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COMMON FIXED POINTS IN METRICALLY CONVEX PARTIAL METRIC SPACES

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Abstract. In this study, we extend some common fixed points theorems for mappings in metrically convex metric spaces into partial metric spaces. We generalize earlier results by Imdad *et al.* We also provide an illustrative example.

1. Introduction and Preliminaries

The existing literature contains a number of fixed point theorems for self mappings in partial metric spaces. These include theorems by Mathews [7], Oltra and Valero [8] and Karapinar *et al.* [6]. However fixed point theorems for non-self mappings in partial metric spaces are not often discussed.

The study of fixed point for non-self mappings in metrically convex metric spaces was introduced by Assad and Kirk [1]. The concept of metrical convexity has been used by several authors including Gajić and Rakočević [2], Imdad and Kumar [5] and Hadzić [3] to develop fixed point theorems for non-self mappings.

In the proofs of our main results, we require the following definitions and lemmas.

The partial metric space is a generalization of the metric space introduced by Mathews [7] in 1994 as a part of a study of denotational semantics of dataflow networks.. It is defined as follows:

Definition 1.1. [7] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$:
(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(P2) $p(x, x) \leq p(x, y)$,
(P3) $p(x, y) = p(y, x)$,
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

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From (P1) and (P2) we have

$$p(x,y)=0 \Rightarrow p(x,y)=p(x,x)=p(y,y) \Rightarrow x=y. \tag{1.1}$$

From (P4) we have

$$p(x,y) \le p(x,z)+p(z,y). \tag{1.2}$$

An example of a partial metric space (X,p) is when $X=\mathbb{R}_+$, the set of all non-negative real numbers, and $p(x,y)=\max\{x,y\}$ for all $x,y\in X$.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x,\varepsilon): x\in X, \varepsilon>0\}$ where
$B_p(x,\varepsilon)=\{y\in X: p(x,y)<p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

Definition 1.2. [7]. Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then

(i) $\{x_n\}$ converges to a point $x\in X$ if and only if $p(x,x)=\lim_{n\to+\infty} p(x,x_n)$.

(ii) $\{x_n\}$ is called a Cauchy sequence if only if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$.

(iii) A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to+\infty} p(x_n,x_m)$.

The metric derived from a partial metric is defined as follows:.

Lemma 1.1. [7] *If p is a partial metric on X, then the function $p^s: X\times X\to\mathbb{R}_+$ given by*

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y) \tag{1.3}$$

defines a metric on X.

We take note of the following lemma.

Lemma 1.2. [7]. *Let (X,p) be a partial metric space.*

(i) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(ii) (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore $\lim_{n\to+\infty} p(x_n,x)=0$ if and only if $p(x,x)=\lim_{n\to+\infty} p(x_n,x)=\lim_{n,m\to+\infty} p(x_n,x_m)=0$.

A metrically convex metric space is defined as follows:

Definition 1.3. [1].
A complete metric space (X,d) is said to be (metrically) convex if X has the property that for each $x,y\in X$ with $x\neq y$ there exists $z\in X, x\neq z\neq y$, such that $d(x,z)+d(z,y)=d(x,y)$.

If (X,d) is a metrically convex metric space, and $x,y\in X$, we term

$$seg[x,y]:=\{z\in X: d(x,y)=d(x,z)+d(z,y)\}.$$

The following lemma is obtained from Assad and Kirk [1].

Lemma 1.3. *Let K be a closed subset of the complete and convex metric space X. If $x\in K$ and $y\notin K$, then there exists a point $z\in\partial K$ (the boundary of K) such that*

$$d(x,z)+d(z,y)=d(x,y).$$

We introduce the metrically convex partial metric space.

Definition 1.4. A partial metric space (X, p) is said to be metrically convex if the corresponding metric space (X, p^s) is metrically convex in the sense of Lemma 1.1, where $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$.

As an example, the partial metric space $(\mathbb{R}_+, p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$ is metrically convex because (X, p^s) where $p^s(x, y) = |x - y|$ is the metric derived from the partial metric p, is metrically convex.

Lemma 1.4. *Let (X, p) be a metrically convex partial metric space. Let $x, y \in X$. If $z \in \operatorname{seg}[x, y]$ then:*
(i) $p(x, y) + p(z, z) = p(x, z) + p(z, y)$,
(ii) $p(x, y) \geq p(x, z)$.

Proof. Applying (1.3) to Definition 1.3, if $z \in \operatorname{seg}[x, y]$, then we have:

$$\begin{aligned} & p^s(x, y) = p^s(x, z) + p^s(z, y) \\ & \Rightarrow 2p(x, y) - p(x, x) - p(y, y) = 2p(x, z) - p(x, x) - p(z, z) \\ & \qquad\qquad + 2p(z, y) - p(z, z) - p(y, y) \\ & \Rightarrow p(x, y) = p(x, z) + p(z, y) - p(z, z) \\ & \Rightarrow p(x, y) + p(z, z) = p(x, z) + p(z, y). \end{aligned}$$

As $p(z, y) - p(z, z) \geq 0$, from (P2) of Definition 1.1 we have

$$p(x, y) \geq p(x, z).$$

□

Lemma 1.5. *Let K be a non-empty subset of a metrically convex partial metric space (X, p) which is closed in (X, p^s). If $x \in K$ and $y \in X \backslash K$, then there exists a point $z \in \partial K$ (the boundary of K with respect to (X, p^s)) such that*

$$p(x, y) + p(z, z) = p(x, z) + p(z, y).$$

Proof. From Definition 1.4, if the partial metric space (X, p) is metrically convex, then (X, p^s) is metrically convex. From Lemma 1.3 (ii), this means that if $x \in K$ and $y \in X \backslash K$ then there exists z in ∂K, (the boundary of K), such that $p^s(x, y) = p^s(x, z) + p^s(z, y)$. Using (1.3), this means

$$\begin{aligned} p^s(x, y) &= p^s(x, z) + p^s(z, y) \\ \Rightarrow 2p(x, y) - p(x, x) - p(y, y) &= 2p(x, z) - p(x, x) - p(z, z) \\ &\quad + 2p(z, y) - p(z, z) - p(y, y) \\ \Rightarrow 2p(x, y) &= 2p(x, z) + 2p(z, y) - 2p(z, z) \\ \Rightarrow p(x, y) + p(z, z) &= p(x, z) + p(z, y) \\ \Rightarrow p(x, z) + p(z, y) &= p(x, y) + p(z, z). \end{aligned}$$

□

We now prove the following lemma, which is modified from Theorem 1 of Assad and Kirk [1], and is necessary for our work.

Lemma 1.6. *Consider a sequence* $\{w_n\}_{n\in\mathbb{N}} \in \mathbb{R}_+$ *such that, for all* $n \geq 2$ *we have*

$$w_n \leq k \max\{w_{n-2}, w_{n-1}\}, k \in (0,1), \tag{1.4}$$

then

$$w_n \leq k^{n/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.5}$$

Proof. We prove the lemma by the induction. First we show that Lemma 1.6 holds for $n = 2$.

We note that $k \in (0,1)$ implies $k < k^{1/2}$. Hence if $n = 2$, then (1.4) leads to

$$w_2 \leq k \max\{w_0, w_1\} \leq k^{1/2} \max\{w_0, w_1\} = k^{2/2} k^{-1/2} \max\{w_0, w_1\}. \tag{1.6}$$

We then show that the lemma holds for $n = 3$. If $n = 3$, then (1.4) leads to $w_3 \leq k \max\{w_1, w_2\}$. If $w_1 \geq w_2$, then we get

$$\begin{aligned} w_3 &\leq k \max\{w_1, w_2\} \\ \Rightarrow w_3 &\leq k w_1 \\ &\leq k \max\{w_0, w_1\} \\ &= k^{3/2} \cdot k^{-1/2} \max\{w_0, w_1\}. \end{aligned}$$

If however $w_1 < w_2$, we get

$$\begin{aligned} w_3 &\leq k \max\{w_1, w_2\} \\ \Rightarrow w_3 &\leq k w_2 \\ \Rightarrow w_3 &\leq k \times k^{2/2} k^{-1/2} \max\{w_0, w_1\}, \text{ from (1.6)} \\ &\leq k^{3/2} \max\{w_0, w_1\} \\ &\leq k^{3/2} \cdot k^{-1/2} \max\{w_0, w_1\}, \text{ because } k^{-1/2} \geq 1. \end{aligned}$$

We now show that, if Lemma 1.6 holds for $1 \leq n \leq j$ where $j \geq 2$, then it must be hold for $j + 1$. Hence we have from (1.4)

$$w_{j+1} \leq k \max\{w_{j-1}, w_j\}. \tag{1.7}$$

We consider two cases.

Case (i): Suppose $w_{j-1} \leq w_j$. Then (1.7) becomes

$$\begin{aligned} w_{j+1} &\leq k w_j \\ &\leq k \cdot k^{j/2} k^{-1/2} \max\{w_0, w_1\} \text{ from (1.5)} \\ &= k^{(j+2)/2} k^{-1/2} \max\{w_0, w_1\}. \end{aligned} \tag{1.8}$$

Case (ii): Suppose $w_{j-1} > w_j$. Then (1.7) becomes

$$\begin{aligned} w_{j+2} &\leq k w_{j-1} \\ &\leq k \cdot k^{(j-1)/2} k^{-1/2} \max\{w_0, w_1\} \text{ from (1.5)} \\ &= k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}. \end{aligned} \tag{1.9}$$

We note that for $j \geq 2$ and $k \in (0,1)$ we have $k^{(j+1)/2} > k^{(j+2)/2}$. Hence (1.8) and (1.9) imply that

$$w_{j+1} \leq k^{(j+1)/2} k^{-1/2} \max\{w_0, w_1\}.$$

□

Here we introduce a relation that will be used in our proof.

Remark 1.1. Let $s, t, \alpha \geq 0$ with $s < t$. Then

$$\frac{s}{t} \leq \frac{s+\alpha}{t+\alpha}. \tag{1.10}$$

We prove Remark 1.1 here:

$$\frac{s}{t} - \frac{s+\alpha}{t+\alpha} = \frac{\alpha(s-t)}{t(t+\alpha)} \leq 0 \text{ because } s < t$$
$$\Rightarrow \frac{s}{t} \leq \frac{s+\alpha}{t+\alpha}.$$

We also note that if $T : K \to X$ is a continuous mapping and $\{x_n\}$ is a convergent sequence with all $x_n \in K$, where K is closed, then we have

$$T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n.$$

The following definition is modified from Hadzić [3].

Definition 1.5. Let K be a non-empty subset of a partial metric space (X, p) and $F, T : K \to X$. The pair (F, T) is said to be weakly p-commuting if for every $x \in K$ with $Fx, Tx \in K$ we have

$$p(TFx, FTx) \leq p(Fx, Tx).$$

Let K be a non-empty subset of a partial metric space (X, p) and $F, T : K \to X$. The point $v \in K$ is called a *coincidence point* of F and T if $Fv = Tv = w$. If also $v = w$, then v is called a *common fixed point* of F and T. We say also that F and T are *coincidentally commuting* if at every coincidence point $v \in K$, we have $FTv = TFv$.

The following definition is found in Imdad *et al.* [4].

Definition 1.6. [4] Let K be a non-empty subset of a metric space (X, d) and let $F, G, S, T : K \to X$ satisfy the condition

$$\begin{aligned} d(Fx, Gy) \leq a \max\big\{d(Tx, Sy)/2, d(Tx, Fx), d(Sy, Gy)\big\} \\ + b\big[d(Tx, Gy) + d(Fx, Sy)\big]. \end{aligned}$$

for all $x, y \in K$ with $x \neq y, a, b \geq 0$ and $a + 2b < 1$. Then (F, G) is called a generalized (S, T) contraction on K.

The following result is established in Imdad *et al.* [4].

Theorem 1.1. [4] *Let (X, d) be a complete metrically convex metric space and K a non-empty closed subset of X and let $F, G, S, T : K \to X$. If (F, G) is a generalized (S, T) contraction of K satisfying*
(i) $\partial K \subseteq SK \cap TK$, $FK \cap K \subseteq SK$, $GK \cap K \subseteq TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$, $Sx \in \partial K \Rightarrow Gx \in K$,
(iii) (F, T) and (G, S) are weakly commuting pairs and
(iv) one of F, G, S and T is continuous in K.
Then there exists a unique point $z \in K$ such that $Fz = Gz = Sz = Tz = z$. Furthermore z remains a unique common fixed point of both pairs separately.

In this study, we seek to extend Theorem 1.1 from metric spaces into partial metric spaces.

2. Main Results

We begin with the following definition which is an extension of Definition 1.6 into partial metric spaces.

Definition 2.1. Let K be a non-empty subset of a partial metric space (X, p) and let $F, G, S, T : K \to X$ satisfying the condition

$$p(Fx, Gy) \leq a \max\{p(Tx, Sy)/2, p(Tx, Fx), p(Sy, Gy)\} + b[p(Tx, Gy) + p(Fx, Sy)].$$

for all $x, y \in K$ with $x \neq y, a, b \geq 0$ and $a + 2b < \frac{1}{2}$. Then (F, G) is called a generalized (S, T) p-contraction on K.

The following relations for $a, b \geq 0$ and $a + 2b \leq \frac{1}{2}$ will assist us in developing our result.

Let

$$\frac{a+b}{1-2b} = h < \frac{1}{2}. \tag{2.1}$$

From (2.1), we have

$$\frac{a+b}{1-b} < \frac{a+b}{1-2b} = h. \tag{2.2}$$

We now make use of (1.10) on (2.2), and get

$$\frac{b}{1-a-b} \leq \frac{a+b}{1-b} < h. \tag{2.3}$$

We intend to prove the following theorem.

Theorem 2.1. *Let (X, p) be a complete metrically convex partial metric space and K a non empty closed subset of X, the closure being with respect to (X, p^s). Let ∂K, the boundary of K with respect to (X, p^s), be non-empty. Also let $F, G, S, T : K \to X$. If (F, G) is a generalized (S, T) p-contraction of K satisfying*
(i) $\partial K \subseteq SK \cap TK$, $FK \cap K \subseteq SK$, $GK \cap K \subseteq TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$, $Sx \in \partial K \Rightarrow Gx \in K$,
(iii) (F, T) and (G, S) are weakly p-commuting pairs and
(iv) one of F, G, S and T is continuous in K.
Then there exists a unique point $z \in K$ such that $Fz = Gz = Sz = Tz = z$ and $p(z, z) = 0$.

Proof. We form two sequences $\{x_n\}$ and $\{w_n\}$ in the following way.

We commence with an arbitrary point $w_0 \in \partial K$. From assumption (i) there is a point $x_0 \in K$ such that $w_0 = Tx_0$. From (ii), $Fx_0 \in K$. According to (i), we can choose $x_1 \in K$ such that $w_1 = Sx_1 = Fx_0$. We locate Gx_1. We consider the following scenarios.

If $Gx_1 \in K$, then, using (i), we can choose $x_2 \in K$ such that $Tx_2 = Gx_1 = w_2$.

If however $Gx_1 \notin K$, by Lemma 1.5, we can choose $w_2 \in \partial K$ such that $w_2 \in \text{seg}[w_1, Gx_1]$. As $w_2 \in \partial K$, from (ii), we can find $x_2 \in K$ such that $Tx_2 = w_2$. We then find Fx_2.

We proceed inductively as follows:

We set $w_{2n} = Tx_{2n}$. If $Fx_{2n} \in K$, then by (i), we can choose $x_{2n+1} \in K$ such that $Fx_{2n} = Sx_{2n+1} = w_{2n+1}$.

If however $Fx_{2n} \notin K$, by Lemma 1.5, we can choose $w_{2n+1} \in \partial K$ such that $w_{2n+1} \in \text{seg}[w_{2n}, Fx_{2n}]$. As $w_{2n+1} \in \partial K$, then by (ii), we can find $x_{2n+1} \in K$ such that $Sx_{2n+1} = w_{2n+1}$.

We then find Gx_{2n+1}.

In a similar vein, we set $w_{2n+1} = Gx_{2n+1}$. If $Gx_{2n+1} \in K$, then by (i), we can choose $x_{2n+2} \in K$ such that $Gx_{2n+1} = Tx_{2n+2} = w_{2n+2}$. On the other hand, if $Gx_{2n+1} \notin K$, by Lemma 1.5, we can choose $w_{2n+2} \in \partial K$ such that $w_{2n+2} \in \text{seg}[w_{2n+1}, Gx_{2n+1}]$. As $w_{2n+2} \in \partial K$, then by (ii), we can find $x_{2n+2} \in K$ such that $Tx_{2n+2} = w_{2n+2}$.

We then find Fx_{n+2}.

We define two sets, P and Q, forming a partition of the set $\{w_n\}$.

Let $P = \big\{w_{2n} \in \{w_n\} : w_{2n} = Gx_{2n-1}\big\} \cup \big\{w_{2n+1} \in \{w_n\} : w_{2n+1} = Fx_{2n}\big\}$ and $Q = \big\{w_{2n} \in \{w_n\} : w_{2n} \neq Gx_{2n-1}\big\} \cup \big\{w_{2n+1} \in \{w_n\} : w_{2n+1} \neq Fx_{2n}\big\}$. Note that
$w_{2n} \in P \Rightarrow w_{2n} = Tx_{2n} = Gx_{2n-1}$ for $n \geq 1$ and
$w_{2n+1} \in P \Rightarrow w_{2n+1} = Sx_{2n+1} = Fx_{2n}$. Note also that
$w_{2n} \in Q \Rightarrow w_{2n} = Tx_{2n} \in \partial K$ and $w_{2n} \in \text{seg}[w_{2n-1}, Gx_{2n-1}]$. Similarly, $w_{2n+1} \in Q \Rightarrow w_{2n+1} = Sx_{2n+1} \in \partial K$ and $w_{2n+1} \in \text{seg}[w_{2n}, Fx_{2n}]$.

We investigate three cases.

Case 1: Let $(w_m, w_{m+1}) \in P \times P$. Let us assume m is odd, meaning $m = 2n+1$ for some $n \in \mathbb{N}$. Then

$$
\begin{aligned}
&p(w_m, w_{m+1}) = p(w_{2n+1}, w_{2n+2}) = p(Sx_{2n+1}, Tx_{2n+2}) = p(Fx_{2n}, Gx_{2n+1})\\
&\leq a \max\left\{\frac{1}{2}p(Tx_{2n}, Sx_{2n+1}), p(Tx_{2n}, Fx_{2n}), p(Sx_{2n+1}, Gx_{2n+1})\right\}\\
&\qquad + b\left[p(Tx_{2n}, Gx_{2n+1}) + p(Fx_{2n}, Sx_{2n+1})\right]\\
&= a \max\left\{\frac{1}{2}p(Tx_{2n}, Sx_{2n+1}), p(Tx_{2n}, Sx_{2n+1}), p(Sx_{2n+1}, Tx_{2n+2})\right\}\\
&\qquad + b\left[p(Tx_{2n}, Tx_{2n+2}) + p(Sx_{2n+1}, Sx_{2n+1})\right]\\
&= a \max\left\{\frac{1}{2}p(w_{2n}, w_{2n+1}), p(w_{2n}, w_{2n+1}), p(w_{2n+1}, w_{2n+2})\right\}\\
&\qquad + b[p(w_{2n}, w_{2n+2}) + p(w_{2n+1}, w_{2n+1})].
\end{aligned}
$$

We note that from (P4) of Definition 1.1, we have

$$
p(w_{2n}, w_{2n+2}) + p(w_{2n+1}, w_{2n+1}) \leq p(w_{2n}, w_{2n+1}) + p(w_{2n+1}, w_{2n+2}).
$$

Hence we have

$$
\begin{aligned}
p(w_m, w_{m+1}) &= p(w_{2n+1}, w_{2n+2})\\
&\leq a \max\left\{\frac{1}{2}p(w_{2n}, w_{2n+1}), p(w_{2n}, w_{2n+1}), p(w_{2n+1}, w_{2n+2})\right\}\\
&\qquad + b[p(w_{2n}, w_{2n+1}) + p(w_{2n+1}, w_{2n+2})]\\
&= a \max\left\{p(w_{2n}, w_{2n+1}), p(w_{2n+1}, w_{2n+2})\right\}\\
&\qquad + b[p(w_{2n}, w_{2n+1}) + p(w_{2n+1}, w_{2n+2})].
\end{aligned} \tag{2.4}
$$

If $p(w_{2n}, w_{2n+1}) \le p(w_{2n+1}, w_{2n+2})$, then (2.4) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &= p(w_{2n+1}, w_{2n+2}) \\ &\le ap(w_{2n+1}, w_{2n+2}) + b[p(w_{2n}, w_{2n+1}) + p(w_{2n+1}, w_{2n+2})] \\ &\le \frac{b}{1-a-b}\, p(w_{2n}, w_{2n+1}) \\ &\le h\, p(w_{2n}, w_{2n+1}), \text{ (in view of (2.3)).} \end{aligned}$$

This is a contradiction because $h < \frac{1}{2}$. Hence
$p(w_{2n}, w_{2n+1}) > p(w_{2n+1}, w_{2n+2})$. Thus (2.4) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &= p(w_{2n+1}, w_{2n+2}) \\ &\le ap(w_{2n}, w_{2n+1}) + b[p(w_{2n}, w_{2n+1}) + p(w_{2n+1}, w_{2n+2})] \\ &\le \frac{a+b}{1-b} p(w_{2n}, w_{2n+1}) \\ &= \frac{a+b}{1-b} p(w_{m-1}, w_m) \\ &\le h\, p(w_{m-1}, w_m), \text{ (in view of (2.2)).} \end{aligned}$$

A similar argument shows that when $m = 2n$, we have

$$p(w_{2n}, w_{2n+1}) \le hp(w_{2n-1}, w_{2n}).$$

Thus, in general, if $(w_m, w_{m+1}) \in P \times P$, then

$$p(w_m, w_{m+1}) \le hp(w_{m-1}, w_m),\ h < \frac{1}{2}. \tag{2.5}$$

Case 2: Let $(w_m, w_{m+1}) \in P \times Q$. If m is odd, then $m = 2n+1$ for some $n \in \mathbb{N}$. As $w_{m+1} = w_{2n+2} \in Q$, it means $w_{2n+2} \in \text{seg}[w_{2n+1}, Gx_{2n+1}]$. From Lemma 1.4 (ii), this means $p(w_{2n+1}, w_{2n+2}) \le p(w_{2n+1}, Gx_{2n+1})$. We note that, in this case, $w_{2n+1} = Sx_{2n+1} = Fx_{2n}$. Hence

$$\begin{aligned} p(w_m, w_{m+1}) &\le p(w_{2n+1}, Gx_{2n+1}) \\ &= p(Fx_{2n}, Gx_{2n+1}) \\ &\le hp(w_{2n}, w_{2n+1}) \\ &= hp(w_{m-1}, w_m), \end{aligned}$$

using an argument similar to that of Case 1.

We obtain a similar result when m is even. Hence, when $(w_m, w_{m+1}) \in P \times Q$, we have

$$p(w_m, w_{m+1}) \le hp(w_{m-1}, w_m),\ h < \frac{1}{2}. \tag{2.6}$$

Case 3: Let $(w_m, w_{m+1}) \in Q \times P$.

We show that $w_m \in Q, m \ge 1$ implies that $w_{m-1} \in P$.

Now suppose $w_{m-1} \in Q$. This means $w_{m-1} \in \partial K$. By (ii), this implies that $w_m \in P$ which is a contradiction. Hence $w_{m-1} \in P$.

Suppose m is odd, implying $m = 2n + 1$ for some $n \in \mathbb{N}$.

As $w_{2n+1} \in \text{seg}[w_{2n}, Fx_{2n}]$, we have

$$\begin{aligned} p(w_m, w_{m+1}) &= p(w_{2n+1}, w_{2n+2}) \\ &\leq p(w_{2n+1}, Fx_{2n}) + p(Fx_{2n}, w_{2n+2}), \text{ from (1.2)} \\ &\leq p(w_{2n}, Fx_{2n}) + p(Fx_{2n}, w_{2n+2}), \text{ from Lemma 1.4.} \end{aligned} \tag{2.7}$$

We note that, from the construction of proof, $w_{2n} = Gx_{2n-1}$ and $w_{2n+2} = Gx_{2n+1}$. Thus (2.7) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &\leq p(Gx_{2n-1}, Fx_{2n}) + p(Fx_{2n}, Gx_{2n+1}) \\ &= p(Fx_{2n}, Gx_{2n-1}) + p(Fx_{2n}, Gx_{2n+1}) \\ \Rightarrow p(w_m, w_{m+1}) &\leq 2\max\{p(Fx_{2n}, Gx_{2n-1}), p(Fx_{2n}, Gx_{2n+1})\} \end{aligned} \tag{2.8}$$

We consider two subcases.

Subcase 3.1 Suppose $p(Fx_{2n}, Gx_{2n-1}) \leq p(Fx_{2n}, Gx_{2n+1})$. Then (2.8) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &\leq 2p(Fx_{2n}, Gx_{2n+1}) \\ &\leq 2a\max\left\{\frac{1}{2}p(Tx_{2n}, Sx_{2n+1}), p(Tx_{2n}, Fx_{2n}), p(Sx_{2n+1}, Gx_{2n+1})\right\} \\ &\quad + 2b\left[p(Tx_{2n}, Gx_{2n+1}) + p(Fx_{2n}, Sx_{2n+1})\right] \\ &\leq 2a\max\left\{\frac{1}{2}p(Tx_{2n}, Sx_{2n+1}), p(Tx_{2n}, Fx_{2n}), p(Sx_{2n+1}, Tx_{2n+2})\right\} \\ &\quad + 2b\left[p(Tx_{2n}, Tx_{2n+2}) + p(Fx_{2n}, Sx_{2n+1})\right]. \end{aligned} \tag{2.9}$$

We note that as $Sx_{2n+1} \in \text{seg}\{Tx_{2n}, Fx_{2n}\}$, we have from Lemma 1.4 (ii)

$$p(Tx_{2n}, Fx_{2n}) \geq p(Tx_{2n}, Sx_{2n+1}) \geq \frac{1}{2}p(Tx_{2n}, Sx_{2n+1}).$$

We also have from (P4) of Definition 1.1

$$\begin{aligned} &p(Tx_{2n}, Tx_{2n+2}) + p(Fx_{2n}, Sx_{2n+1}) \\ &\quad \leq p(Tx_{2n}, Sx_{2n+1}) + p(Sx_{2n+1}, Tx_{2n+2}) - p(Sx_{2n+1}, Sx_{2n+1}) \\ &\qquad + p(Fx_{2n}, Sx_{2n+1}) \\ &\quad = p(Sx_{2n+1}, Tx_{2n+2}) + p(Tx_{2n}, Fx_{2n}), \text{by Lemma 1.4 (i).} \end{aligned}$$

From the construction of sequence, we have $(w_m, w_{m+1}) = (Sx_{2n+1}, Tx_{2n+2})$. Hence (2.9) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &= p(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq 2a\max\{p(Tx_{2n}, Fx_{2n}), p(Sx_{2n+1}, Tx_{2n+2})\} \\ &\quad + 2b\left[p(Sx_{2n+1}, Tx_{2n+2}) + p(Tx_{2n}, Fx_{2n})\right]. \end{aligned} \tag{2.10}$$

If $p(Tx_{2n}, Fx_{2n}) \leq p(Sx_{2n+1}, Tx_{2n+2})$ then (2.10) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &= p(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq 2ap(Sx_{2n+1}, Tx_{2n+2}) \\ &\quad + 2b\left[p(Sx_{2n+1}, Tx_{2n+2}) + p(Tx_{2n}, Fx_{2n})\right] \\ &\leq 2ap(Sx_{2n+1}, Tx_{2n+2}) \\ &\quad + 2b\left[p(Sx_{2n+1}, Tx_{2n+2}) + p(Sx_{2n+1}, Tx_{2n+2})\right], \\ &\qquad \text{because } p(Tx_{2n}, Fx_{2n}) \leq p(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq 2(a+2b)p(Sx_{2n+1}, Tx_{2n+2}) \\ &< p(Sx_{2n+1}, Tx_{2n+2}) \text{ because } 2(a+2b) < 1. \end{aligned}$$

This is a contradiction. Hence $p(Tx_{2n}, Fx_{2n}) > p(Sx_{2n+1}, Tx_{2n+2})$. Thus (2.10) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &= p(Sx_{2n+1}, Tx_{2n+2}) \\ &\leq 2ap(Tx_{2n}, Fx_{2n}) \\ &\quad + 2b\left[p(Sx_{2n+1}, Tx_{2n+2}) + p(Tx_{2n}, Fx_{2n})\right] \\ &\leq 2\frac{a+b}{1-2b}p(Tx_{2n}, Fx_{2n}) \\ &\leq 2hp(Tx_{2n}, Fx_{2n}) \\ &\leq 2hp(Gx_{2n-1}, Fx_{2n}) \text{ because } Gx_{2n-1} = Tx_{2n} \\ &\leq 2h \times hp(w_{m-1}, w_m), \text{as per Case 2} \end{aligned}$$

$$\Rightarrow p(w_m, w_{m+1}) \leq hp(w_{m-1}, w_m) \text{ as } 2h < 1. \tag{2.11}$$

Subcase 3.2 Now let us consider when $p(Fx_{2n}, Gx_{2n-1}) > p(Fx_{2n}, Gx_{2n+1})$. Then (2.8) becomes

$$\begin{aligned} p(w_m, w_{m+1}) &\leq 2p(Fx_{2n}, Gx_{2n-1}) \\ &\leq 2hp(w_{2n}, w_{2n-1}) \\ &= 2hp(w_{2n-1}, w_{2n}), \text{ by Case 2} \end{aligned}$$

$$\Rightarrow p(w_m, w_{m+1}) = 2hp(w_{m-2}, w_{m-1}), \text{ because } m = 2n+1. \tag{2.12}$$

From (2.11), and (2.12), we conclude that when m is an odd natural number, and $(w_m, w_{m+1}) \in Q \times P$, we have

$$p(w_m, w_{m+1}) \leq 2h \max\{p(w_{m-2}, w_{m-1}), p(w_{m-1}, w_m)\}. \tag{2.13}$$

We get the same result (2.13) when m is an even number.

The case $p(w_m, w_{m+1}) \in Q \times Q$ is not possible.

Referring to (2.5), (2.6) and (2.13), we conclude that for all possible cases: Case 1, Case 2 and Case 3, we have

$$p(w_m, w_{m+1}) \leq 2h \max\{p(w_{m-2}, w_{m-1}), p(w_{m-1}, w_m)\}. \tag{2.14}$$

We note that for $h \geq 0$, we have

$$\begin{aligned} 2h < 1 &\Rightarrow \frac{2(a+b)}{1-2b} < 1 \\ &\Rightarrow 2a + 2b < 1 - 2b \\ &\Rightarrow a + 2b < \frac{1}{2}, \end{aligned}$$

as required by the assumption.

We apply Lemma 1.6 to (2.14) and get

$$p(w_n, w_{n+1}) < (2h)^{n/2}\delta, \tag{2.15}$$

where $\delta = (2h)^{1/2} \max\{p(w_0, w_1), p(w_1, w_2)\}$.

Suppose $n, m \in \mathbb{N}, n > m$. Using (1.2) inductively, we have

$$\begin{aligned} p(w_m, w_n) &\leq \sum_{i=m}^{n-1} p(w_i, w_{i+1}) \\ &\leq \sum_{i=m}^{n-1} (2h)^{i/2}\delta \text{ from (2.15)}, \\ &\leq \delta(2h)^{m/2} \frac{1-(2h)^{(n-m)/2}}{1-(2h)^{1/2}} \\ &\leq \delta(2h)^{m/2} \frac{1}{1-(2h)^{1/2}}. \end{aligned}$$

Taking $n, m \to \infty$ we get

$$\lim_{n,m\to\infty} p(w_m, w_n) = 0 < +\infty.$$

This makes $\{w_n\} \in K$ a Cauchy sequence.

From the assumption, K is a closed in (X, p^s). This makes K complete in (X, p^s) and hence complete in (X, p), $\big($see Lemma 1.2 (ii)$\big)$. This means that there is $z \in K$ such that

$$\lim_{m,n\to\infty} p(w_m, w_n) = \lim_{m,n\to\infty} p(w_n, z) = p(z, z) = 0.$$

This means $w_n \to z$ in (X, p).

Consider a subsequence $\{w_{2n_k}\}$ of $\{w_n\}$ which is contained in P. For all $k \in \mathbb{N}$ we have

$$w_{2n_k} = Tx_{2n_k} = Gx_{2n_k-1}.$$

Hence

$$\begin{aligned} &\lim_{k\to\infty} w_{2n_k} = z \\ &\Rightarrow \lim_{k\to\infty} Tx_{2n_k} = \lim_{k\to\infty} Gx_{2n_k-1} = z \\ &\Rightarrow \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Gx_{2n-1} = z. \end{aligned}$$

Let us now consider the subsequence $\{w_{2n_k+1}\}$ of $\{w_n\}$ which is contained in P. Using a similar argument, we get

$$\lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Fx_{2n} = z.$$

Thus we have shown that

$$\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Gx_{2n-1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Fx_{2n} = z. \tag{2.16}$$

Let us now consider the subsequence $\{w_{2n_k}\} = \{Tx_{2n_k}\}$ of $\{w_n\}$ which is contained in P. If we assume that S is continuous, then
$\lim_{k\to\infty} STx_{2n_k} = Sz$. As the pair $\{G, S\}$ are weakly p-commutative, we have

$$\begin{aligned} p(STx_{2n_k}, GSx_{2n_k-1}) &= p(SGx_{2n_k-1}, GSx_{2n_k-1}) \\ &\leq p(Gx_{2n_k-1}, Sx_{2n_k-1}). \end{aligned}$$

Taking $k \to \infty$ and applying (2.16), we get

$$p(Sz, \Gamma) \leq p(z, z) = 0 \Rightarrow p(Sz, \Gamma) = 0 \Rightarrow \Gamma = Sz, \tag{2.17}$$

where $\Gamma := \lim_{k\to\infty} GSx_{2n_k-1}$.
To show that $Sz = z$, we consider

$$\begin{aligned} &p(Fx_{2n_k}, GSx_{2n_k-1}) \\ &\leq a \max\{p(Tx_{2n_k}, SSx_{2n_k-1})/2, p(Tx_{2n_k}, Fx_{2n_k}), p(SSx_{2n_k-1}, GSx_{2n_k-1})\} \\ &\qquad + b[p(Tx_{2n_k}, GSx_{2n_k-1}) + p(Fx_{2n_k}, SSx_{2n_k-1})]. \end{aligned}$$

Taking $k \to \infty$ and making use of (2.8) (with $\Gamma = Sz$), we get

$$\begin{aligned} p(z, Sz) &\leq a \max\{p(z, Sz)/2, p(z, z), p(Sz, Sz)\} \\ &\qquad + b[p(z, Sz) + p(z, Sz)] \\ &\leq a \max\{p(z, Sz), p(Sz, Sz)\} + 2bp(z, Sz) \\ &= ap(z, Sz) + 2bp(z, Sz) \\ \Rightarrow p(z, Sz) &\leq (a + 2b)p(z, Sz). \end{aligned} \tag{2.18}$$

From the assumption, we have $a + 2b < \frac{1}{2}$. Hence (2.18) implies

$$p(z, Sz) = 0 \Rightarrow Sz = z.$$

Now consider

$$\begin{aligned} p(Fx_{2n}, Gz) &\leq a \max\{p(Tx_{2n}, Sz)/2, p(Tx_{2n}, Fx_{2n}), p(Sz, Gz)\} \\ &\qquad + b[p(Tx_{2n}, Gz) + p(Fx_{2n}, Sz]. \end{aligned}$$

Taking $k \to \infty$ and noting that $p(z, Sz) = 0$, we get

$$\begin{aligned} p(z, Gz) &\leq a \max\{p(z, Sz)/2, p(z, z), p(z, Gz)\} \\ &\qquad + b[p(z, Gz) + p(z, Sz)] \\ &= ap(z, Gz) + bp(z, Gz) \\ &= (a + b)p(z, Gz). \end{aligned}$$

From the assumption $a + 2b < \frac{1}{2}$ implying $a + b \leq \frac{1}{2}$. This means

$$p(z, Gz) = 0 \Rightarrow Gz = z. \tag{2.19}$$

If we expand $p(Fz, Gx_{2n-1})$ and use a similar argument, we will arrive at

$$Fz = z. \tag{2.20}$$

As $Gz = z \in K$, by (i) in the assumption, there is $u \in K$ such that $Tu = Gz = z$. Hence

$$\begin{aligned} p(Fu, z) &= p(Fu, Gz) \\ &\leq a \max\{p(Tu, Sz)/2, p(Tu, Fu), p(Sz, Gz)\} \\ &\quad + b[p(Tu, Gz) + p(Fu, Sz] \\ &= a \max\{p(z, z)/2, p(z, Fu), p(z, z)\} + b[p(z, z) + p(Fu, z)] \\ &= (a + b)p(Fu, z) \\ \Rightarrow p(Fu, z) &\leq p(Fu, z), \text{ because } a + b < 1 \\ \Rightarrow p(Fu, z) &= 0 \end{aligned}$$

$$\Rightarrow Fu = z = Gz = Tu. \tag{2.21}$$

Using the weak p-compatibility of (F, T), together with (2.21), we have

$$\begin{aligned} p(FTu, TFu) &\leq p(Fu, Tu) = p(z, z) = 0 \\ \Rightarrow p(Fz, Tz) &= 0 \\ \Rightarrow Fz &= Tz = z. \end{aligned}$$

Hence we have proved that

$$Fz = Gz = Sz = Tz = z,\ p(z, z) = 0.$$

To prove that z is unique, suppose z' is also a common fixed point v of F, G, S and T. Then

$$\begin{aligned} p(z', z) &= p(Fz', Gz) \\ &\leq a \max\{p(Tz', Sz)/2, p(Tz', Fz'), p(Sz, Gz)\} \\ &\quad + b[p(Tz', Gz) + p(Fz', Sz)] \\ &= a \max\{p(z', z)/2, p(z', z'), p(z, z)\} + b[p(z', z) + p(z', z)] \\ &\leq a \max\{p(z', z), p(z', z'), p(z, z)\} + 2b\, p(z', z) \\ &= ap(z', z) + 2bp(z', z) \\ &= (a + 2b)p(z', z). \end{aligned}$$

As $a + 2b < \dfrac{1}{2}$, this leads to

$$p(z', z) = 0 \Rightarrow z' = z.$$

We have shown that z is unique and proved Theorem 2.1 for S being a continuous mapping. A similar argument applies for continuity in at least one of the mappings F, T or G. □

Remark 2.1. We get special cases of Theorem 2.1 if Definition 2.1 is adjusted by putting

(i) $S = T$;
(ii) $S = T = I$, where I is the identity mapping;
(iii) $S = I$;
(iv) $a = 0$ or
(v) $b = 0$.

Theorem 2.1 remains true if the condition of weak p-commutativity and continuity of one of the mappings is replaced by closedness of TK and SK (or FK and GK), together with coincidental commutativity of (F,T) (or (G,S)). Doing so, we get the following theorem:

Theorem 2.2. *Let (X,p) be a complete metrically convex partial metric space and let K be a closed non-empty subset of X, the closure being with respect to (X,p^s). Let K, the boundary of K with respect to (X,p^s), be non-empty. Also let $F,G,S,T : K \to X$. If (F,G) is a generalized (S,T) p-contraction of K satisfying*

(i) $\partial K \subset SK \cap TK$, $FK \cap K \subseteq SK$, $GK \cap K \subseteq TK$,
(ii) $Tx \in \partial K \Rightarrow Fx \in K$, $Sx \in \partial K \Rightarrow Gx \in K$,
(iii) TK and SK (or FK and GK) are closed,
(iv) (F,T) and (G,S) have coincidence points and
(v) the pairs (F,T) and (G,S) are coincidentally commuting.

Then there exists a point $z \in K$ such that $Fz = Gz = Sz = Tz = z$. Furthermore z remains a unique common fixed point of both pairs separately and $p(z,z) = 0$.

Proof: The proof sequence is the same as that used for proving Theorem 2.1 until we reach the equation (2.16). From there we proceed as follows. Let us consider the subsequence $w_{2n_k} = \{Tx_{2n_k}\}$ contained in P. Assuming TK is closed, we have z as defined in (2.16) being an element of TK. Hence, there is $u \in K$ such that $Tu = z$. Now consider

$$\begin{aligned} p(Fu, Tx_{2n_k}) &= p(Fu, Gx_{2n_k-1}) \\ &\le a \max\{p(Tu, Sx_{2n_k-1})/2, p(Tu, Fu), p(Sx_{2n_k-1}, Gx_{2n_k-1})\} \\ &\quad + b[p(Tu, Gx_{2n_k-1}) + p(Fu, Sx_{2n_k-1})]. \end{aligned}$$

Taking $k \to \infty$, we get

$$\begin{aligned} p(Fu, z) &\le a \max\{p(z,z)/2, p(z, Fu), p(z,z)\} \\ &\quad + b[p(z,z) + p(Fu,z)] \\ &= ap(Fu,z) + b[p(z,z) + p(Fu,z)] \\ &\le \frac{b}{1-a-b}\, p(z,z) \\ &\le p(z,z). \end{aligned}$$

This implies $p(Fu,z) = 0$ and hence, $Fu = z = Tu$, so that u is a coincidence point of F and T. Because (F,T) are coincidentally commuting, we have $FTu = TFu \Rightarrow Fz = Tz$.

Similarly, we consider the subsequence $w_{2n_k+1} = \{Sx_{2n_k+1}\}$ contained in P. As SK is closed, there is $v \in K$ such that $Sv = z$. Expanding $p(Sx_{2n_k+1}, Gv)$, then taking $k \to \infty$, and using the coincidental commutativity of (S,G), we get the similar result that

$$Sv = z = Gv \text{ and } Sz = Gz.$$

To show that $Fz = z$, we proceed as follows:

$$\begin{aligned}
p(Fz, z) &= p(Fz, Gv) \\
&\leq a \max\{p(Tz, Sv)/2, p(Tz, Fz), p(Sv, Gv)\} \\
&\qquad + b[p(Tz, Gv) + p(Fz, Sv)] \\
&= a \max\{p(Fz, z)/2, p(Fz, Fz), p(z, z)\} \\
&\qquad + b[p(Fz, z) + p(Fz, z)] \\
&\leq a \max\{p(Fz, z), p(Fz, Fz), p(z, z)\} \\
&\qquad + b[p(Fz, z) + p(Fz, z)] \\
&= ap(Fz, z) + 2bp(Fz, z) \\
&= (a + 2b)p(Fz, z).
\end{aligned}$$

As $(a + 2b) < \frac{1}{2}$, this means

$$p(Fz, z) = 0 \Rightarrow Fz = z = Tz.$$

If we consider $p(z, Gz)$ and use the same argument as above, we reach a similar conclusion that $Gz = z = Sz$. The uniqueness of z follows easily.

Thus we have z as a unique common fixed point of F, G, S and T and $p(z, z) = 0$.

We get the same result if we consider FK and GK as closed. We have completed the proof of Theorem 2.2.

Remark 2.2. We get special cases of Theorem 2.2 if Definition 2.1 is adjusted by putting

(i) $S = T$;
(ii) $S = T = I$, where I is the identity mapping;
(iii) $S = I$;
(iv) $a = 0$ or
(v) $b = 0$.

We provide an example on the use of Theorem 2.2.

Example 2.1. Consider the metrically complete convex partial metric space $(\mathbb{R}_+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$. The derived metric for p is $p^s(x, y) = |x - y|$, and it is metrically convex. This makes the partial metric space (R_+, p) also metrically convex by Definition 1.4.

Let $K = [0, 2]$, a closed subset of $\mathbb{R}_+$. Let the mappings $F, T, G, S : K \to \mathbb{R}_+$ be defined as

$$Fx = x,\ Tx = 3x,\ Gx = 2x,\ Sx = 10x.$$

We have $\partial K = \{0, 2\}, SK = [0, 20], TK = [0, 6]$ implying $\partial K \subset SK \cap TK$. We also $FK = [0, 2], GK = [0, 4]$ implying $FK \cap K \subset SK$ and $GK \cap K \subset TK$.

When $Tx \in \partial K$, we have $x = 0$ or $x = \frac{2}{3}$. This makes $Fx \in \left\{0, \frac{2}{3}\right\}$ implying $F_x \in K$. Similarly, when $Sx \in \partial K$, we have $x = 0$ or $x = \frac{2}{10}$. This makes $Gx \in \left\{0, \frac{2}{5}\right\}$ implying $F_x \in K$.

We have the pairs F, T and G, S coincidentally commuting at their coincidence point $z = 0$ because $FT(0) = TF(0)$ and $GS(0) = SG(0)$. Finally $z = 0$ is a common fixed point of all the four mappings and $p(z, z) = 0$.

We set $a = \dfrac{1}{4}, b = \dfrac{1}{32}$ and note that $a + 2b = \dfrac{5}{16} < \dfrac{1}{2}$.

We consider the case of $x \leq y$ which leads to

$$\begin{aligned} p(Fx, Gy) &= p(x, 2y) = \max\{x, 2y\} = 2y \\ &\leq \frac{1}{4} \times 10y = \frac{1}{4}\max\{10y, 2y\} = \frac{1}{4}\max\{Sy, Gy\} \\ &= \frac{1}{4}p(Sy, Gy). \end{aligned}$$

Hence

$$\begin{aligned} p(Fx, Gy) \leq \frac{1}{4}\max\big\{p(Tx, Sy)/2, p(Tx, Fx), p(Sy, Gy)\big\} \\ + \frac{1}{32}\big[p(Tx, Gy) + p(Fx, Sy)\big]. \end{aligned}$$

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