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ON THE GROWTH PROPERTIES OF GENERALIZED ITERATED ENTIRE FUNCTIONS

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Abstract. In this paper, we study some growth properties of generalized iterated entire functions to generalize some earlier results.

1. INTRODUCTION AND DEFINITIONS

If f and g be two transcendental entire functions defined in the open complex plane $\mathbb{C}$, then Clunie [4] proved that $\lim_{r\to\infty} \frac{T(r,f\circ g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r,f\circ g)}{T(r,g)} = \infty$. In [10] Singh proved some comparative growth properties of $\log T(r, f\circ g)$ and $T(r,f)$ and raised the problem of investigating the comparative growth properties of $\log T(r, f\circ g)$ and $T(r,g)$. After this several authors {see [3], [7] etc.,} made close investigation on comparative growth of $\log T(r, f\circ g)$ and $T(r,g)$ by imposing certain restrictions on orders of f and g. In the present paper, we study such growth properties for generalized iterated entire functions.

Definition 1.1. Let f be a meromorphic function and $T(r,f)$ be its Nevanlinna's characteristic function. Then the numbers $\rho(f)$, $\lambda(f)$ defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and $\lambda(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$ are respectively called order and lower order of f.

Definition 1.2. ([3]) Let f be a meromorphic function. Then the numbers $\rho_p(f)$, $\lambda_p(f)$ defined by

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}$$

and $\lambda_p(f) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log r}$, where $p = 1, 2, 3, ...$

are respectively called p-th order and p-th lower order of f.

For $p = 1$, the above definition coincides with the classical definition of order and lower order.

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If f is entire one can easily verify that

$$\rho_p(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}$$

and $$\lambda_p(f) = \liminf_{r\to\infty} \frac{\log^{[p+1]} M(r,f)}{\log r}, \text{ where } p = 1, 2, 3, ... \quad .$$

Definition 1.3. ([3]) Let f be a meromorphic function. Then the numbers $\overline{\rho}_p(f)$, $\overline{\lambda}_p(f)$ defined by

$$\overline{\rho}_p(f) = \limsup_{r\to\infty} \frac{\log^{[p+1]} T(r,f)}{\log r}$$

and $$\overline{\lambda}_p(f) = \liminf_{r\to\infty} \frac{\log^{[p+1]} T(r,f)}{\log r}, \text{ where } p = 1, 2, 3, ...$$

are respectively called pth hyper order and pth hyper lower order of f.

If f is entire one can easily verify that

$$\overline{\rho}_p(f) = \limsup_{r\to\infty} \frac{\log^{[p+2]} M(r,f)}{\log r}$$

and $$\overline{\lambda}_p(f) = \liminf_{r\to\infty} \frac{\log^{[p+2]} M(r,f)}{\log r}, \text{ where } p = 1, 2, 3, ... \quad .$$

Definition 1.4. ([3]) Let f be a meromorphic function of order zero. Then the numbers $\rho_p^*(f)$ and $\lambda_p^*(f)$ are defined as follows

$$\rho_p^*(f) = \limsup_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[2]} r}$$

and $$\lambda_p^*(f) = \liminf_{r\to\infty} \frac{\log^{[p]} T(r,f)}{\log^{[2]} r}, \text{ where } p = 1, 2, 3, ... \quad .$$

Definition 1.5. ([7]) A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f if

i) $\lambda_f(r)$ is non negative and continuous for $r \geq r_0$ say;

ii) $\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda'_f(r-0)$ and $\lambda'_f(r+0)$ exist;

iii) $\lim\limits_{r\to\infty} \lambda_f(r) = \lambda(f) < \infty$;

iv) $\lim\limits_{r\to\infty} r\lambda'_f(r) \log r = 0$; and

v) $\liminf\limits_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1.$

Definition 1.6. A real valued function $\varphi(r)$ is said to have the property P_1 if

i) $\varphi(r)$ is non negative and continuous for $r \geq r_0$ say;

ii) $\varphi(r)$ is strictly increasing and $\varphi(r) \to \infty$ as $r \to \infty$;

iii) $\log \varphi(r) \leq \delta\varphi(\frac{r}{4})$ holds for every $\delta > 0$ and for all sufficiently large values of r.

Remark 1.1. If $\varphi(r)$ satisfies the property P_1 then it is clear that $\log^{[p]} \varphi(r) \leq \delta\varphi(\frac{r}{4})$ holds for every $p \geq 1$.

Definition 1.7. ([1]) Let f and g be two non-constant entire functions and α be any real number satisfying $0 < \alpha \leq 1$. Then the generalized iteration of f with respect to g is defined as follows:

$$f_{1,g}(z) = (1-\alpha)z + \alpha f(z)$$
$$f_{2,g}(z) = (1-\alpha)g_{1,f}(z) + \alpha f(g_{1,f}(z))$$
$$f_{3,g}(z) = (1-\alpha)g_{2,f}(z) + \alpha f(g_{2,f}(z))$$

$$\begin{aligned} & .... \qquad .... \qquad .... \\ & f_{n,g}(z) = (1-\alpha)g_{n-1,f}(z) + \alpha f(g_{n-1,f}(z)) \end{aligned}$$

and so are

$$\begin{aligned} g_{1,f}(z) &= (1-\alpha)z + \alpha g(z) \\ g_{2,f}(z) &= (1-\alpha)f_{1,g}(z) + \alpha g(f_{1,g}(z)) \\ g_{3,f}(z) &= (1-\alpha)f_{2,g}(z) + \alpha g(f_{2,g}(z)) \\ & .... \qquad .... \qquad .... \\ g_{n,f}(z) &= (1-\alpha)f_{n-1,g}(z) + \alpha g(f_{n-1,g}(z)). \end{aligned}$$

Definition 1.8. ([3]) Let a be a complex number, finite or infinite. The Valiron deficiency $\delta(a,f)$ of a with respect to a meromorphic function f is defined as:

$$\begin{aligned} \delta(a,f) &= 1 - \liminf_{r\to\infty} \tfrac{N(r,a;f)}{T(r,f)} \\ &= \limsup_{r\to\infty} \tfrac{m(r,a;f)}{T(r,f)}. \end{aligned}$$

We do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are available in [5] and [11]. Throughout we assume $f,\ g$ etc., are non-constant entire functions such that maximum modulus functions of $f,\ g$ and all of their generalized iterated functions satisfy property P_1.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *([5]) If $f(z)$ be regular in $|z| \le R$, then for $0 \le r < R$*

$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f)$.

In particular, if f be non-constant entire, then for all large values of r

$T(r,f) \le \log M(r,f) \le 3T(2r,f)$.

Lemma 2.2. *([7]) Let f be a meromorphic function. Then for $\delta > 0$ the function $r^{\lambda(f)+\delta-\lambda_f(r)}$ is an increasing function of r.*

Lemma 2.3. *([8]) Let f be an entire function of finite lower order. If there exist entire functions $a_i (i = 1,2,3,...m; m \le \infty)$ satisfying $T(r,a_i) = o\{T(r,f)\}$ and $\sum\limits_{i=1}^{m} \delta(a_i,f) = 1$ then*

$$\lim_{r\to\infty} \tfrac{T(r,f)}{\log M(r,f)} = \tfrac{1}{\pi}.$$

Lemma 2.4. *([2]) If f is meromorphic and g is entire then for all large values of r*

$T(r, f\circ g) \le (1+o(1)) \frac{T(r,g)}{\log M(r,g)} T(M(r,g),f)$.

Since g is entire so using Lemma 2.1, we have

$T(r, f\circ g) \le (1+o(1)) T(M(r,g),f)$.

Lemma 2.5. *([9]) Let f and g be transcendental entire functions with $\rho(g) < \infty$, η be a constant satisfying $0 < \eta < 1$ and δ be a positive number. Then*

$$T(r, f \circ g) + O(1) \geq N(r, 0; f \circ g)$$
$$\geq (\log \tfrac{1}{\eta})\left[\frac{N(M((\eta r)^{\frac{1}{1+\delta}}, g), 0; f)}{\log(M((\eta r)^{\frac{1}{1+\delta}}, g) - O(1))} - O(1)\right]$$

as $r \to \infty$ through all values.

Lemma 2.6. *Let f and g be two non-constant entire functions. Then $M(r, f \circ g) \leq M(M(r, g), f)$ holds for all large values of r.*

Lemma 2.7. *([3]) For a meromorphic function f of finite lower order, lower proximate order exists.*

3. MAIN THEOREMS

In this section, we present the main results of this paper.

Theorem 3.1. *Let $f(z)$ and $g(z)$ be two entire functions such that $\lambda_p(f)$ and $\rho_p(g)$ are finite and $\lambda_p(g) > 0$. Then for even n*

i) $$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{T(r, g)} \leq 3\rho_p(f)2^{\lambda(g)}$$

ii) $$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{T(r, g)} \geq \frac{\lambda_p(f)}{2.4^{(n-1)\lambda(g)}}.$$

Proof**.** If $\lambda(g) = \infty$, then (i) and (ii) are obvious. So we suppose that $\lambda(g) < \infty$. If $\rho_p(f) = \infty$ then (i) is obvious. So we suppose that $\rho_p(f) < \infty$. Since f and g are non-constants so

$$M(r, f) \geq \mu r \text{ and } M(r, g) \geq \mu r \quad \text{for some } 0 < \mu < 1. \tag{3.1}$$

Now by Lemma 2.1 we get for all large values of r and arbitrary $\epsilon > 0$

$$\begin{aligned} T(r, f_{n,g}) &\leq \log M(r, f_{n,g}) \\ &= \log M(r, (1-\alpha)g_{n-1,f} + \alpha f(g_{n-1,f})) \\ &\leq \log\{(1-\alpha)\tfrac{1}{\mu}M(M(r, g_{n-1,f}), f) + \tfrac{1}{\mu}\alpha M(M(r, g_{n-1,f}), f)\}, \\ &\qquad \text{using (3.1) and Lemma 2.6} \end{aligned}$$

$$= \log M(M(r, g_{n-1,f}), f) + O(1) \tag{3.2}$$

$$\begin{aligned} \text{or, } \log^{[p]} T(r, f_{n,g}) &\leq \log^{[p+1]} M(M(r, g_{n-1,f}), f) + O(1) \\ &< (\rho_p(f) + \epsilon)\log M(r, g_{n-1,f}) + O(1) \end{aligned}$$

$$\begin{aligned} \text{or, } \log^{[2p]} T(r, f_{n,g}) &< \log^{[p]} \log M(r, g_{n-1,f}) + O(1) \\ &< \log^{[p]}\{\log M(M(r, f_{n-2,g}), g)\} + O(1), \qquad \text{using (3.2)} \\ &< (\rho_p(g) + \epsilon)\log M(r, f_{n-2,g}) + O(1). \end{aligned}$$

So, $\log^{[3p]} T(r, f_{n,g}) < (\rho_p(f) + \epsilon)\log M(r, g_{n-3,f}) + O(1)$.

Proceeding similarly after some steps we get

$$\log^{[(n-2)p]} T(r, f_{n,g}) < (\rho_p(g) + \epsilon)\log M(r, f_{2,g}) + O(1).$$

$$\begin{aligned} \text{So, } \log^{[(n-1)p]} T(r, f_{n,g}) &< (\rho_p(f) + \epsilon)\log M(r, g_{1,f}) + O(1) \\ &= (\rho_p(f) + \epsilon)\log M(r, (1-\alpha)z + \alpha g(z)) + O(1) \\ &\leq (\rho_p(f) + \epsilon)\{\log M(r, z) + \log M(r, g)\} + O(1) \end{aligned}$$

$$= (\rho_p(f) + \epsilon)\{\log r + \log M(r, g)\} + O(1). \tag{3.3}$$

On the other hand, since $\liminf_{r\to\infty} \frac{T(r,g)}{r^{\lambda g(r)}} = 1$, we get for a sequence of values of r tending to infinity

$$T(r,g) < (1+\epsilon) r^{\lambda g(r)} \tag{3.4}$$

and for all large of values of r,

$$T(r,g) > (1-\epsilon) r^{\lambda g(r)}. \tag{3.5}$$

Therefore, for all large values of r, we get from (3.3) and (3.5)

$$\begin{aligned}
\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} &< \frac{(\rho_p(f)+\epsilon)\{\log r + \log M(r,g)\} + O(1)}{(1-\epsilon) r^{\lambda g(r)}} \\
&= \frac{(\rho_p(f)+\epsilon)\log M(r,g)}{(1-\epsilon) r^{\lambda g(r)}} + o(1) \qquad [\text{since } \lim_{r\to\infty} \lambda_g(r) = \lambda(g) > 0] \\
&\le \frac{(\rho_p(f)+\epsilon) 3T(2r,g)}{(1-\epsilon) r^{\lambda g(r)}} + o(1).
\end{aligned}$$

Therefore we get from (3.4) for a sequence of values of r tending to infinity

$$\begin{aligned}
\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} &\le \frac{3(\rho_p(f)+\epsilon)(1+\epsilon)(2r)^{\lambda(g)+\delta}}{(1-\epsilon)(2r)^{\lambda(g)+\delta-\lambda g(2r)} r^{\lambda g(r)}} + o(1) \\
&= \frac{3(\rho_p(f)+\epsilon)(1+\epsilon)}{(1-\epsilon)} 2^{\lambda(g)+\delta} \frac{r^{\lambda(g)+\delta-\lambda g(r)}}{(2r)^{\lambda(g)+\delta-\lambda g(2r)}} + o(1) \\
&\le \frac{3(\rho_p(f)+\epsilon)(1+\epsilon)}{(1-\epsilon)} 2^{\lambda(g)+\delta} + o(1)
\end{aligned}$$

because $r^{\lambda(g)+\delta-\lambda_g(r)}$ is an increasing function of r.

Since $\epsilon > 0$ and $\delta > 0$ are arbitrary we get

$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} \le 3\rho_p(f) 2^{\lambda(g)}$ and (i) is proved.

If $\lambda_p(f) = 0$, then (ii) is obvious. So we suppose that $\lambda_p(f) > 0$. Then we have for all large values of r

$$\begin{aligned}
T(r, f_{n,g}) &= T(r, (1-\alpha) g_{n-1,f} + \alpha f(g_{n-1,f})) \\
&\ge T(r, \alpha f(g_{n-1,f})) - T(r, (1-\alpha) g_{n-1,f}) + O(1) \\
&\ge T(r, f(g_{n-1,f})) - T(r, g_{n-1,f}) + O(1) \qquad [\text{for } \alpha \ne 1] \\
&> \tfrac{1}{3} \exp^{[p-1]} \{\tfrac{1}{9} M(\tfrac{r}{4}, g_{n-1,f})\}^{\lambda_p(f)-\epsilon} - T(r, g_{n-1,f}) + O(1), \\
&\qquad\qquad \text{see [10], page 100}\}
\end{aligned}$$

$$\begin{aligned}
\text{or, } \log^{[p]} T(r, f_{n,g}) &> \log\{\tfrac{1}{9} M(\tfrac{r}{4}, g_{n-1,f})\}^{\lambda_p(f)-\epsilon} - \log^{[p]} T(r, g_{n-1,f}) + O(1) \\
&\ge (\lambda_p(f)-\epsilon) \log M(\tfrac{r}{4}, g_{n-1,f}) - \tfrac{1}{2}(\lambda_p(f)-\epsilon) \log M(\tfrac{r}{4}, g_{n-1,f}) \\
&\qquad + O(1), \\
&\qquad \text{using property } P_1 \text{ and Lemma 2.1} \\
&= \tfrac{1}{2}(\lambda_p(f)-\epsilon) \log M(\tfrac{r}{4}, g_{n-1,f}) + O(1) \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
\text{or, } \log^{[2p]} T(r, f_{n,g}) &> \log^{[p]} \{\log M(\tfrac{r}{4}, g_{n-1,f})\} + O(1) \\
&\ge \log^{[p]} T(\tfrac{r}{4}, g_{n-1,f}) + O(1), \qquad \text{using Lemma 2.1} \\
&> \tfrac{1}{2}(\lambda_p(g)-\epsilon) \log M(\tfrac{r}{4^2}, f_{n-2,g}) + O(1). \qquad \text{using (3.6)}
\end{aligned}$$

Proceeding similarly after some steps we get

$$\log^{[(n-2)p]} T(r, f_{n,g}) > \tfrac{1}{2}(\lambda_p(g)-\epsilon) \log M(\tfrac{r}{4^{n-2}}, f_{2,g}) + O(1). \tag{3.7}$$

$$\begin{aligned}
\text{So, } \log^{[(n-1)p]} T(r, f_{n,g}) &> \tfrac{1}{2}(\lambda_p(f)-\epsilon) \log M(\tfrac{r}{4^{n-1}}, g_{1,f}) + O(1) \\
&= \tfrac{1}{2}(\lambda_p(f)-\epsilon) \log M(\tfrac{r}{4^{n-1}}, (1-\alpha)z + \alpha g(z)) + O(1) \\
&\ge \tfrac{1}{2}(\lambda_p(f)-\epsilon)\{\log M(\tfrac{r}{4^{n-1}}, g) - \log M(\tfrac{r}{4^{n-1}}, z)\} + O(1) \qquad (3.8) \\
&\ge \tfrac{1}{2}(\lambda_p(f)-\epsilon)\{T(\tfrac{r}{4^{n-1}}, g) - \log \tfrac{r}{4^{n-1}}\} + O(1). \qquad (3.9)
\end{aligned}$$

From (3.4), (3.5) and (3.9) we get for a sequence of values of r tending to infinity

$$\begin{aligned}
\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} &> \frac{\frac{1}{2}(\lambda_p(f)-\epsilon)\{T(\frac{r}{4^{n-1}}, g) - \log \frac{r}{4^{n-1}}\} + O(1)}{(1+\epsilon) r^{\lambda g(r)}} \\
&= \frac{\frac{1}{2}(\lambda_p(f)-\epsilon) T(\frac{r}{4^{n-1}}, g)}{(1+\epsilon) r^{\lambda g(r)}} + o(1) \qquad \{\text{since } \lim_{r\to\infty} \lambda_g(r) = \lambda(g) > 0\} \\
&> \frac{\frac{1}{2}(\lambda_p(f)-\epsilon)(1-\epsilon)(\frac{r}{4^{n-1}})^{\lambda g(\frac{r}{4^{n-1}})}}{(1+\epsilon) r^{\lambda g(r)}} + o(1)
\end{aligned}$$

$$= \frac{\frac{1}{2}(\lambda_p(f)-\epsilon)(1-\epsilon)}{(1+\epsilon)}\left(\frac{1}{4^{n-1}}\right)^{\lambda(g)+\delta} \frac{r^{\lambda(g)+\delta-\lambda g(r)}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda(g)+\delta-\lambda g(\frac{r}{4^{n-1}})}} + o(1)$$
$$\geq \frac{\frac{1}{2}(\lambda_p(f)-\epsilon)(1-\epsilon)}{(1+\epsilon)4^{(n-1)(\lambda(g)+\delta)}} + o(1)$$

because $r^{\lambda(g)+\delta-\lambda_g(r)}$ is ultimately an increasing function of r.

Since $\epsilon > 0$ and $\delta > 0$ are arbitrary, so we have from above that

$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} \geq \frac{\lambda_p(f)}{2.4^{(n-1)\lambda(g)}}$ and (ii) is proved.

Theorem 3.2. *Let $f(z)$ and $g(z)$ be two entire functions such that $\lambda_p(g)$ and $\rho_p(f)$ are finite and $\lambda_p(f) > 0$. Then for odd n*

i) $\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,f)} \leq 3\rho_p(g)2^{\lambda(f)}$

ii) $\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,f)} \geq \frac{\lambda_p(g)}{2.4^{(n-1)\lambda(f)}}$.

Theorem 3.3. *Let $f(z)$ and $g(z)$ be two entire functions such that $\lambda_p(g) > 0$. Also suppose that there exist entire functions $a_i (i = 1, 2, 3, ..., m; m \leq \infty)$ such that $T(r, a_i) = o\{T(r, g)\}$ as $r \to \infty (i = 1, 2, 3, ..., m)$ and $\sum_{i=1}^{m} \delta(a_i, g) = 1$. Then for even n*

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} \geq \frac{\pi\lambda_p(f)}{2.4^{(n-1)\lambda(g)}}.$$

Proof**.** If $\lambda(g) = \infty$ or $\lambda_p(f) = 0$, then the theorem is obvious. So we suppose that $\lambda(g) < \infty$ and $\lambda_p(f) > 0$.

For $0 < \epsilon < \min\{\lambda_p(f), \lambda_p(g), 1\}$ we get from (3.8)

$$\log^{[(n-1)p]} T(r, f_{n,g}) > \frac{1}{2}(\lambda_p(f) - \epsilon)\{\log M(\tfrac{r}{4^{n-1}}, g) - \log \tfrac{r}{4^{n-1}}\} + O(1)$$

Therefore,
$$\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} > \frac{\frac{1}{2}(\lambda_p(f)-\epsilon)\{\log M(\frac{r}{4^{n-1}},g)-\log\frac{r}{4^{n-1}}\}+O(1)}{T(r,g)}$$
$$= \frac{\frac{1}{2}(\lambda_p(f)-\epsilon)\log M(\frac{r}{4^{n-1}},g)}{T(r,g)} + o(1)$$
$$= \frac{1}{2}(\lambda_p(f) - \epsilon)\frac{\log M(\frac{r}{4^{n-1}},g)}{T(\frac{r}{4^{n-1}},g)}\frac{T(\frac{r}{4^{n-1}},g)}{T(r,g)} + o(1).$$

But from (3.4) and (3.5) we get for a sequence of values of r tending to infinity and for $\delta > 0$

$$\frac{T(\frac{r}{4^{n-1}},g)}{T(r,g)} > \frac{(1-\epsilon)}{(1+\epsilon)} \frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda(g)+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda(g)+\delta-\lambda g(\frac{r}{4^{n-1}})}} \frac{1}{r^{\lambda g(r)}}$$
$$\geq \frac{(1-\epsilon)}{(1+\epsilon)} \frac{1}{(4^{n-1})^{\lambda(g)+\delta}}$$

because $r^{\lambda(g)+\delta-\lambda_g(r)}$ is an increasing function of r.

Since $\epsilon(> 0)$ and $\delta(> 0)$ are arbitrary, so we have from Lemma 2.3 and above that

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,g)} \geq \frac{\pi\frac{1}{2}\lambda_p(f)}{4^{(n-1)\lambda(g)}}$$
$$= \frac{\pi\lambda_p(f)}{2.4^{(n-1)\lambda(g)}}.$$

Theorem 3.4. *Let $f(z)$ and $g(z)$ be two entire functions such that $\lambda_p(f) > 0$. Also suppose that there exist entire functions $a_i (i = 1, 2, 3, ..., m; m \leq \infty)$ such that $T(r, a_i) = o\{T(r, f)\}$ as $r \to \infty (i = 1, 2, 3, ..., m)$ and $\sum_{i=1}^{m} \delta(a_i, f) = 1$. Then for odd n*

$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(r,f)} \geq \frac{\pi\lambda_p(g)}{2.4^{(n-1)\lambda(f)}}$.

Theorem 3.5. *Let $f(z)$ be an entire function and $g(z)$ be a transcendental entire function such that $\rho_p(f)$, $\lambda(g)$ and $\rho_p(g)$ are finite. Also suppose that there exist entire functions $a_i (i = 1, 2, 3, ..., m; m \leq \infty)$ such that $T(r, a_i) = o\{T(r, g)\}$ as $r \to \infty (i = 1, 2, 3, ..., m)$ and $\sum_{i=1}^{m} \delta(a_i, g) = 1$. Then for even n*

$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(2^{n-2}r,g)} \leq \pi\lambda_p(f)$.

Proof**.** We have for all large values of r

$$\begin{aligned}
T(r, f_{n,g}) &= T(r, (1-\alpha)g_{n-1,f} + \alpha f(g_{n-1,f})) \\
&\leq T(r, g_{n-1,f}) + T(r, f(g_{n-1,f})) + O(1) \\
&\leq T(r, g_{n-1,f}) + (1+o(1))T(M(r, g_{n-1,f}), f) + O(1), \text{ using Lemma 2.4}
\end{aligned}$$

$$\begin{aligned}
\text{or, } \log^{[p]} T(r, f_{n,g}) &\leq \log^{[p]} T(r, g_{n-1,f}) + \log^{[p]} T(M(r, g_{n-1,f}), f) + O(1) \\
&< \log^{[p]} T(r, g_{n-1,f}) + (\rho_p(f)+\epsilon) \log M(r, g_{n-1,f}) + O(1) \\
&\leq T(2r, g_{n-1,f}) + (\rho_p(f)+\epsilon)3T(2r, g_{n-1,f}) + O(1), \\
&\qquad \text{using Lemma 2.1} \\
&= \{3(\rho_p(f)+\epsilon)+1\}T(2r, g_{n-1,f}) + O(1) \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
\text{or, } \log^{[2p]} T(r, f_{n,g}) &< \log^{[p]} T(2r, g_{n-1,f}) + O(1) \\
&< \{3(\rho_p(g)+\epsilon)+1\}T(2^2r, f_{n-2,g}) + O(1), \qquad \text{using (3.10)}
\end{aligned}$$

$\text{or, } \log^{[3p]} T(r, f_{n,g}) < \log^{[p]} T(2^2r, f_{n-2,g}) + O(1)$.

Proceeding similarly after some steps we get

$$\begin{aligned}
\log^{[(n-1)p]} T(r, f_{n,g}) &< \log^{[p]} T(2^{n-2}r, f_{2,g}) + O(1) \\
&= \log^{[p]} T(2^{n-2}r, (1-\alpha)g_{1,f} + \alpha f(g_{1,f})) + O(1) \\
&\leq \log^{[p]} T(2^{n-2}r, g_{1,f}) + \log^{[p]} T(2^{n-2}r, f(g_{1,f})) + O(1) \\
&\leq \log^{[p]} T(2^{n-2}r, g_{1,f}) + \log^{[p]} T(M(2^{n-2}r, g_{1,f}), f) + O(1). \qquad (3.11) \\
&\qquad \text{using Lemma 2.4}
\end{aligned}$$

Therefore, for a sequence of values of r tending to infinity

$$\begin{aligned}
\log^{[(n-1)p]} T(r, f_{n,g}) &< \log^{[p]} T(2^{n-2}r, g_{1,f}) + (\lambda_p(f)+\epsilon) \log M(2^{n-2}r, g_{1,f}) + O(1) \\
&= \log^{[p]} T(2^{n-2}r, (1-\alpha)z + \alpha g) + (\lambda_p(f)+\epsilon) \\
&\qquad \times \log M(2^{n-2}r, (1-\alpha)z + \alpha g) + O(1) \\
&\leq \log^{[p]} T(2^{n-2}r, z) + \log^{[p]} T(2^{n-2}r, g) + (\lambda_p(f)+\epsilon)\{\log M(2^{n-2}r, z) \\
&\qquad + \log M(2^{n-2}r, g)\} + O(1) \\
&\leq \log^{[p+1]}(2^{n-2}r) + \log^{[p]} T(2^{n-2}r, g) + (\lambda_p(f)+\epsilon)\{\log(2^{n-2}r) \\
&\qquad + \log M(2^{n-2}r, g)\} + O(1).
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, } \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(2^{n-2}r,g)} &< \frac{\log^{[p]} T(2^{n-2}r,g) + (\lambda_p(f)+\epsilon) \log M(2^{n-2}r,g) + O(1)}{T(2^{n-2}r,g)} \\
&= (\lambda_p(f)+\epsilon) \frac{\log M(2^{n-2}r,g)}{T(2^{n-2}r,g)} + o(1).
\end{aligned}$$

Since $\epsilon(> 0)$ is arbitrary, we get using Lemma 2.3 that

$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(2^{n-2}r,g)} \leq \pi\lambda_p(f)$.

Remark 3.1. Under the hypothesis of Theorem 3.5 we have also

$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{T(2^{n-2}r,g)} \leq \pi\rho_p(f)$.

Theorem 3.6. *Let $f(z)$ be a transcendental entire function and $g(z)$ be an entire function such that $\rho_p(f)$, $\lambda(f)$ and $\rho_p(g)$ are finite. Also suppose that there exist entire functions $a_i (i = 1, 2, 3, ..., m; m \leq \infty)$ satisfying $T(r, a_i) = o(T(r, f))$ as $r \to \infty (i = 1, 2, 3, ..., m)$ and $\sum_{i=1}^{m} \delta(a_i, f) = 1$. Then for odd n*

$$\liminf_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{T(2^{n-2}r, f)} \leq \pi\lambda_p(g).$$

Remark 3.2. Under the hypothesis of Theorem 3.6 we have also

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{T(2^{n-2}r, f)} \leq \pi\rho_p(g).$$

Theorem 3.7. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_p(f) \leq \rho_p(f) < \infty$ and $0 < \lambda_p(g) \leq \rho_p(g) < \infty$. Then for even n*

$$\frac{\overline{\lambda}_p(g)}{\rho_p(g)} \leq \liminf_{r\to\infty} \frac{\log^{[np+1]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[np+1]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})} \leq \frac{\overline{\rho}_p(g)}{\lambda_p(g)}$$

for $k = 0, 1, 2, ...$.

Proof**.** We have for all large values of r from (3.9)

$$\log^{[(n-1)p]} T(r, f_{n,g}) > \tfrac{1}{2}(\lambda_p(f) - \epsilon)\{T(\tfrac{r}{4^{n-1}}, g) - \log \tfrac{r}{4^{n-1}}\} + O(1)$$

or,

$$\log^{[np]} T(r, f_{n,g}) > \log^{[p]} T(\tfrac{r}{4^{n-1}}, g) - \log^{[p+1]}(\tfrac{r}{4^{n-1}}) + O(1) \tag{3.12}$$

or,

$$\log^{[np+1]} T(r, f_{n,g}) > \log^{[p+1]} T(\tfrac{r}{4^{n-1}}, g) - \log^{[p+2]}(\tfrac{r}{4^{n-1}}) + O(1). \tag{3.13}$$

Since $\limsup_{r\to\infty} \frac{\log^{[p]} T(r, g^{(k)})}{\log r} = \rho_p(g)$ so for all large values of r we obtain

$$\log^{[p]} T(r, g^{(k)}) < (\rho_p(g) + \epsilon) \log r. \tag{3.14}$$

Now from (3.13) and (3.14)

$$\frac{\log^{[np+1]} T(r, f_{n,g})}{\log^{[p]} T(r, g^k)} > \frac{\log^{[p+1]} T(\frac{r}{4^{n-1}}, g) - \log^{[p+2]}(\frac{r}{4^{n-1}}) + O(1)}{(\rho_p(g) + \epsilon) \log r}$$

$$= \frac{1}{(\rho_p(g) + \epsilon)} \frac{\log^{[p+1]} T(\frac{r}{4^{n-1}}, g)}{\log(\frac{r}{4^{n-1}})} \frac{\log(\frac{r}{4^{n-1}})}{\log r} + o(1).$$

Since ϵ (> 0) was arbitrary, by Definition 1.3

$$\frac{\overline{\lambda}_p(g)}{\rho_p(g)} \leq \liminf_{r\to\infty} \frac{\log^{[np+1]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})}. \tag{3.15}$$

From (3.3) for all large values of r and arbitrary $\epsilon > 0$

$$\log^{[(n-1)p]} T(r, f_{n,g}) < (\rho_p(f) + \epsilon)\{\log r + \log M(r, g)\} + O(1)$$

or,

$$\log^{[np]} T(r, f_{n,g}) < \log^{[p+1]} r + \log^{[p+1]} M(r, g) + O(1) \tag{3.16}$$

or, $\log^{[np+1]} T(r, f_{n,g}) < \log^{[p+2]} r + \log^{[p+2]} M(r, g) + O(1)$.

Therefore,

$$\frac{\log^{[np+1]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})} < \frac{\log^{[p+2]} M(r, g)}{\log^{[p]} T(r, g^{(k)})} + o(1). \tag{3.17}$$

Since $\liminf_{r\to\infty} \frac{\log^{[p]} T(r, g^{(k)})}{\log r} = \lambda_p(g)$, it follows for all large values of r

$$\log^{[p]} T(r, g^{(k)}) > (\lambda_p(g) - \epsilon) \log r. \tag{3.18}$$

Now from (3.17) and (3.18)

$$\frac{\log^{[np+1]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})} < \frac{\log^{[p+2]} M(r, g)}{\log r.(\lambda_p(g) - \epsilon)} + o(1).$$

Since $\epsilon (> 0)$ is arbitrary, we have

$$(3.19) \qquad \limsup_{r\to\infty} \frac{\log^{[np+1]} T(r,f_{n,g})}{\log^{[p]} T(r,g^{(k)})} \le \frac{\overline{\rho}_p(g)}{\lambda_p(g)}.$$

The theorem follows from (3.15) and (3.19).

Theorem 3.8. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_p(f) \le \rho_p(f) < \infty$ and $0 < \lambda_p(g) \le \rho_p(g) < \infty$. Then for odd n*

$$\frac{\overline{\lambda}_p(f)}{\rho_p(f)} \le \liminf_{r\to\infty} \frac{\log^{[np+1]} T(r,f_{n,g})}{\log^{[p]} T(r,f^{(k)})} \le \limsup_{r\to\infty} \frac{\log^{[np+1]} T(r,f_{n,g})}{\log^{[p]} T(r,f^{(k)})} \le \frac{\overline{\rho}_p(f)}{\lambda_p(f)}$$

for $k = 0, 1, 2, \ldots$.

Theorem 3.9. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_p(f) \le \rho_p(f) < \infty$, $0 < \lambda_p(g) \le \rho_p(g) < \infty$ and $\lambda(g) < \infty$. Then for even n*

$$\frac{\lambda_p(g)}{\rho_p(g)} \le \liminf_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} \le 1 \le \limsup_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} \le \frac{\rho_p(g)}{\lambda_p(g)}.$$

Proof**.** From (3.12) we get for all large values of r

$$\frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} > \frac{\log^{[p]} T(\frac{r}{4^{n-1}},g) - \log^{[p+1]}(\frac{r}{4^{n-1}}) + O(1)}{\log^{[p]} T(r,g)}$$

$$= \frac{\log^{[p]} T(\frac{r}{4^{n-1}},g)}{\log(\frac{r}{4^{n-1}})} \frac{\log r - \log 4^{n-1}}{\log^{[p]} T(r,g)} + o(1)$$

$$(3.20) \qquad = \frac{\log^{[p]} T(\frac{r}{4^{n-1}},g)}{\log(\frac{r}{4^{n-1}})} \frac{\log r}{\log^{[p]} T(r,g)} + o(1).$$

Since $\limsup_{r\to\infty} \frac{\log^{[p]} T(r,g)}{\log r} = \rho_p(g)$, for all large values of r, we obtain

$$(3.21) \qquad \log^{[p]} T(r,g) < (\rho_p(g) + \epsilon) \log r.$$

Since $\epsilon(> 0)$ is arbitrary, we get from (3.20) and (3.21)

$$(3.22) \qquad \frac{\lambda_p(g)}{\rho_p(g)} \le \liminf_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)}.$$

From (3.16) we get for all large values of r

$$(3.23) \qquad \log^{[np]} T(r,f_{n,g}) < \log^{[p+1]} r + \log^{[p+1]} M(r,g) + O(1).$$

Again from Lemma 2.1 and (3.4) we get for a sequence of values of r tending to infinity and for $\delta > 0$

$$\log M(r,g) < 3(1+\epsilon)(2r)^{\lambda_g(2r)}$$

$$= 3(1+\epsilon)\frac{(2r)^{\lambda(g)+\delta}}{(2r)^{\lambda(g)+\delta-\lambda_g(2r)}}$$

$$= 3(1+\epsilon)2^{\lambda(g)+\delta} \frac{r^{\lambda(g)+\delta-\lambda_g(r)}}{(2r)^{\lambda(g)+\delta-\lambda_g(2r)}} r^{\lambda_g(r)}$$

$$\le 3(1+\epsilon)2^{\lambda(g)+\delta} r^{\lambda_g(r)}$$

because $r^{\lambda(g)+\delta-\lambda_g(r)}$ is an increasing function of r.

Using (3.5) we get for a sequence of values of r tending to infinity

$$\log M(r,g) < \frac{3(1+\epsilon)}{1-\epsilon} 2^{\lambda(g)+\delta} T(r,g).$$

Therefore, $\log^{[p+1]} M(r,g) < \log^{[p]} T(r,g) + O(1)$.

So, from (3.23) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} < 1 + o(1).$$

So,

$$(3.24) \qquad \liminf_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} \le 1.$$

Also from (3.16) we get for all large values of r

$$\frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} < \frac{\log^{[p+1]} r + \log^{[p+1]} M(r,g) + O(1)}{\log^{[p]} T(r,g)}$$

$$= \frac{\log^{[p+1]} M(r,g)}{\log^{[p]} T(r,g)} + o(1)$$

$$= \frac{\log^{[p+1]} M(r,g)}{\log r} \frac{\log r}{\log^{[p]} T(r,g)} + o(1). \tag{3.25}$$

Since $\liminf_{r\to\infty} \frac{\log^{[p]} T(r,g)}{\log r} = \lambda_p(g)$, it follows for all large values of r

$$\log^{[p]} T(r,g) > (\lambda_p(g) - \epsilon)\log r. \tag{3.26}$$

Since $\epsilon(>0)$ is arbitrary, we get from (3.25) and (3.26)

$$\limsup_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} \leq \frac{\rho_p(g)}{\lambda_p(g)}. \tag{3.27}$$

From (3.12) we get for all large values of r

$$\frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} > \frac{\log^{[p]} T(\frac{r}{4^{n-1}},g) - \log^{[p+1]}(\frac{r}{4^{n-1}}) + O(1)}{\log^{[p]} T(r,g)}$$

$$= \frac{\log^{[p]} T(\frac{r}{4^{n-1}},g)}{\log^{[p]} T(r,g)} + o(1). \tag{3.28}$$

Now from (3.5) we get for all large values of r

$$T(\tfrac{r}{4^{n-1}},g) > (1-\epsilon)(\tfrac{r}{4^{n-1}})^{\lambda_g(\frac{r}{4^{n-1}})}$$

$$= (1-\epsilon)(\tfrac{1}{4^{n-1}})^{\lambda_g+\delta} \frac{r^{\lambda(g)+\delta-\lambda_g(r)}}{(\frac{r}{4^{n-1}})^{\lambda(g)+\delta-\lambda_g(\frac{r}{4^{n-1}})}} r^{\lambda_g(r)}$$

$$\geq (1-\epsilon)(\tfrac{1}{4^{n-1}})^{\lambda_g+\delta} r^{\lambda_g(r)}$$

because $r^{\lambda(g)+\delta-\lambda_g(r)}$ is an increasing function of r.

So, by (3.4) we get for a sequence of values of r tending to infinity

$$T(\tfrac{r}{4^{n-1}},g) > (1-\epsilon)(\tfrac{1}{4^{n-1}})^{\lambda(g)+\delta} . \tfrac{T(r,g)}{1+\epsilon}.$$

So,

$$\log^{[p]} T(\tfrac{r}{4^{n-1}},g) > \log^{[p]} T(r,g) + O(1). \tag{3.29}$$

Therefore by (3.28) and (3.29) we get for a sequence of values of r tending to infinity

$$\frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} > \frac{\log^{[p]} T(r,g)}{\log^{[p]} T(r,g)} + o(1).$$

Hence,

$$\limsup_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} \geq 1. \tag{3.30}$$

The theorem follows from (3.22), (3.24), (3.27) and (3.30).

Remark 3.3. If in addition to the condition of Theorem 3.9, we suppose that $\rho_p(g) = \lambda_p(g)$ then for even n

$$\lim_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} = 1.$$

Remark 3.4. The conditions $\lambda_p(f) > 0$ or $\rho_p(f) < \infty$ cannot be omitted in Theorem 3.9 and Remark 3.3 which are evident from the following examples.

Example 3.1. Let $f(z) = z$, $g(z) = \exp z$, $p = 1$ and $\alpha = 1$.

Then $\rho_p(f) = \lambda_p(f) = 0$, $0 < 1 = \rho_p(g) = \lambda_p(g) < \infty$ and $f_{n,g}(z) = \exp^{[\frac{n}{2}]} z$ for even n.

Now,

$$\log^{[np]} T(r,f_{n,g}) = \log^{[n]} T(r, \exp^{[\frac{n}{2}]} z)$$
$$\leq \log^{[n]}(\log M(r, \exp^{[\frac{n}{2}]} z))$$
$$= \log^{[\frac{n}{2}+1]} r.$$

Therefore, $\lim_{r\to\infty} \frac{\log^{[np]} T(r,f_{n,g})}{\log^{[p]} T(r,g)} = 0.$

Example 3.2. Let $f(z) = \exp^{[2]} z$, $g(z) = \exp z$, $p = 1$ and $\alpha = 1$.

Then $\rho_p(f)=\lambda_p(f)=\infty$, $\rho_p(g)=\lambda_p(g)=1$ and $f_{n,g}(z)=exp^{[\frac{3n}{2}]}z$ for even n.

Now, $$\log^{[np]}T(r,f_{n,g})=\log^{[n]}T(r,\exp^{[\frac{3n}{2}]}z)$$
$$\geq \log^{[n]}(\tfrac{1}{3}\log M(\tfrac{r}{2},\exp^{[\frac{3n}{2}]}z))$$
$$=\exp^{[\frac{n}{2}-1]}(\tfrac{r}{2})+O(1).$$

Therefore, $\lim\limits_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,g)}=\infty$.

Theorem 3.10. *Let $f(z)$ and $g(z)$ be two entire functions such that $0<\lambda_p(f)\leq\rho_p(f)<\infty$, $0<\lambda_p(g)\leq\rho_p(g)<\infty$ and $\lambda(f)<\infty$. Then for odd n*
$$\frac{\lambda_p(f)}{\rho_p(f)}\leq\liminf_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}\leq 1\leq\limsup_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}\leq\frac{\rho_p(f)}{\lambda_p(f)}.$$

Remark 3.5. If in addition to the condition of Theorem 3.10, we suppose that $\rho_p(f)=\lambda_p(f)$ then for odd n
$$\lim_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}=1.$$

Remark 3.6. Similarly the conditions $\lambda_p(g)>0$ or $\rho_p(g)<\infty$ cannot be omitted in Theorem 3.10 and Remark 3.5, which are evident from the following examples.

Example 3.3. Let $f(z)=\exp z$, $g(z)=z$, $p=1$ and $\alpha=1$.

Then $\rho_p(g)=\lambda_p(g)=0$, $0<1=\rho_p(f)=\lambda_p(f)<\infty$ and $f_{n,g}(z)=\exp^{[\frac{n+1}{2}]}z$ for odd n.

Now, $$\log^{[np]}T(r,f_{n,g})=\log^{[n]}T(r,\exp^{[\frac{n+1}{2}]}z)$$
$$\leq\log^{[n]}(\log M(r,\exp^{[\frac{n+1}{2}]}z))$$
$$=\log^{[\frac{n+1}{2}]}r.$$

Therefore, $\lim\limits_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f)}=0$.

Example 3.4. Let $f(z)=\exp z$, $g(z)=\exp^{[2]}z$, $p=1$ and $\alpha=1$.

Then $\rho_p(f)=\lambda_p(f)=1$, $\rho_p(g)=\lambda_p(g)=\infty$ and $f_{n,g}(z)=\exp^{[1+\frac{3(n-1)}{2}]}z=\exp^{[\frac{3n-1}{2}]}z$ for odd n.

Now, $$\log^{[np]}T(r,f_{n,g})=\log^{[n]}T(r,\exp^{[\frac{3n-1}{2}]}z)$$
$$\geq\log^{[n]}(\tfrac{1}{3}\log M(\tfrac{r}{2},\exp^{[\frac{3n-1}{2}]}z))$$
$$=\exp^{[\frac{n-3}{2}]}(\tfrac{r}{2})+O(1).$$

Therefore, $\lim\limits_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,g)}=\infty$.

Theorem 3.11. *Let $f(z)$ and $g(z)$ be two entire functions such that $0<\lambda_p(f)\leq\rho_p(f)<\infty$ and $0<\lambda_p(g)\leq\rho_p(g)<\infty$. Then for even n*
$$\frac{\lambda_p(g)}{\rho_p(f)}\leq\liminf_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f^{(k)})}\leq\limsup_{r\to\infty}\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f^{(k)})}\leq\frac{\rho_p(g)}{\lambda_p(f)}$$
for $k=0,1,2,3,...$.

Proof**.** From (3.12) we get for all large values of r
$$\frac{\log^{[np]}T(r,f_{n,g})}{\log^{[p]}T(r,f^{(k)})}>\frac{\log^{[p]}T(\frac{r}{4^{n-1}},g)-\log^{[p+1]}(\frac{r}{4^{n-1}})+O(1)}{\log^{[p]}T(r,f^{(k)})}$$

$$= \frac{\log^{[p]} T(\frac{r}{4^{n-1}}, g)}{\log(\frac{r}{4^{n-1}})} . \frac{\log r - \log 4^{n-1}}{\log^{[p]} T(r, f^{(k)})} + o(1)$$

$$= \frac{\log^{[p]} T(\frac{r}{4^{n-1}}, g)}{\log(\frac{r}{4^{n-1}})} . \frac{\log r}{\log^{[p]} T(r, f^{(k)})} + o(1). \tag{3.31}$$

Since $\limsup_{r\to\infty} \frac{\log^{[p]} T(r,f^{(k)})}{\log r} = \rho_p(f)$, so for all large values of r

$$\log^{[p]} T(r, f^{(k)}) < (\rho_p(f) + \epsilon) \log r. \tag{3.32}$$

From (3.31) and (3.32)

$$\frac{\log^{[np]} T(r, f_{n,g})}{\log^{[p]} T(r, f^{(k)})} > \frac{\lambda_p(g) - \epsilon}{\rho_p(f) + \epsilon} + o(1).$$

Since $\epsilon (> 0)$ is arbitrary

$$\frac{\lambda_p(g)}{\rho_p(f)} \leq \liminf_{r\to\infty} \frac{\log^{[np]} T(r, f_{n,g})}{\log^{[p]} T(r, f^{(k)})}. \tag{3.33}$$

Also from (3.16) for all large values of r

$$\frac{\log^{[np]} T(r, f_{n,g})}{\log^{[p]} T(r, f^{(k)})} < \frac{\log^{[p+1]} r + \log^{[p+1]} M(r, g) + O(1)}{\log^{[p]} T(r, f^{(k)})}$$

$$= \frac{\log^{[p+1]} M(r, g)}{\log r} \frac{\log r}{\log^{[p]} T(r, f^{(k)})} + o(1). \tag{3.34}$$

Since $\liminf_{r\to\infty} \frac{\log^{[p]} T(r,f^{(k)})}{\log r} = \lambda_p(f)$, it follows for all large values of r

$$\log^{[p]} T(r, f^{(k)}) > (\lambda_p(f) - \epsilon) \log r. \tag{3.35}$$

Since $\epsilon (> 0)$ is arbitrary, we get from (3.34) and (3.35)

$$\limsup_{r\to\infty} \frac{\log^{[np]} T(r, f_{n,g})}{\log^{[p]} T(r, f^{(k)})} \leq \frac{\rho_p(g)}{\lambda_p(f)}. \tag{3.36}$$

The theorem follows from (3.33) and (3.36).

Theorem 3.12. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_p(f) \leq \rho_p(f) < \infty$ and $0 < \lambda_p(g) \leq \rho_p(g) < \infty$. Then for odd n*

$$\frac{\lambda_p(f)}{\rho_p(g)} \leq \liminf_{r\to\infty} \frac{\log^{[np]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[np]} T(r, f_{n,g})}{\log^{[p]} T(r, g^{(k)})} \leq \frac{\rho_p(f)}{\lambda_p(g)}$$

for $k = 0, 1, 2, 3, \ldots$.

Theorem 3.13. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_p(f) \leq \rho_p(f) < \infty$ and $\rho_p(g) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{\log^{[p-1]} T(\exp^{[p]}(2^{n-2} r), f^{(k)})} = 0 \quad \text{for } k = 0, 1, 2, 3, \ldots \ .$$

Proof**.** First suppose that n is even. Suppose $0 < \epsilon < \lambda_p(f)$.
From (3.11) we have for all large values of r

$$\begin{aligned}
\log^{[(n-1)p]} T(r, f_{n,g}) &< \log^{[p]} T(2^{n-2} r, g_{1,f}) + \log^{[p]} T(M(2^{n-2} r, g_{1,f}), f) + O(1) \\
&< \log^{[p]} T(2^{n-2} r, g_{1,f}) + (\rho_p(f) + \epsilon) \log M(2^{n-2} r, g_{1,f}) + O(1) \\
&= \log^{[p]} T(2^{n-2} r, (1-\alpha) z + \alpha g) + (\rho_p(f) + \epsilon) \\
&\qquad \times \log M(2^{n-2} r, (1-\alpha) z + \alpha g) + O(1) \\
&\leq \log^{[p]} T(2^{n-2} r, z) + \log^{[p]} T(2^{n-2} r, g) + (\rho_p(f) + \epsilon)\{\log M(2^{n-2} r, z) \\
&\qquad + \log M(2^{n-2} r, g)\} + O(1) \\
&< \log^{[p+1]}(2^{n-2} r) + (\rho_p(g) + \epsilon) \log(2^{n-2} r) + (\rho_p(f) + \epsilon) \log(2^{n-2} r) \\
&\qquad + (\rho_p(f) + \epsilon) \exp^{[p-1]}(2^{n-2} r)^{\rho_p(g) + \epsilon} + O(1).
\end{aligned} \tag{3.37}$$

On the other hand we get for all large values of r

$\frac{\log^{[p]} T(r,f^{(k)})}{\log r} > \lambda_p(f) - \epsilon$

or, $\log^{[p-1]} T(r, f^{(k)}) > r^{\lambda_p(f)-\epsilon}$.

Therefore,

$$\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r), f^{(k)}) > (\exp^{[p]}(2^{n-2}r))^{\lambda_p(f)-\epsilon}. \tag{3.38}$$

From (3.37) and (3.38) we have for all large values of r

$$\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r),f^{(k)})} < \frac{(\rho_p(f)+\epsilon)exp^{[p-1]}(2^{n-2}r)^{\rho_p(g)+\epsilon}}{(\exp^{[p]}(2^{n-2}r))^{\lambda_p(f)-\epsilon}} + o(1).$$

and hence, $\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r),f^{(k)})} = 0$ and the theorem is proved for even n.

Also for odd n we get as in (3.37)

$$\log^{[(n-1)p]} T(r, f_{n,g}) < \log^{[p+1]}(2^{n-2}r)+(\rho_p(f)+\epsilon)\log(2^{n-2}r)+(\rho_p(g)+\epsilon)\log(2^{n-2}r) +(\rho_p(g)+\epsilon)\exp^{[p-1]}(2^{n-2}r)^{\rho_p(f)+\epsilon}+O(1)$$

and consequently the theorem follows immediately.

Remark 3.7. The condition $\rho_p(g) < \infty$ cannot be omitted in Theorem 3.13 which is evident from the following example.

Example 3.5. Let $f(z) = \exp z$, $g(z) = \exp^{[3]} z$, $p = 1$ and $\alpha = 1$.

Then $\rho_p(f) = \lambda_p(f) = 1$, $\rho_p(g) = \infty$ and

$$\begin{aligned} f_{n,g}(z) &= \exp^{[2n]} z \text{ when } n \text{ is even.} \\ &= \exp^{[2n-1]} z \text{ when } n \text{ is odd.} \end{aligned}$$

Therefore for even n

$$\begin{aligned} \log^{[(n-1)p]} T(r, f_{n,g}) &= \log^{[n-1]} T(r, \exp^{[2n]} z) \\ &\geq \log^{[n-1]}[\tfrac{1}{3}\log M(\tfrac{r}{2}, \exp^{[2n]} z)] \\ &= \exp^{[n]}(\tfrac{r}{2}) + O(1), \end{aligned}$$

and for odd n

$$\begin{aligned} \log^{[(n-1)p]} T(r, f_{n,g}) &= \log^{[n-1]} T(r, \exp^{[2n-1]} z) \\ &\geq \log^{[n-1]}[\tfrac{1}{3}\log M(\tfrac{r}{2}, \exp^{[2n-1]} z)] \\ &= \exp^{[n-1]}(\tfrac{r}{2}) + O(1). \end{aligned}$$

Also, $\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r), f^{(k)}) = T(\exp(2^{n-2}r), f^{(k)}) = \frac{\exp(2^{n-2}r)}{\pi}$.

Thus it follows that for any $n \geq 2$

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r),f^{(k)})} = \infty.$$

Theorem 3.14. *Let $f(z)$ and $g(z)$ be two entire functions such that $0 < \lambda_p(g) \leq \rho_p(g) < \infty$ and $\rho_p(f) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r),g^{(k)})} = 0 \quad \textit{for } k = 0, 1, 2, 3, \ldots \quad .$$

Remark 3.8. The condition $\rho_p(f) < \infty$ cannot be omitted in Theorem 3.14 which is evident from the following example.

Example 3.6. Let $f(z) = \exp^{[3]} z$, $g(z) = \exp z$, $p = 1$ and $\alpha = 1$.

Then $\rho_p(g) = \lambda_p(g) = 1$, $\rho_p(f) = \infty$ and

$f_{n,g}(z) = \exp^{[2n]} z$ when n is even.

$= \exp^{[2n+1]} z \qquad$ when n is odd.

Therefore as in Example 3.5 we get for even n

$$\log^{[(n-1)p]} T(r, f_{n,g}) \geq \exp^{[n]}(\tfrac{r}{2}) + O(1),$$

and for odd n

$$\log^{[(n-1)p]} T(r, f_{n,g}) \geq \exp^{[n+1]}(\tfrac{r}{2}) + O(1).$$

Also, $\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r), g^{(k)}) = \frac{\exp(2^{n-2}r)}{\pi}$.

Thus it follows that for any $n \geq 2$

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{\log^{[p-1]} T(\exp^{[p]}(2^{n-2}r), g^{(k)})} = \infty.$$

Theorem 3.15. *Let $f(z)$ and $g(z)$ be two transcendental entire functions such that*
(i) $0 < \lambda_p(g) \leq \rho_p(g) \leq \rho(g) < \infty$;
(ii) $\lambda_p(f) > 0$;
and (iii) $\delta(0; f) < 1$.
Then for any real number A and for even n

$$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r, f_{n,g})}{\log^{[p]} T(r^A, g^{(k)})} = \infty \text{ for } k = 0, 1, 2, 3, ... \quad .$$

Proof. We suppose that $A > 0$, because otherwise the theorem is obvious.
From (3.7) we get for all large values of r

$$\begin{aligned}
\log^{[(n-2)p]} T(r, f_{n,g}) &> \tfrac{1}{2}(\lambda_p(g) - \epsilon) \log M(\tfrac{r}{4^{n-2}}, f_{2,g}) + O(1) \\
&= \tfrac{1}{2}(\lambda_p(g) - \epsilon) \log M(\tfrac{r}{4^{n-2}}, (1-\alpha)g_{1,f} + \alpha f(g_{1,f})) + O(1) \\
&\geq \tfrac{1}{2}(\lambda_p(g) - \epsilon)\{\log M(\tfrac{r}{4^{n-2}}, f(g_{1,f})) - \log M(\tfrac{r}{4^{n-2}}, g_{1,f})\} \\
&\qquad + O(1) \\
&\geq \tfrac{1}{2}(\lambda_p(g) - \epsilon)\{T(\tfrac{r}{4^{n-2}}, f(g_{1,f})) - \log M(\tfrac{r}{4^{n-2}}, g_{1,f})\} + O(1)
\end{aligned}$$

or,

$$\log^{[(n-1)p]} T(r, f_{n,g}) \geq \log^{[p]} T(\tfrac{r}{4^{n-2}}, f(g_{1,f})) - \log^{[p+1]} M(\tfrac{r}{4^{n-2}}, g_{1,f}) + O(1). \tag{3.39}$$

For given $\epsilon(0 < \epsilon < 1 - \delta(0; f))$

$N(r, 0; f) > (1 - \delta(0; f) - \epsilon)T(r, f)$ for all sufficiently large values of r.

So, from Lemma 2.5, for all sufficiently large values of r

$$T(\tfrac{r}{4^{n-2}}, f(g_{1,f})) + O(1) \geq (\log \tfrac{1}{\eta})[\frac{(1-\delta(0;f)-\epsilon)T\{M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}), f\}}{\log M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}) - O(1)} - O(1)]$$

or, $$\log^{[p]} T(\tfrac{r}{4^{n-2}}, f(g_{1,f})) \geq \log^{[p]} T(M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}), f) - \log^{[p+1]} M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}) + O(1)$$

$$= \log^{[p]} T(M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}), f) + O(\log r). \tag{3.40}$$

Again
$$\begin{aligned}
\log^{[p+1]} M(\tfrac{r}{4^{n-2}}, g_{1,f}) &= \log^{[p+1]} M(\tfrac{r}{4^{n-2}}, (1-\alpha)z + \alpha g) \\
&\geq \log^{[p+1]} M(\tfrac{r}{4^{n-2}}, g) - \log^{[p+1]} M(\tfrac{r}{4^{n-2}}, z) \\
&> (\lambda_p(g) - \epsilon) \log(\tfrac{r}{4^{n-2}}) - \log^{[p+1]} \tfrac{r}{4^{n-2}} \\
&= O(\log r). \qquad (3.41)
\end{aligned}$$

Therefore from (3.39), (3.40) and (3.41) for all sufficiently large values of r

$$\begin{aligned}
\log^{[(n-1)p]} T(r, f_{n,g}) &> \log^{[p]} T(M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}), f) + O(\log r) \\
&> (\lambda_p(f) - \epsilon) \log M((\eta r)^{\frac{1}{1+\gamma}}, g_{1,f}) + O(\log r) \\
&= (\lambda_p(f) - \epsilon) \log M((\eta r)^{\frac{1}{1+\gamma}}, (1-\alpha)z + \alpha g(z)) + O(\log r) \\
&\geq (\lambda_p(f)-\epsilon)(\log M((\eta r)^{\frac{1}{1+\gamma}}, g) - \log M((\eta r)^{\frac{1}{1+\gamma}}, z)) + O(\log r) \\
&> (\lambda_p(f)-\epsilon)(\exp^{[p-1]}(\eta r)^{\frac{1}{1+\gamma}(\lambda_p(g)-\epsilon)} - \log(\eta r)^{\frac{1}{1+\gamma}}) + O(\log r)
\end{aligned}$$

$$= (\lambda_p(f) - \epsilon)\exp^{[p-1]}(\eta r)^{\frac{1}{1+\gamma}(\lambda_p(g)-\epsilon)} + O(\log r). \tag{3.42}$$

Also,

$$\log^{[p]} T(r^A, g^{(k)}) < A(\rho_p(g) + \epsilon)\log r \tag{3.43}$$

for all sufficiently large values of r.

So from (3.42) and (3.43) for all sufficiently large values of r

$$\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p]} T(r^A,g^{(k)})} > \frac{O(\log r)}{A(\rho_p(g)+\epsilon)\log r} + \frac{(\lambda_p(f)-\epsilon)\exp^{[p-1]}(\eta r)^{\frac{1}{1+\gamma}(\lambda_p(g)-\epsilon)}}{A(\rho_p(g)+\epsilon)\log r}.$$

Therefore, $\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p]} T(r^A,g^{(k)})} = \infty.$

Theorem 3.16. *Let $f(z)$ and $g(z)$ be two transcendental entire functions such that*
(i) $0 < \lambda_p(f) \le \rho_p(f) \le \rho(f) < \infty$;
(ii) $\lambda_p(g) > 0$;
and (iii) $\delta(0; g) < 1$.
Then for any real number A and for odd n
$\limsup_{r\to\infty} \frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log^{[p]} T(r^A,f^{(k)})} = \infty$ *for $k = 0, 1, 2, 3, ...$* .

Theorem 3.17. *Let $f(z)$ and $g(z)$ be two entire functions such that $\rho_p(f) = 0$, $\rho_p^*(f) < \infty$ and $\rho(g) < \infty$. Then for even n, $\rho_{(n-1)p}(f_{n,g}) < \infty$.*

Proof**.** To prove the theorem we first prove that $\rho_p(g_{1,f}) < \infty$ for any $p \ge 1$.
We have $g_{1,f}(z) = (1-\alpha)z + \alpha g(z)$, $\rho(z) = 0$ and $\rho(g) < \infty$.
So, $\rho(g_{1,f}) \le \max\{\rho(z), \rho(g)\}$.
Therefore, $\rho(g_{1,f}) < \infty$.
Again $\rho_p(g_{1,f}) \le \rho(g_{1,f}) < \infty$.
From (3.11) for all large values of r

$$\begin{aligned}\frac{\log^{[(n-1)p]} T(r,f_{n,g})}{\log r} &\le \frac{\log^{[p]} T(2^{n-2}r,g_{1,f})}{\log r} + \frac{\log^{[p]} T(M(2^{n-2}r,g_{1,f}),f)}{\log r} + o(1)\\ &= \frac{\log^{[p]} T(2^{n-2}r,g_{1,f})}{\log(2^{n-2}r)}\frac{\log 2^{n-2}+\log r}{\log r} + \frac{\log^{[p]} T(M(2^{n-2}r,g_{1,f}),f)}{\log\log M(2^{n-2}r,g_{1,f})}\\ &\qquad\times\frac{\log\log M(2^{n-2}r,g_{1,f})}{\log r} + o(1)\end{aligned}$$

Therefore, $\rho_{(n-1)p}(f_{n,g}) < \infty$.

Theorem 3.18. *Let $f(z)$ and $g(z)$ be two entire functions such that $\rho_p(g) = 0$, $\rho_p^*(g) < \infty$ and $\rho(f) < \infty$. Then for odd n, $\rho_{(n-1)p}(f_{n,g}) < \infty$.*

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