Konuralp Journal of Mathematics
Volume 4 No. 2 pp. 116–123 (2016) ©KJM

THE ANALYSIS OF THE EFFECT OF THE NORMS IN THE STEP SIZE SELECTION FOR THE NUMERICAL INTEGRATION

GÜLNUR ÇELİK KIZILKAN, AHMET DUMAN, AND KEMAL AYDIN

Abstract. In scientific studies involving norm calculations, the choice of the norm affects the obtained results. We have aimed to examine the behavior of the step sizes using different norms and norm inequalities in step size strategy obtained in [1] for linear Cauchy problems.

1. Introduction

Selection of step size is an important concept for the convergence of the numerical solution to exact solution in numerical integration of differential equation systems. For the use constant step size, it must be investigated how should be selected the step size in the first step of numerical integration. Also, if the solution is changing slowly in some regions and it is changing rapidly in some another regions then it is not practical to use constant step size in numerical integration. So, we should use small step sizes in the region where the solution changes rapidly and we should choose larger step size in the region where the solution changes slowly. In literature, step size strategies have been given for the numerical integration. Consider the Cauchy problem

$$X' = F(t, X), X(t_0) = X_0$$

on the region $D = \{(t, X) : |t - t_0| \leq T, |x_j - x_{j0}| \leq b_j\}$, where $X(t) = (x_j(t))$, $X_0 = (x_{j0})$; $x_{j0} = x_j(t_0)$, $F(t, X) = (f_j)$; $f_j = f_j(t, x_1, x_2, ..., x_N)$, $F(t, X) \in C^1([t_0 - T, t_0 + T] \times R^N)$, $X(t)$, X_0 and $b = (b_j) \in R^N$. In [1, 2] a step size strategy for $F(t, X) = AX$ is proposed by

$$h_i \leq \frac{1}{\alpha \sqrt[4]{N^5}} \left(\frac{2\delta_L}{\beta_{i-1}}\right)^{\frac{1}{2}} \tag{1.1}$$

such that the local error $||LE_i|| \leq \delta_L$. Strategy given in (1.1) is the generalization of the strategy in [3, 4].

2010 *Mathematics Subject Classification.* 67F35, 67L05, 97N30.
Key words and phrases. Step size strategy, linear systems, norms.

The above-mentioned step size strategies are based on matrix and vector norms. As in all the scientific studies involving norm calculations, the choice of the norm affects the obtained results in step size strategies.

The aim of this paper to examine the behavior of the step sizes using different norms and norm inequalities in step size strategy obtained in [1] for linear Cauchy problems. In section 2, we have introduced the step size strategy based on error analysis for the linear systems (SSS). We have reminded commonly used vector and matrix norms. In section 3, we have investigated the effects of choice of the norms on step size strategy. Finally, we have analyzed the all strategies with numerical examples.

2. The step size strategy and norms

2.1. **The Step Size Strategy (SSS).** Let us consider the Cauchy problem

$$X' = AX, X(t_0) = X_0. \tag{2.1}$$

Following inequality is given

$$||LE_i|| \leq \frac{h_i^2}{2}||A||^2||Z(\tau_i)||, \tau_i \in [t_{i-1}, t_i) \tag{2.2}$$

for the local error of the Cauchy problem (2.1) in i-th step of the numerical integration. According to equation (2.2), the upper bound of local error for the system (2.1) is given by

$$||LE_i|| \leq (\frac{1}{2}\alpha^2\beta_{i-1})\sqrt{N^5}h_i^2, \tag{2.3}$$

where

$$||A|| \leq N \max_{i,j} |a_{ij}| = N\alpha,$$
$$||Z|| \leq \sqrt{N} \max_j |z_j| \leq \sqrt{N} \max_j(\sup_{\tau_i} |z_j(\tau_i)|) \leq \sqrt{N}\beta_{i-1}.$$

From the inequality(2.3)in the step$i$, the step size is calculated by

$$h_i \leq (\frac{1}{\alpha\sqrt[4]{N^5}})(\frac{2\delta_L}{\beta_{i-1}})^{\frac{1}{2}} \tag{2.4}$$

such that the local error $||LE_i|| \leq \delta_L$ where δ_L is the error level that is determined by user ([1, 2]).

While formulating the step sizes (2.4), a more practical way is obtained for calculations by using the upper bound (2.3) instead of the upper bound (2.2) of the local error. The effects of the calculation errors resulting from floating point arithmetic are reduced in doing so.

2.2. **Vector and Matrix Norms and Relations between Matrix Norms.** A norm is a real valued function that provides a measure of the size of vectors and matrices. For $X = (x_j) \in R^N$, some commonly used norms are given below. The l_2 norm (Euclidean norm) is defined by

$$||X||_2 = (\sum_{j=1}^N x_j^2)^{\frac{1}{2}}.$$

The l_1 norm (sum norm) is given as

$$||X||_1 = \sum_{j=1}^N |x_j|.$$

Another norm is formulated by

$$||X||_\infty = \max_j |x_j|,$$

which is called as l_∞ norm (maximum norm).

For $A = (a_{ij}) \in R^{M\times N}$, the most frequently used matrix norms are the l_1 (maximum column) norm

$$||A||_1 = \max_j \sum_{i=1}^{M} |a_j|,$$

the l_∞ (maximum row) norm

$$||A||_\infty = \max_i \sum_{j=1}^{N} |a_j|,$$

the l_2 (spectral) norm

$$||A||_2 = \sqrt{\lambda_{max}(A^T A)},$$

where $\lambda_{max}(A^T A)$ is the maximum eigenvalue of the matrix $A^T A$, Frobenius norm

$$||A||_F = (\sum_{i=1}^{M}\sum_{j=1}^{N} |a_{ij}|^2)^{\frac{1}{2}},$$

and the maximum norm

$$||A||_{\max} = \max_{i,j} |a_{ij}|.$$

We have used in our study the relations

$$||A||_2 \le ||A||_F,\ ||A||_F \le \sqrt{N}||A||_2,\ ||A||_2 \le N||A||_{\max},\ ||A||_2 \le \sqrt{||A||_1||A||_\infty}$$

which hold for all matrices $A = (a_{ij}) \in R^{N\times N}$. And we have also used the compatible norms in this study.

For all information about norms in this section, you can see for example [5, 6, 7, 8, 9, 10].

3. An Analysis on the Effect of the Norms in the Step Size Selection

3.1. **The Effect of Choice of Norm to Step Size Strategy.** The inequality (2.4) given in [1, 2] gives step sizes based on matrix and vector norms in the i-th step of numerical integration of the Cauchy problem (2.1) such that local error is smaller than δ_L error level. Different formulations are obtained for the step size according to the choice of the norms in the inequality (2.2). Changes that occur in step sizes may be significant. Now, let investigate the effect of the norms to step sizes. In calculations consider that

$$||Z(\tau_i)||_k \le sup_{\tau_i}||Z(\tau_i)||_k \le \beta_{k,i-1},\ k = 1, 2, \infty.$$

Strategy 1 (SSS1) The step sizes given by

$$h_i \le \frac{1}{||A||_2}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.1}$$

are obtained from the inequality (2.2) according to l_2 norm.

Strategy 2 (SSS2) The step sizes given by

$$h_i \le \frac{1}{||A||_1}(\frac{2\delta_L}{\beta_{1,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.2}$$

are obtained from the inequality (2.2) according to l_1 norm.

Strategy 3 (SSS3) The step sizes given by

$$h_i \le \frac{1}{||A||_\infty}(\frac{2\delta_L}{\beta_{\infty,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.3}$$

from the inequality (2.2) according to l_∞ norm.

Strategy 4 (SSS4) The step sizes given by

$$h_i \le \frac{1}{||A||_F}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.4}$$

Ex.	INPUT		Step number with SSSLS and SSS*k* (*k*=1,2,3,4,5,6,7)							
	A	*T*	SSS	SSS1	SSS2	SSS3	SSS4	SSS5	SSS6	SSS7
1	$\begin{pmatrix} 0.2 & 1 \\ 0 & -0.1 \end{pmatrix}$	10	168	72	81	84	72	103	142	93
2	$\begin{pmatrix} 50 & 1 \\ -0.01 & 0 \end{pmatrix}$	0.1	134	54	55	55	54	78	112	2986
3	$\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$	25	34	26	31	24	27	28	35	43

TABLE 1. Step number with the strategies in numerical integration

from the inequality (2.2) according to l_2 norm.

Strategy 5 (SSS5) By using inequality $||A||_2 \leq ||A||_F \leq \sqrt{N}||A||_2$, the step sizes are calculated by

$$h_i \leq \frac{1}{||A||_2}(\frac{2\delta_L}{N\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.5}$$

from the inequality (2.2) according to l_2 norm.

Strategy 6 (SSS6) By using inequality $||A||_2 \leq N||A||_{\max}$, the step sizes obtained by

$$h_i \leq \frac{1}{N||A||_{\max}}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.6}$$

from the inequality (2.2).

Strategy 7 (SSS7) The step sizes are given as follows

$$h_i \leq \frac{1}{||A||_1||A||_\infty}(\frac{2\delta_L}{\beta_{2,i-1}})^{\frac{1}{2}}, \tau_i \in [t_{i-1}, t_i) \tag{3.7}$$

by considering the inequality $||A||_2 \leq \sqrt{||A||_1||A||_\infty}$.

3.2. **Analysis of the Strategies with Numerical Examples.** Consider $X'(t) = AX(t)$, $X(t_0) = X_0$ on the region $D = \{(t, X) : |t - t_0| \leq T, |x_j - x_{j0}| \leq b_j\}$. Let $t_0 = 0$, $b_j = 5$, $x_{j0} = 1$ and $\delta_L = 10^{-1}$.

Following figures give us an idea about the step sizes obtained from strategies. The values and numbers of the step sizes depend on the choice of norm.

The main strategy SSS usually generates little step sizes which cause an expensive computation as shown in Figure 1 and Figure 3. However, no matter how the matrix, SSS provides ease of calculation for the step sizes. Because, calculation the parameters α and β_{i-1} of SSS in inequalities (2.4) is easier to obtain the parameters of the other strategies.

As we can see from Figure 1, Figure 2 and Figure 3, SSS1 gives the largest step sizes than other strategies. But, in this case local errors may occur very close to error level δ_L in calculations (see, Figure 4.(b)). The calculation errors may cause to be $||LE_i|| > \delta_L$ on some steps in numerical integration because of the effects of the floating-point arithmetic (Remark 3.1. in [2], and Remark 1. in [1]). If the situation that the occurred errors exceeds desired error level is not so important, then SSS1 is the most suitable strategy for the numerical integration. Because it always provides quite cheap computations.

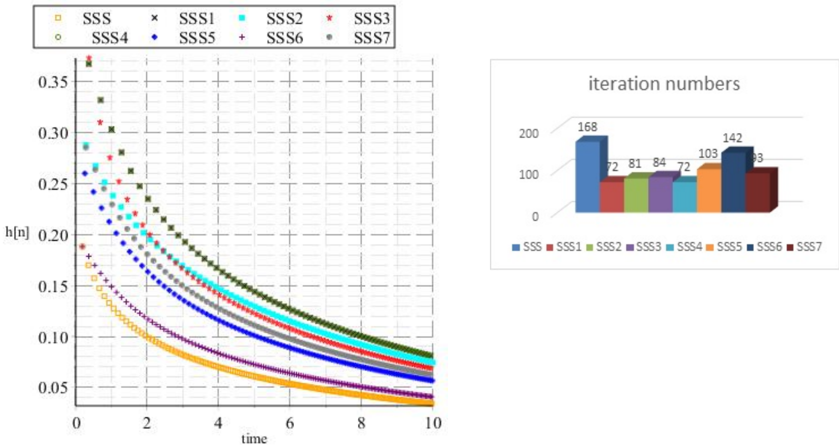


FIGURE 1. Step sizes and iteration numbers for Example 1.

For SSSk (k=2,3,4,5), almost similar results have been obtained as SSS1. So, we think that it will be enough to comment only SSS1.

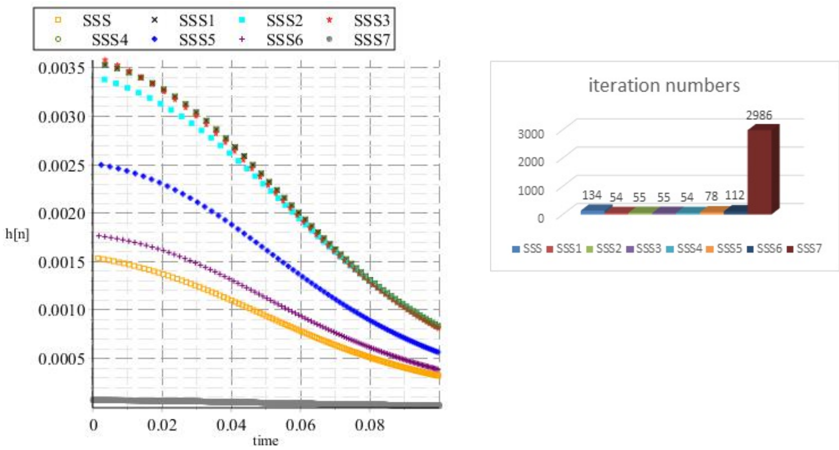


FIGURE 2. Step sizes and iteration numbers for Example 2.

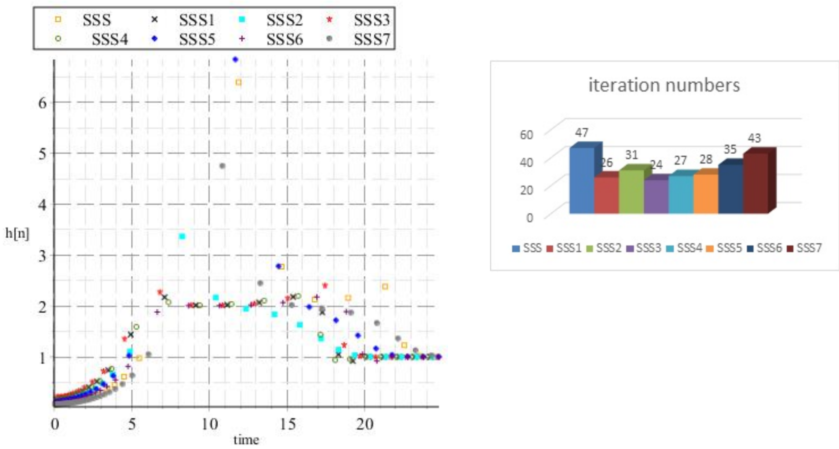


FIGURE 3. Step sizes and iteration numbers for Example 3.

SSS6 completes the calculation process a little less step when compared with SSS. The step sizes are partially calculated more easily with SSS6 than SSSk (k=1,2,3,4,5,7) because of the term $||A||_{max}$. But the calculation of the step sizes with SSS is easiest of among all the strategies.

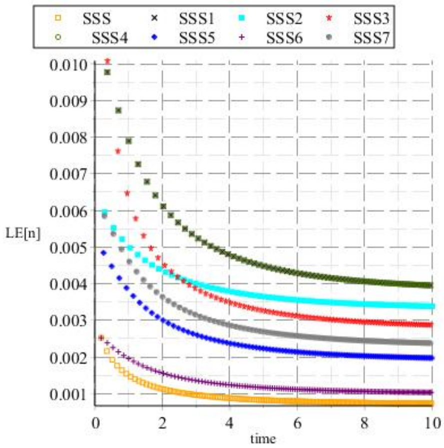
It is not practical to compare SSS7 directly with the other strategies regarding largeness of calculated step sizes and the number of iterations. For instance, iteration has taken 2986 steps in Example 2, but it has taken 43 steps in Example 3 as we can see in Figure 2 and Figure 3. That is, it may calculate the largest or the smallest step sizes according to given coefficient matrix. Even one of the elements of the coefficient matrix is large, the number of iterations increases in the calculation. The term $||A||_1||A||_\infty$ in SSS7 causes the becoming smaller of the step sizes. So, if the elements of the matrix is not very large, SSS7 should be used.

Figure 4 shows the local errors calculated by the strategies for Example 1, Example 2 and Example 3.

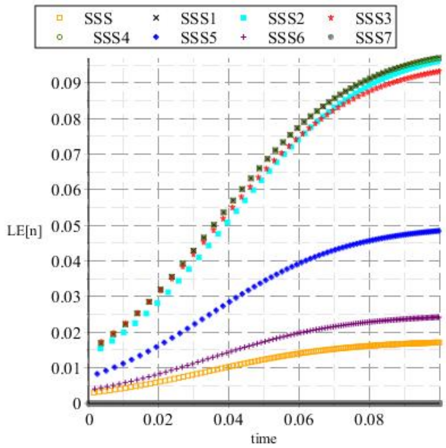
4. Conclusion

In this paper, the effects of choice of the norms have been examined in the calculation of the step sizes. It has been seen that some norms and norm inequalities provide ease of calculation for step size.

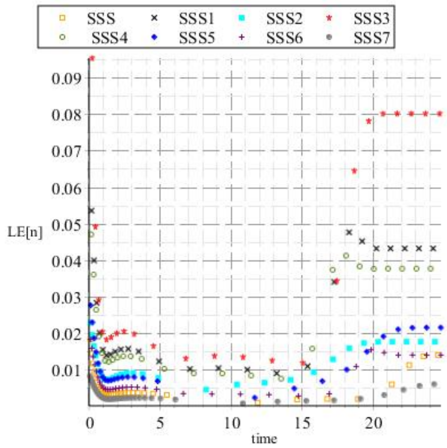
SSS1 gives the larger step sizes than other strategies. So, SSS1 completes the numerical integration in less time and fewer steps. It provides quite cheap computations. Although it is advantageous with this aspect, local errors may occur very close to error level δ_L in calculations. As all computations are done with floating-point arithmetic on computer, the calculation errors may cause to be $||LE_i|| > \delta_L$ on some steps in numerical integration. If this situation is unimportant, users



(a)



(b)



(c)

FIGURE 4. Local errors for Example 1, Example 2 and Example 3.

should prefer SSS1 (or SSSk (k=2,3,4,5) that have similar properties) for cheap computations.

However the effects of floating point arithmetic does not considered in this study, it has emphasized that SSS is given to reduce these effects. SSS usually generates little step sizes which cause an expensive computation, but even so, it allows us to easier calculations for the step sizes. SSS should be used for ease of calculations.

SSS4 may be suggested if the elements of the matrix is not very large. If the coefficient matrix has at least one large element, it may calculate too small step sizes according to coefficient matrix.

Consequently, the choice of the norm should be considered as an important part of the step size strategy.

References

[1] Çelik Kızılkan, G. and Aydın, K., Step Size Strategies Based on Error Analysis for The Linear Systems, SDU Journal of Science (e-journal), Vol:6, No.2 (2011), 149-159.

[2] Çelik Kızılkan, G.,*Step size strategies on the numerical integration of the systems of differential equations*,Ph.D. Thesis, Graduate Natural and Applied Sciences, Seluk University, Konya, 2009 (in Turkish).

[3] Çelik Kızılkan, G.,*On the finding of step size in the numerical integration of initial value problem*,Master Thesis, Graduate Natural and Applied Sciences, Seluk University, Konya, 2004 (in Turkish).

[4] Çelik Kızılkan, G. and Aydın, K.,Step Size Strategy Based on Error Analysis, SUFEFD, Vol: 25 (2005), 79-86 (in Turkish).

[5] Golub, G. H. and Van Loan, C.F., *Matrix Computations*, third edition, The Johns Hopkins University Press, London, 1996.

[6] Quarteroni, A., Sacco, R. and Saleri, F.,*Numerical Mathematics*, Springer Verlag, New York, 2000.

[7] Bulgak, A. and Bulgak, H.,*Linear Algebra*, Seluk University Applied Mathematics Research Center Publications, Konya, 2001.

[8] Taşcı, D.,*Linear Algebra*, Gazi Bookstore, Ankara, 2005.

[9] Chapra, S.C. and Canale, R.P., *Numerical methods for engineers*, sixth edition, McGraw Hill, New York, 2010.

[10] Acton, A.C., A brief overview of the condition number (25 October 2008). http://ansys.net/ansys/tips/acton20080825-condition_number.pdf (Access date: 17.04.2016)

Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Science, Konya, Turkey
E-mail address: `gckizilkan@konya.edu.tr`

Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Science, Konya, Turkey
E-mail address: `aduman@konya.edu.tr`

Selçuk University, Faculty of Science, Department of Mathematics, Konya, Turkey
E-mail address: `kaydin@selcuk.edu.tr`