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# THE $(q,k)$-EXTENSION OF SOME GAMMA FUNCTION INEQUALITIES

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Abstract. In this paper, the authors establish some inequalities for the $(q,k)$-extension of the classical Gamma function. The procedure utilizes a monotonicity property of the $(q,k)$-extension of the psi function. As an application, some previous results are recovered as special cases of the results of this paper.

## 1. Introduction and Discussion

Let us begin by recalling the following basic definitions concerning our results.

The classical Euler's Gamma function, $\Gamma(x)$ and the classical psi or digamma function, $\psi(x)$ are usually defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dx=\lim_{n\to\infty}\frac{n!n^x}{x(x+1)(x+2)\dots(x+n)}$$

and

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}.$$

The $p$-extension (also known as $p$-analogue, $p$-deformation or $p$-generalization) of the Gamma function, $\Gamma_p(x)$ is defined (see [2]) for $p\in\mathbb{N}$ and $x>0$ by

$$\Gamma_p(x)=\frac{p!p^x}{x(x+1)\dots(x+p)}=\frac{p^x}{x(1+\frac{x}{1})\dots(1+\frac{x}{p})}$$

where $\lim_{p\to\infty}\Gamma_p(x)=\Gamma(x)$.

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Also, the $q$-extension of the Gamma function, $\Gamma_q(x)$ is defined (see [7]) for $q \in (0,1)$ and $x > 0$ by

$$\Gamma_q(x) = (1-q)^{1-x} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+x}} = (1-q)^{1-x} \prod_{n=1}^{\infty} \frac{1-q^{n}}{1-q^{n+x}}$$

where $\lim_{q\to 1} \Gamma_q(x) = \Gamma(x)$ .

The $k$-extension of the Gamma function, $\Gamma_k(t)$ is similarly defined (see [5]) for $k > 0$ and $x \in \mathbb{C}\backslash k\mathbb{Z}^-$ as

$$\Gamma_k(x) = \int_0^{\infty} e^{-\frac{t^k}{k}} t^{x-1}\, dx = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

where $(x)_{n,k} = x(x+k)(x+2k)\dots(x+(n-1)k)$ is the $k$-Pochhammer symbol and $\lim_{k\to 1} \Gamma_k(x) = \Gamma(x)$.

Krasniqi and Merovci [8] also defined the $(p,q)$-extension of the Gamma for $p \in \mathbb{N}$, $q \in (0,1)$ and $x > 0$ by

$$\Gamma_{p,q}(x) = \frac{[p]_q^x [p]_q!}{[x]_q [x+1]_q \dots [x+p]_q}$$

where $[p]_q = \frac{1-q^p}{1-q}$, and $\Gamma_{p,q}(x) \to \Gamma(x)$, as $p \to \infty$ and $q \to 1$.

Furthermore, Díaz and Teruel [6] also defined the $(q,k)$-extension of the Gamma for $q \in (0,1)$, $k > 0$ and $x > 0$ as follows.

$$\Gamma_{q,k}(x) = \frac{(1-q^k)_{q,k}^{\frac{x}{k}-1}}{(1-q)^{\frac{x}{k}-1}} = \frac{(1-q^k)_{q,k}^{\infty}}{(1-q^x)_{q,k}^{\infty}(1-q)^{\frac{x}{k}-1}}$$

where
(i) $(x+y)_{q,k}^n := \prod_{i=0}^{n-1}(x+q^{ik}y)$ and
(ii) $(1+x)_{q,k}^t := \frac{(1+x)_{q,k}^{\infty}}{(1+q^{kt}x)_{q,k}^{\infty}}$, for $x, y, t \in \mathbb{R}$ and $n \in \mathbb{N}$.
By using the relations (i) and (ii), the function $\Gamma_{q,k}(x)$ takes the following form.

$$\Gamma_{q,k}(x) = \frac{1}{(1-q)^{\frac{x}{k}-1}} \prod_{n=0}^{\infty} \frac{1-q^{(n+1)k}}{1-q^{nk+x}} \tag{1.1}$$

It satisfies the following identities.

$$\Gamma_{q,k}(x+k) = [x]_q \Gamma_{q,k}(x) \tag{1.2}$$

$$\Gamma_{q,k}(k) = 1 \tag{1.3}$$

Similarly, the $(q,k)$-extension of the psi fuction is defined as

$$\psi_{q,k}(x) = \frac{d}{dx} \ln \Gamma_{q,k}(x) = \frac{\Gamma'_{q,k}(x)}{\Gamma_{q,k}(x)}$$

where $\Gamma_{q,k}(x) \to \Gamma(x)$, $\psi_{q,k}(x) \to \psi(x)$ as $q \to 1$, $k \to 1$.

The function $\psi_{q,k}(x)$ as defined above exhibits the following series representations .

$$\psi_{q,k}(x) = -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{nk+x}}{1-q^{nk+x}}, \tag{1.4}$$

$$= -\frac{1}{k}\ln(1-q) + (\ln q)\sum_{n=0}^{\infty}\frac{q^{(n+1)x}}{1-q^{(n+1)k}} \tag{1.5}$$

By taking the logarithmic derivative of (1.1), we obtain (1.4), and the relation (1.5) was established in [4].
In 2008, Shabani [14] proved the following inequality generalizing the previous results of [1], [3], [12] and [13].

$$\frac{\Gamma(a+b)^{\mu}}{\Gamma(c+d)^{\lambda}} \leq \frac{\Gamma(a+bx)^{\mu}}{\Gamma(c+dx)^{\lambda}} \leq \frac{\Gamma(a)^{\mu}}{\Gamma(c)^{\lambda}}, \quad x \in [0,1] \tag{1.6}$$

where $a, b, c, d, \mu$ and $\lambda$ are real numbers such that $a + bx > 0$, $c + dx > 0$, $a + bx \leq c + dx$, $0 < \mu b \leq \lambda d$ and $\psi(a + bx) > 0$ or $\psi(c + dx) > 0$.

Since then, this inequality has attracted the attention of several researchers. For instance, in [9], [15], [11] and [8], the authors respectively proved the results for the $p$-extension, $q$-extension, $k$-extension and $(p, q)$-extension of the gamma function.

Also, in 2009, Vinh and Ngoc [16] proved the following related results by using the Dirichlet's integral.

$$\frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i)}{\Gamma(\beta+\sum_{i=1}^{n}\alpha_i)} \leq \frac{\prod_{i=1}^{n}\Gamma(1+\alpha_i x)}{\Gamma(\beta+\sum_{i=1}^{n}\alpha_i x)} \leq \frac{1}{\Gamma(\beta)} \tag{1.7}$$

where $x \in [0, 1]$, $\beta \geq 1$, $\alpha_i > 0$, $n \in \mathbb{N}$.

In addition, in the papers [11] and [17], the authors by using different procedures, separately proved the $k$-extension of (1.7) together with other results.

In this paper, the objective is to prove the $(q, k)$-extension of the inequalities (1.6) and (1.7) by using similar techniques as in [11] and [14]. We present our results in the following section.

## 2. Results

In order to present our results, we need the following Lemmas.

**Lemma 2.1.** *Let $x > 0$, $y > 0$ with $x \leq y$, then for $q \in (0, 1)$ and $k > 0$,*

$$\psi_{q,k}(x) \leq \psi_{q,k}(y). \tag{2.1}$$

*Proof.* By differentiating the series representation (1.4), we obtain

$$\psi'_{q,k}(x) = (\ln q)^2\sum_{n=0}^{\infty}\frac{q^{nk+x}}{(1-q^{nk+x})^2} \geq 0.$$

Thus, $\psi_{q,k}(x)$ is increasing for $x > 0$. Then, for $0 < x \leq y$, we have $\psi_{q,k}(x) \leq \psi_{q,k}(y)$ concluding the proof. See also [10]. □

*Remark* 2.1. The fact that $\psi_{q,k}(x)$ is increasing for $x > 0$, can alternatively be established by using identity (1.2). That is, by taking the logarithmic derivative of (1.2), we obtain $\psi_{q,k}(x+k) - \psi_{q,k}(x) = -\frac{(\ln q)q^x}{1-q} \geq 0$.

**Lemma 2.2.** *Let $a$, $b$, $c$ and $d$ be real numbers such that $0 < a + bx \leq c + dx$. Then for $q \in (0,1)$ and $k > 0$, we have*

$$\psi_{q,k}(a+bx) \leq \psi_{q,k}(c+dx).$$

*Proof.* Follows directly from Lemma 2.1. □

**Lemma 2.3.** *Let $a$, $b$, $c$, $d$, $\mu$ and $\lambda$ be positive real numbers such that $\mu b \leq \lambda d$ and $a + bx \leq c + dx$. For $q \in (0,1)$ and $k > 0$, if either:*
*(i) $\psi_{q,k}(a+bx) > 0$ or*
*(ii) $\psi_{q,k}(c+dx) > 0$,*
*then,*

$$\mu b \psi_{q,k}(a+bx) - \lambda d \psi_{q,k}(c+dx) \leq 0.$$

**Lemma 2.4.** *Let $a$, $b$, $c$, $d$, $\mu$ and $\lambda$ be positive real numbers such that $\mu b \geq \lambda d$ and $a + bx \leq c + dx$. For $q \in (0,1)$ and $k > 0$, if either:*
*(i) $\psi_{q,k}(a+bx) < 0$ or*
*(ii) $\psi_{q,k}(c+dx) < 0$,*
*then,*

$$\mu b \psi_{q,k}(a+bx) - \lambda d \psi_{q,k}(c+dx) \leq 0.$$

The proofs of Lemmas 2.3 and 2.4 follow from Lemma 2.2, and are in a similar method as those in [14]. As a result, we omit them.

**Lemma 2.5.** *Let $x \geq 0$, $\alpha_i > 0$, $\beta \geq k > 0$ and $q \in (0,1)$. Then, $k + \alpha_i x \leq \beta + \sum_{i=1}^{n} \alpha_i x$, $n \in \mathbb{N}$ implies*

$$\psi_{q,k}(k + \alpha_i x) \leq \psi_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x).$$

*Proof.* Follows from Lemma 2.1. □

**Theorem 2.1.** *Define a function $G$ for $x \geq 0$, $q \in (0,1)$ and $k > 0$ by*

$$G(x) = \frac{\Gamma_{q,k}(a+bx)^{\mu}}{\Gamma_{q,k}(c+dx)^{\lambda}}$$

*where $a$, $b$, $c$, $d$, $\mu$ and $\lambda$ are positive real numbers such that $a + bx \leq c + dx$, $\mu b \leq \lambda d$ and either $\psi_{q,k}(a+bx) > 0$ or $\psi_{q,k}(c+dx) > 0$. Then $G$ is a decreasing function of $x$ and the following inequalities hold true:*

$$\frac{\Gamma_{q,k}(a+b)^{\mu}}{\Gamma_{q,k}(c+d)^{\lambda}} \leq \frac{\Gamma_{q,k}(a+bx)^{\mu}}{\Gamma_{q,k}(c+dx)^{\lambda}} \leq \frac{\Gamma_{q,k}(a)^{\mu}}{\Gamma_{q,k}(c)^{\lambda}} \tag{2.2}$$

*for $x \in [0,1]$, and*

$$\frac{\Gamma_{q,k}(a+bx)^{\mu}}{\Gamma_{q,k}(c+dx)^{\lambda}} < \frac{\Gamma_{q,k}(a+b)^{\mu}}{\Gamma_{q,k}(c+d)^{\lambda}} \tag{2.3}$$

*for $x \in (1,\infty)$.*

*Proof.* Let $u(x) = \ln G(x)$ . Then,

$$u(x) = \ln \frac{\Gamma_{q,k}(a+bx)^{\mu}}{\Gamma_{q,k}(c+dx)^{\lambda}} = \mu \ln \Gamma_{q,k}(a+bx) - \lambda \ln \Gamma_{q,k}(c+dx)$$

That implies,

$$\begin{aligned} u'(x) &= \mu b \frac{\Gamma'_{q,k}(a+bx)}{\Gamma_{q,k}(a+bx)} - \lambda d \frac{\Gamma'_{q,k} c+dx)}{\Gamma_{q,k}(c+dx)} \\ &= \mu b \psi_{q,k}(a+bx) - \lambda d \psi_{q,k}(c+dx) \leq 0. \quad \text{(i.e. by Lemma 2.3)} \end{aligned}$$

That implies $u$ is decreasing on $x \in [0, \infty)$. Consequently, $G$ is also decreasing on $x \in [0, \infty)$. Then for $x \in [0, 1]$ we have,

$$G(1) \leq G(x) \leq G(0)$$

yielding the result (2.2). Also, for $x \in (1, \infty)$, we have $G(x) < G(1)$ yielding the result (2.3). □

*Remark* 2.2. In Theorem 2.1, if $\mu b \geq \lambda d$ and either $\psi_{q,k}(a+bx) < 0$ or $\psi_{q,k}(c+dx) < 0$, then by Lemma 2.4, $G$ is decreasing and the double inequalities (2.2) and (2.3) still hold true.

**Theorem 2.2.** *Define a function $H$ for $x \geq 0$, $q \in (0, 1)$ and $k > 0$ by*

$$H(x) = \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)}$$

*where $\beta \geq k$, $\alpha_i > 0$, $n \in \mathbb{N}$. Then $H$ is a decreasing function of $x$ and the following inequalities are valid:*

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)} \leq \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)} \leq \frac{1}{\Gamma_{q,k}(\beta)} \tag{2.4}$$

*for $x \in [0, 1]$, and*

$$\frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)} < \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i)} \tag{2.5}$$

*for $x \in (1, \infty)$.*

*Proof.* Let $v(x) = \ln H(x)$. Then,

$$\begin{aligned} v(x) &= \ln \frac{\prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x)}{\Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x)} \\ &= \ln \prod_{i=1}^{n} \Gamma_{q,k}(k + \alpha_i x) - \ln \Gamma_{q,k}(\beta + \sum_{i=1}^{n} \alpha_i x) \end{aligned}$$

implying that,

$$\begin{aligned} v'(x) &= \sum_{i=1}^{n}\left(\alpha_i \frac{\Gamma'_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(k+\alpha_i x)}\right) - \left(\sum_{i=1}^{n}\alpha_i\right)\frac{\Gamma'_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x)}{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x)} \\ &= \sum_{i=1}^{n}(\alpha_i\psi_{q,k}(k+\alpha_i x)) - \left(\sum_{i=1}^{n}\alpha_i\right)\psi_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x) \\ &= \sum_{i=1}^{n}\alpha_i\left[\psi_{q,k}(k+\alpha_i x) - \psi_{q,k}(\beta+\sum_{j=1}^{n}\alpha_j x)\right] \leq 0. \quad \text{(by Lemma 2.5)} \end{aligned}$$

That implies $v$ and as a result $H$ are decreasing on $x \in [0,\infty)$. Then for $x \in [0,1]$ we obtain

$$H(1) \leq H(x) \leq H(0)$$

concluding the proof of (2.4). Also, for $x \in (1,\infty)$, we have $H(x) < H(1)$ establishing the result (2.5). □

By a similar method, it is easy to prove the following results.

**Theorem 2.3.** *Let $Q$ be defined for $x \geq 0$, $q \in (0,1)$ and $k > 0$ by*

$$Q(x) = \frac{\prod_{i=1}^{n}\Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x)}$$

*where $\beta \geq k$, $\alpha_i < 0$, $n \in \mathbb{N}$ such that $0 < k+\alpha_i x \leq \beta + \sum_{i=1}^{n}\alpha_i x$. Then $Q$ is increasing and the following inequalities are satisfied:*

$$\frac{1}{\Gamma_{q,k}(\beta)} \leq \frac{\prod_{i=1}^{n}\Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x)} \leq \frac{\prod_{i=1}^{n}\Gamma_{q,k}(k+\alpha_i)}{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i)} \tag{2.6}$$

*for $x \in [0,1]$, and*

$$\frac{\prod_{i=1}^{n}\Gamma_{q,k}(k+\alpha_i x)}{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i x)} > \frac{\prod_{i=1}^{n}\Gamma_{q,k}(k+\alpha_i)}{\Gamma_{q,k}(\beta+\sum_{i=1}^{n}\alpha_i)} \tag{2.7}$$

*for $x \in (1,\infty)$.*

## 3. Concluding Remarks

*Remark* 3.1. By setting $k = 1$ in Theorem 2.1, we obtain the $q$-extension of (1.6) as presented in [15].

*Remark* 3.2. By allowing $q \to 1$ in Theorem 2.1, we obtain the $k$-extension of (1.6) as presented in [11].

*Remark* 3.3. By allowing $q \to 1$ whiles $k \to 1$ in Theorem 2.1, we obtain (1.6).

*Remark* 3.4. By allowing $q \to 1$ in Theorem 2.2, we obtain the $k$-extension of (1.7) as presented in [11] and [17].

*Remark* 3.5. By setting $k = 1$ in Theorem 2.2, we obtain the $q$-extension of (1.7).

*Remark* 3.6. By allowing $q \to 1$ whiles $k \to 1$ in Theorem 2.2, we obtain (1.7).

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