, because

$$\max(\mu_B(x), \nu_A(x) - \varepsilon), \min(\nu_B(x), \mu_A(x) + \eta) \in [0,1]$$

and if $\mu_B(x) \geq \nu_A(x) - \varepsilon$, then

$$\max(\mu_B(x), \nu_A(x) - \varepsilon) + \min(\nu_B(x), \mu_A(x) + \eta)$$
$$= \mu_B(x) + \min(\nu_B(x), \mu_A(x) + \eta)$$
$$\leq \mu_B(x) + \nu_B(x) \leq 1;$$

and if $\mu_B(x) < \nu_A(x) - \varepsilon$, then

$$\max(\mu_B(x), \nu_A(x) - \varepsilon) + \min(\nu_B(x), \mu_A(x) + \eta)$$
$$= \nu_A(x) - \varepsilon + \min(\nu_B(x), \mu_A(x) + \eta)$$
$$\leq \mu_A(x) + \nu_A(x) + \eta - \varepsilon \leq 1.$$

Fifth, we check directly that

$$O^* \rightarrow_{\varepsilon,\eta} O^* = \{\langle x, \max(0, 1-\varepsilon), \min(1,\eta)\rangle | x \in E\} = \{\langle x, 1-\varepsilon, \eta\rangle | x \in E\};$$
$$O^* \rightarrow_{\varepsilon,\eta} U^* = \{\langle x, \max(0, 1-\varepsilon), \min(0,\eta)\rangle | x \in E\} = \{\langle x, 1-\varepsilon, 0\rangle | x \in E\};$$
$$O^* \rightarrow_{\varepsilon,\eta} E^* = \{\langle x, \max(1, 1-\varepsilon), \min(0,\eta)\rangle | x \in E\} = \{\langle x, 1, 0\rangle | x \in E\} = E^*;$$
$$U^* \rightarrow_{\varepsilon,\eta} O^* = \{\langle x, \max(0, -\varepsilon), \min(1,\eta)\rangle | x \in E\} = \{\langle x, 0, \eta\rangle | x \in E\};$$
$$U^* \rightarrow_{\varepsilon,\eta} U^* = \{\langle x, \max(0, -\varepsilon), \min(0,\eta)\rangle | x \in E\} = \{\langle x, 0, 0\rangle | x \in E\} = U^*;$$
$$U^* \rightarrow_{\varepsilon,\eta} E^* = \{\langle x, \max(1, -\varepsilon), \min(0,\eta)\rangle | x \in E\} = \{\langle x, 1, 0\rangle | x \in E\} = E^*;$$
$$E^* \rightarrow_{\varepsilon,\eta} O^* = \{\langle x, \max(0, -\varepsilon), \min(1, 1+\eta)\rangle | x \in E\} = \{\langle x, 0, 1\rangle | x \in E\} = O^*;$$
$$E^* \rightarrow_{\varepsilon,\eta} U^* = \{\langle x, \max(0, -\varepsilon), \min(0, 1+\eta)\rangle | x \in E\} = \{\langle x, 0, 0\rangle | x \in E\} = U^*;$$
$$E^* \rightarrow_{\varepsilon,\eta} E^* = \{\langle x, \max(1, -\varepsilon), \min(0, 1+\eta)\rangle | x \in E\} = \{\langle x, 1, 0\rangle | x \in E\} = E^*.$$

4. Basic properties of the new (ε, η)-negation and (ε, η)-implication

In [22], George Klir and Bo Yuan discussed the following nine axioms related to fuzzy implications.

Axiom 1 $(\forall x, y)(x \le y \to (\forall z)(I(x, z) \ge I(y, z))$.
Axiom 2 $(\forall x, y)(x \le y \to (\forall z)(I(z, x) \le I(z, y))$.
Axiom 3 $(\forall y)(I(0, y) = 1)$.
Axiom 4 $(\forall y)(I(1, y) = y)$.
Axiom 5 $(\forall x)(I(x, x) = 1)$.
Axiom 6 $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.
Axiom 7 $(\forall x, y)(I(x, y) = 1$ iff $x \le y)$.
Axiom 8 $(\forall x, y)(I(x, y) = I(N(y), N(x)))$.
Axiom 9 I is a continuous function,

where I and N denote "implication" and "negation", respectively.

We must mention that **Axiom 3** will be valid for the new implication if it has the form

Axiom 3' $(\forall y)(I(0, y)$ is an IFTS) iff $\varepsilon = \eta = 0$.
Another modification of this axiom is
Axiom 3* $(\forall y)(I(0, y)$ is an IFTS) if $\varepsilon + \eta \le 1$.

Another modification of **Axiom 5** is
Axiom 5* $(\forall x)(I(x, x)$ is an IFTS).

Theorem 1. Implication $\to_{\varepsilon,\eta}$ and negation $\neg_{\varepsilon,\eta}$ satisfy axioms **A1, A2, A3', A3*, A4, A6, A7, A9** and do not satisfy axioms **A5, A5*, A8**.

Proof. First, we must denote that in the case of IFSs, in the above axioms, the relations $\le$ and $\ge$ are changed with relations $\subseteq$ and $\supseteq$, respectively, and constants 0 and 1 are changed with with sets O^* and E^*, respectively.

Let $A \subseteq B$, i.e., for each $x \in E$: $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$. Then

$$A \to_{\varepsilon,\eta} C = \{\langle x, \max(\mu_C(x), \nu_A(x) - \varepsilon), \min(\nu_C(x), \mu_A(x) + \eta))\rangle | x \in E\}$$
$$\supseteq \{\langle x, \max(\mu_C(x), \nu_B(x) - \varepsilon), \min(\nu_C(x), \mu_B(x) + \eta))\rangle | x \in E\} = B \to_{\varepsilon,\eta} C.$$

Therefore, **Axiom 1** is valid.

$$C \to_{\varepsilon,\eta} A = \{\langle x, \max(\mu_A(x), \nu_C(x) - \varepsilon), \min(\nu_A(x), \mu_C(x) + \eta))\rangle | x \in E\}$$
$$\subseteq \{\langle x, \max(\mu_B(x), \nu_C(x) - \varepsilon), \min(\nu_B(x), \mu_C(x) + \eta))\rangle | x \in E\} = C \to_{\varepsilon,\eta} B.$$

Therefore, **Axiom 2** is valid.

$$O^* \to_{\varepsilon,\eta} B = \{\langle x, \max(\mu_B(x), 1 - \varepsilon), \min(\nu_B(x), \eta))\rangle | x \in E\}.$$

When $\varepsilon > 0$ and/or $\eta > 0$, then $O^* \to_{\varepsilon,\eta} B$ is not a tautological set, while when $\varepsilon = \eta = 0$, it is, i.e., **Axiom 3** is not always valid, while **Axiom 3'** – is valid. Also, when $\varepsilon + \eta \le 1$:

$$\max(\mu_B(x), 1 - \varepsilon) - \min(\nu_B(x), \eta)) \ge 1 - \varepsilon - \eta \ge 0,$$

i.e., **Axiom 3*** is valid.

$$E^* \to_{\varepsilon,\eta} B = \{\langle x, \max(\mu_B(x), 0 - \varepsilon), \min(\nu_B(x), 1 + \eta))\rangle | x \in E\} = B,$$

i.e., **Axiom 4** is valid.

For **Axiom 5** we obtain

$$A \to_{\varepsilon,\eta} A = \{\langle x, \max(\mu_A(x), \nu_A(x) - \varepsilon), \min(\nu_A(x), \mu_A(x) + \eta))\rangle | x \in E\}.$$

Now, we see immediately that, e.g., for $\mu_A(x) = 0.4, \nu_A(x) = 0.5, \varepsilon = \eta = 0.3$, $A \to_{\varepsilon,\eta} A$ is neither a tautological set, nor an IFTS, i.e., **Axiom 5** and **Axiom 5*** are not valid.

$$\begin{aligned}
&A \to_{\varepsilon,\eta} (B \to_{\varepsilon,\eta} C)\\
&= A \to_{\varepsilon,\eta} \{\langle x, \max(\mu_C(x), \nu_B(x) - \varepsilon), \min(\nu_C(x), \mu_B(x) + \eta)\rangle | x \in E\}\\
&= \{\langle x, \max(\mu_C(x), \nu_B(x) - \varepsilon, \nu_A(x) - \varepsilon), \min(\nu_C(x), \mu_B(x) + \eta, \mu_A(x) + \eta)\rangle | x \in E\}\\
&= B \to_{\varepsilon,\eta} \{\langle x, \max(\mu_C(x), \nu_A(x) - \varepsilon), \min(\nu_C(x), \mu_A(x) + \eta)\rangle | x \in E\}\\
&= B \to_{\varepsilon,\eta} (A \to_{\varepsilon,\eta} C).
\end{aligned}$$

Therefore, **Axiom 6** is valid.

Let $A \to_{\varepsilon,\eta} B$ be a tautological set, i.e.,

$$\max(\mu_B(x), \nu_A(x) - \varepsilon) = 1,$$

$$\min(\nu_B(x), \mu_A(x) + \eta) = 0.$$

Hence, if $\varepsilon, \eta > 0$, then for each $x \in E$:

$$\mu_B(x) = 1 \geq \mu_A(x),$$

$$\nu_B(x) = 0 \leq \nu_A(x)$$

i.e., $A \subseteq B$. If $\varepsilon = \eta = 0$, then there exists another possibility:

$$\mu_A(x) = 0 \leq \mu_B(x),$$

$$\nu_A(x) = 1 \geq \nu_B(x).$$

Therefore, **Axiom 7** is valid.

Now, we calculate

$$\begin{aligned}
&\neg_{\varepsilon,\eta} B \to_{\varepsilon,\eta} \neg_{\varepsilon,\eta} A\\
&= \{\langle x, \max(0, \nu_B(x) - \varepsilon), \min(1, \mu_B(x) + \eta)\rangle | x \in E\}\\
&\to_{\varepsilon,\eta} \{\langle x, \max(0, \nu_A(x) - \varepsilon), \min(1, \mu_A(x) + \eta)\rangle | x \in E\}\\
&= \{\langle x, \max(0, \nu_A(x) - \varepsilon, \min(1, \mu_B(x) + \eta) - \varepsilon),\\
&\min(1, \mu_A(x) + \eta, \max(0, \nu_B(x) - \varepsilon) + \eta)\rangle | x \in E\}.
\end{aligned}$$

Axiom 8 will be valid, iff for each $x \in E$

$$\max(0, \nu_A(x) - \varepsilon, \min(1, \mu_B(x) + \eta) - \varepsilon) = \max(\mu_B(x), \nu_A(x) - \varepsilon),$$

$$\min(1, \mu_A(x) + \eta, \max(0, \nu_B(x) - \varepsilon) + \eta) = \min(\nu_B(x), \mu_A(x) + \eta)$$

that, is not always valid, i.e., this axiom is not valid. Finally, **Axiom 9** is valid because the degrees of the implication components are continuous functions. □

Theorem 2. For each IFS A

(a) $\neg_{\varepsilon,\eta}(A \cap B) = \neg_{\varepsilon,\eta} A \cup \neg_{\varepsilon,\eta} B$,

(b) $\neg_{\varepsilon,\eta}(A \cup B) = \neg_{\varepsilon,\eta} A \cap \neg_{\varepsilon,\eta} B$.

Proof. For (a) we obtain

$$\neg_{\varepsilon,\eta}(A \cap B) = \neg_{\varepsilon,\eta}(\{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\})$$

$$= \{\langle x, \max(0, \max(\nu_A(x), \nu_B(x)) - \varepsilon), \min(1, \min(\mu_A(x), \mu_B(x)) + \eta)\rangle | x \in E\}$$

$$= \{\langle x, \max(0, \nu_A(x) - \varepsilon, \nu_B(x) - \varepsilon), \min(1, \mu_A(x) + \eta, \mu_B(x) + \eta)\rangle | x \in E\}$$

$$= \{\langle x, \max(0, \nu_A(x) - \varepsilon), \min(1, \mu_A(x) + \eta)\rangle | x \in E\}$$

$$\cup\{\langle x, \max(0, \nu_B(x) - \varepsilon), \min(1, \mu_B(x) + \eta)\rangle | x \in E\}$$

$$= \neg_{\varepsilon,\eta} A \cup \neg_{\varepsilon,\eta} B.$$

(b) is proved by the same manner. □

It is important to mention that in the same way we can check that the equalities

$$A \cap B = \neg_{\varepsilon,\eta}(\neg_{\varepsilon,\eta} A \cup \neg_{\varepsilon,\eta} B)$$

and

$$A \cup B = \neg_{\varepsilon,\eta}(\neg_{\varepsilon,\eta} A \cap \neg_{\varepsilon,\eta} B)$$

are not valid. On the other hand side, the following equalities, formulated in Theorem 3, are valid.

Theorem 3. For each IFS A

(a) $\neg_{\varepsilon,\eta}(\neg_{\varepsilon,\eta} A \cap \neg_{\varepsilon,\eta} B) = \neg_{\varepsilon,\eta}\neg_{\varepsilon,\eta} A \cup \neg_{\varepsilon,\eta}\neg_{\varepsilon,\eta} B$,
(b) $\neg_{\varepsilon,\eta}(\neg_{\varepsilon,\eta} A \cup \neg_{\varepsilon,\eta} B) = \neg_{\varepsilon,\eta}\neg_{\varepsilon,\eta} A \cap \neg_{\varepsilon,\eta}\neg_{\varepsilon,\eta} B$.

We must mention that similar equalities are valid for a lot of the non-classical intuitionistic fuzzy negations (see, e.g., [5, 6]).

Theorem 4. For every three IFSs A, B, C:

(a) $(A \cap B) \rightarrow_{\varepsilon,\eta} C = (A \rightarrow_{\varepsilon,\eta} C) \cup (B \rightarrow_{\varepsilon,\eta} C)$,
(b) $(A \cup B) \rightarrow_{\varepsilon,\eta} C = (A \rightarrow_{\varepsilon,\eta} C) \cap (B \rightarrow_{\varepsilon,\eta} C)$,
(c) $C \rightarrow_{\varepsilon,\eta} (A \cap B) = (C \rightarrow_{\varepsilon,\eta} A) \cap (C \rightarrow_{\varepsilon,\eta} B)$,
(d) $C \rightarrow_{\varepsilon,\eta} (A \cup B) = (C \rightarrow_{\varepsilon,\eta} A) \cup (C \rightarrow_{\varepsilon,\eta} B)$.

Proof. For (a) we obtain

$$(A \cap B) \rightarrow_{\varepsilon,\eta} C = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x))\rangle | x \in E\} \rightarrow_{\varepsilon,\eta} C$$

$$= \{\langle x, \max(\mu_C(x), \max(\nu_A(x), \nu_B(x)) - \varepsilon), \min(\nu_C(x), \min(\mu_A(x), \mu_B(x)) + \eta)\rangle | x \in E\}$$

$$= \{\langle x, \max(\mu_C(x), \nu_A(x) - \varepsilon, \nu_B(x) - \varepsilon), \min(\nu_C(x), \mu_A(x) + \eta, \mu_B(x) + \eta)\rangle | x \in E\}$$

$$= \{\langle x, \max(\max(\mu_C(x), \nu_A(x) - \varepsilon), \max(\mu_C(x), \nu_B(x) - \varepsilon)),$$

$$\min(\min(\nu_C(x), \mu_A(x) + \eta), \min(\nu_C(x), \mu_B(x) + \eta))\rangle | x \in E\}$$

$$= (A \rightarrow_{\varepsilon,\eta} C) \cup (B \rightarrow_{\varepsilon,\eta} C).$$

Statements (b) – (d) are proved in the same manner. □

5. Conclusion

In the present paper, we introduced a new intuitionistic fuzzy negation and a new intuitionistic fuzzy implication and we studied some of their basic properties. In future, we will search for connections between them and the other intuitionistic fuzzy negations and implications, with the other intuitionistic fuzzy operations and operators.

With the following assertions, that are proved analogously to the above ones, we will illustrate the first step in this new direction of research in intuitionistic fuzzy sets theory.

Theorem 5. For each IFS A

(a) $\neg_{\varepsilon,\eta}\neg^{\varepsilon,\eta}A \subseteq A \subseteq \neg^{\varepsilon,\eta}\neg_{\varepsilon,\eta}A$,
(b) $\neg_1\neg_{\varepsilon,\eta}\neg_1 = \neg^{\varepsilon,\eta}A$,
(c) $\neg_1\neg^{\varepsilon,\eta}\neg_1 = \neg_{\varepsilon,\eta}A$.

These assertions and especially, the first one (the inequality) give idea to use the two negations $\neg^{\varepsilon,\eta}$ and $\neg_{\varepsilon,\eta}$ for determining of optimistic ($\neg^{\varepsilon,\eta}$) and pessimistic ($\neg_{\varepsilon,\eta}$) negations of given objects (expressions, sentences, variables, facts, etc.) which will be an object of discussion in next research.

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The Declaration of Ethics Committee Approval
This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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