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DERIVATIVES WITH RESPECT TO HORIZONTAL AND VERTICAL LIFTS OF THE CHEEGER-GROMOLL METRIC ^{CG}g ON THE $(1,1)-$TENSOR BUNDLE $T_1^1(M)$.

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ABSTRACT. In this paper, we define the Cheeger-Gromoll metric in the $(1,1)$ $-$tensor bundle $T_1^1(M)$, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the Cheeger-Gromoll metric with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.

1. INTRODUCTION

Let M be a differentiable manifold of class C^∞ and finite dimension n. Then the set $T_1^1(M) = \cup_{P\in M} T_1^1(P)$ is, by definition, the tensor bundle of type $(1,1)$ over M, where $\cup$ denotes the disjoint union of the tensor spaces $T_1^1(P)$ for all $P \in M$. For any point $\tilde{P}$ of $T_1^1(M)$ such that $\tilde{P} \in T_1^1(M)$, the surjective correspondence $\tilde{P} \to P$ determines the natural projection $\pi : T_1^1(M) \to M$. The projection π defines the natural differentiable manifold structure of $T_1^1(M)$, that is, $T_1^1(M)$ is a $C^\infty-$ manifold of dimension $n+n^2$. If x^j are local coordinates in a neighborhood U of $P \in M$, then a tensor t at P which is an element of $T_1^1(M)$ is expressible in the form (x^j, t_j^i), where t_j^i are components of t with respect to the natural base. We may consider $(x^j, t_j^i) = (x^j, x^{\bar{j}}) = (x^J), j = 1, ..., n, \bar{j} = n+1, ..., n+n^2, J = 1, ..., n+n^2$ as local coordinates in a neighborhood $\pi^{-1}(U)$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $A = A_j^i \frac{\partial}{\partial x^i} \otimes dx^j$ be the local expressions in U of a vector field X and a $(1,1)$ tensor field A on M, respectively. Then the vertical lift A^V of A and the horizontal lift X^H of X are given, with respect to the induced coordinates,by

$$^VA = \begin{pmatrix} ^VA^j \\ ^VA^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A_j^i \end{pmatrix} \tag{1.1}$$

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and

$$(1.2)\qquad {}^{H}X=\begin{pmatrix} {}^{H}X^{j} \\ {}^{H}X^{\bar{j}} \end{pmatrix}=\begin{pmatrix} X^{j} \\ X^{s}(\Gamma_{sj}^{m}t_{m}^{i}-\Gamma_{sm}^{i}t_{j}^{m}) \end{pmatrix}$$

where Γ_{ij}^{h} are the coefficient of the connection ∇ on M [9].

Let $\varphi\in\Im_{1}^{1}(M)$. The global vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi\in\Im_{0}^{1}(\Im_{1}^{1}(M))$ are respectively defined by

$$\gamma\varphi=\begin{pmatrix} 0 \\ t_{j}^{m}\varphi_{m}^{i} \end{pmatrix},\tilde{\gamma}\varphi=\begin{pmatrix} 0 \\ t_{m}^{i}\varphi_{j}^{m} \end{pmatrix}$$

with respect to the coordinates $\left(x^{i},x^{\bar{j}}\right)$ in $T_{1}^{1}(M)$, where φ_{j}^{i} are the components of φ [9].

The Lie bracket operation of vertical and horizontal vector fields on $T_{1}^{1}(M)$ is given by

$$(1.3)\qquad \begin{aligned} \left[{}^{H}X,{}^{H}Y\right] &= {}^{H}\left[X,Y\right]+(\tilde{\gamma}-\gamma)R(X,Y) \\ \left[{}^{H}X,{}^{V}A\right] &= {}^{V}(\nabla_{X}A) \\ \left[{}^{V}A,{}^{V}B\right] &= 0 \end{aligned}$$

for any $X,Y\in\Im_{0}^{1}(M)$ and $A,B\in\Im_{1}^{1}(M)$, where R is the curvature tensor field of the connection ∇ on M defined by $R(X,Y)=[\nabla_{X},\nabla_{Y}]-\nabla_{[X,Y]}$ and $(\tilde{\gamma}-\gamma)\,R(X,Y)=\binom{0}{t_{m}^{i}R_{klj}^{m}X^{k}Y^{l}-t_{j}^{m}R_{klm}^{i}X^{k}Y^{l}}$(for details, see [7, 17] and for sufraces [3, 4]).

1.1. **Cheeger-Gromoll type metric on the (1, 1)-tensor bundle.** An n-dimensional manifold M in which a (1,1) tensor field φ satisfying $\varphi^{2}=id$, $\varphi\neq\pm id$ is given is called an almost product manifold. A Riemannian almost product manifold (M,φ,g) is a manifold M with an almost product structure φ and a Riemannian metric g such that [1, 2, 10, 11]

$$(1.4)\qquad g(\varphi X,Y)=g(X,\varphi Y)$$

for all $X,Y\in\Im_{0}^{1}(M)$. Also, the condition (3.1) is referred to as purity condition for g with respect to φ [9]. The almost product structure φ is integrable, i.e. the Nijenhuis tensor N_{φ} determined by

$$N_{\varphi}(X,Y)=[\varphi X,\varphi Y]-\varphi\left[\varphi X,Y\right]-\varphi\left[X,\varphi Y\right]+[X,Y]$$

for all $X,Y\in\Im_{0}^{1}(M)$ is zero then the Riemannian almost product manifold. (M,φ,g) is called a Riemannian product manifold. A locally decomposable Riemannian manifold can be defined as a triple (M,φ,g) which consists of a smooth manifold M endowed with an almost product structure φ and a pure metric g such that $\nabla\varphi=0$,where ∇ is the Levi-Civita connectian of g [9].

Definition 1.1. Let $T_{1}^{1}(M)$ be the $(1,1)-$tensor bundle over a Riemannian manifold (M,g). For each $P\in M$, the extension of scalar product g (marked by G) is defined on the tensor space $\pi^{-1}(P)=T_{1}^{1}(P)$ by $G(A,B)=g_{ij}g^{jl}A_{j}^{i}B_{l}^{t}$ for all $A,B\in\Im_{1}^{1}(P)$. The Cheeger-Gromoll type metric ${}^{CG}g$ is defined on $T_{1}^{1}(M)$ by the following three equations:

$$(1.5)\qquad {}^{CG}g(X^{H},Y^{H})=(g(X,Y))^{V}$$

$$ {}^{CG}g(A^V, Y^H) = 0 \tag{1.6} $$

$$ {}^{CG}g(A^V, B^V) = \frac{1}{\alpha}(G(A,B) + G(A,t)G(B,t))^V \tag{1.7} $$

for any $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$, where $r^2 = G(t,t) = g_{it}g^{jl}t_j^i t_l^t$ and $\alpha = 1 + r^2$ [9].

2. Main Results

Definition 2.1. Let M be an $n-$dimensional diferentiable manifold. Differantial transformation of algebra $T(M)$, defined by

$$ D = \nabla_X : T(M) \to T(M),\ X \in \Im_0^1(M) $$

is called as covariant derivation with respect to vector field X if

$$ \begin{aligned} \nabla_{fX+gY}t &= f\nabla_X t + g\nabla_Y t, \\ \nabla_X f &= Xf, \end{aligned} \tag{2.1} $$

where $\forall f, g \in \Im_0^0(M)$, $\forall X, Y \in \Im_0^1(M)$, $\forall t \in \Im(M)$ (see [13], p.123).
On the other hand, a transformation defined by

$$ \nabla : \Im_0^1(M) \times \Im_0^1(M) \to \Im_0^1(M) $$

is called as an affin connection (see for details [13, 16]).

Definition 2.2. The horizontal lift ${}^H\nabla$ of any connection ∇ on the tensor bundle $T_1^1(M)$ is defined by

$$ \begin{aligned} {}^H\nabla_{{}^V A}\, {}^V B &= 0,\ {}^H\nabla_{{}^V A}\, {}^H Y = 0, \\ {}^H\nabla_{{}^H X}\, {}^V B &= {}^V(\nabla_X B),\ {}^H\nabla_{{}^H X}\, {}^H Y = {}^H(\nabla_X Y) \end{aligned} \tag{2.2} $$

for all vector fields $X, Y \in \Im_0^1(M)$ and $A, B \in \Im_1^1(M)$ (see [8, 14, 15, 17]).

Theorem 2.1. *Let ${}^{CG}g$ be the Cheeger-Gromoll type metric ${}^{CG}g$ defined by (1.5),(1.6),(1.7) and the horizontal lift ${}^H\nabla$ of any connection ∇ on the tensor bundle $T_1^1(M)$ is defined by (2.2). From Definintion 1.1 and Definintion 2.1, we get the following* *results*

$$\begin{aligned}
i)\ ({}^H\nabla_{{}^V C}{}^{CG}g)({}^V A, {}^V B) &= 0,\\
ii)\ ({}^H\nabla_{{}^V C}{}^{CG}g)({}^V A, {}^H Y) &= 0,\\
iii)\ ({}^H\nabla_{{}^V C}\,{}^{CG}g)({}^H X, B^V) &= 0,\\
iv)\ ({}^H\nabla_{{}^V C}{}^{CG}g)({}^H X, {}^H Y) &= 0,\\
v)\ ({}^H\nabla_{{}^H Z}{}^{CG}g)({}^V A, {}^H Y) &= 0,\\
vi)\ ({}^H\nabla_{{}^H Z}{}^{CG}g)({}^H X, {}^V B) &= 0,\\
vii)\ ({}^H\nabla_{{}^H Z}{}^{CG}g)({}^H X, {}^H Y) &= {}^V((\nabla_Z g)(X,Y)),\\
viii)\ ({}^H\nabla_{{}^H Z}{}^{CG}g)({}^V A, {}^V B) &= {}^V(\nabla_Z \frac{1}{\alpha})^V(G(A,B) + G(A,t)G(B,t))\\
&\quad + \frac{1}{\alpha}^V((\nabla_Z G)(A,B)) + \frac{1}{\alpha}^V(\nabla_Z(G(A,t)G(B,t)))\\
&\quad - \frac{1}{\alpha}^V(G((\nabla_Z A),t)G(B,t))\\
&\quad - \frac{1}{\alpha}^V(G(A,t)G((\nabla_Z B),t)),
\end{aligned}$$

where the vertical lift ${}^V A \in \Im^1_0(T^1_1 M)$ of $A \in \Im^1_1(M)$ and the horizontal lifts ${}^H X \in \Im^1_0(T^1_1 M)$ of $X \in \Im^1_0(M)$ defined by (1.1) and (1.2), respectively.

Proof. $i)$

$$\begin{aligned}
({}^H\nabla_{{}^V C}{}^{CG}g)({}^V A, {}^V B) &= {}^H\nabla_{{}^V C}{}^{CG}g({}^V A, {}^V B) - {}^{CG}g({}^H\nabla_{{}^V C}{}^V A, {}^V B)\\
&\quad - {}^{CG}g({}^V A, {}^H\nabla_{{}^V C}{}^V B)\\
&= {}^H\nabla_{{}^V C}\frac{1}{\alpha}^V(G(A,B) + G(A,t)G(B,t))\\
&= 0
\end{aligned}$$

$ii)$

$$\begin{aligned}
({}^H\nabla_{{}^V C}{}^{CG}g)({}^V A, {}^H Y) &= {}^H\nabla_{{}^V C}{}^{CG}g({}^V A, {}^H Y) - {}^{CG}g({}^H\nabla_{{}^V C}{}^V A, {}^H Y)\\
&\quad - {}^{CG}g({}^V A, {}^H\nabla_{{}^V C}{}^H Y)\\
&= - {}^{CG}g({}^V A, {}^H\nabla_{{}^V C}{}^H Y)\\
&= 0
\end{aligned}$$

$iii)$

$$\begin{aligned}
({}^H\nabla_{{}^V C}\,{}^{CG}g)({}^H X, B^V) &= {}^H\nabla_{{}^V C}\,{}^{CG}g({}^H X, {}^V B) - {}^{CG}g({}^H\nabla_{{}^V C}{}^H X, {}^V B)\\
&\quad - {}^{CG}g({}^H X, {}^H\nabla_{{}^V C}{}^V B)\\
&= - {}^{CG}g({}^H\nabla_{{}^V C}{}^H X, {}^V B)\\
&= 0
\end{aligned}$$

iv)

$$\begin{aligned}
({}^H\nabla_{{}^VC}{}^{CG}g)({}^HX,{}^HY) &= {}^H\nabla_{{}^VC}{}^{CG}g({}^HX,{}^HY) -{}^{CG}g({}^H\nabla_{{}^VC}{}^HX,{}^HY) \\
&\quad -{}^{CG}g({}^HX,{}^H\nabla_{{}^VC}{}^HY) \\
&= {}^H\nabla_{{}^VC}{}^V(g(X,Y)) \\
&= {}^VC^V(g(X,Y)) \\
&= 0
\end{aligned}$$

v)

$$\begin{aligned}
({}^H\nabla_{{}^HZ}{}^{CG}g)({}^VA,{}^HY) &= {}^H\nabla_{{}^HZ}{}^{CG}g({}^VA,{}^HY) -{}^{CG}g({}^H\nabla_{{}^HZ}{}^VA,{}^HY) \\
&\quad -{}^{CG}g({}^VA,{}^H\nabla_{{}^HZ}{}^HY) \\
&= {}^{CG}g({}^V(\nabla_ZA),{}^HY) -{}^{CG}g({}^VA,{}^H(\nabla_ZY)) \\
&= 0
\end{aligned}$$

vi)

$$\begin{aligned}
({}^H\nabla_{{}^HZ}{}^{CG}g)({}^HX,{}^VB) &= {}^H\nabla_{{}^HZ}{}^{CG}g({}^HX,{}^VB) -{}^{CG}g({}^H\nabla_{{}^HZ}{}^HX,{}^VB) \\
&\quad -{}^{CG}g({}^HX,{}^H\nabla_{{}^HZ}{}^VB) \\
&= -{}^{CG}g({}^H(\nabla_ZX),{}^VB) -{}^{CG}g({}^HX,{}^V(\nabla_ZB)) \\
&= 0
\end{aligned}$$

vii)

$$\begin{aligned}
({}^H\nabla_{{}^HZ}{}^{CG}g)({}^HX,{}^HY) &= {}^H\nabla_{{}^HZ}{}^{CG}g({}^HX,{}^HY) -{}^{CG}g({}^H\nabla_{{}^HZ}{}^HX,{}^HY) \\
&\quad -{}^{CG}g({}^HX,{}^H\nabla_{{}^HZ}{}^HY) \\
&= {}^H\nabla_{{}^HZ}{}^V(g(X,Y)) -{}^{CG}g({}^H(\nabla_ZX),{}^HY) \\
&\quad -{}^{CG}g({}^HX,{}^H(\nabla_ZY)) \\
&= {}^V(\nabla_Zg(X,Y)) -{}^V(g((\nabla_ZX),Y)) -{}^V(g(X,(\nabla_ZY))) \\
&= {}^V((\nabla_Zg)(X,Y))
\end{aligned}$$

viii)

$$\begin{aligned}
({}^H\nabla_{{}^H Z}{}^{CG}g)({}^V A, {}^V B) &= {}^H\nabla_{{}^H Z}{}^{CG}g({}^V A, {}^V B) - {}^{CG}g({}^H\nabla_{{}^H Z}{}^V A, {}^V B) \\
&\quad - {}^{CG}g({}^V A, {}^H\nabla_{{}^H Z}{}^V B) \\
&= {}^H\nabla_{{}^H Z}\frac{1}{\alpha}^V (G(A,B) + G(A,t)G(B,t)) \\
&\quad - {}^{CG}g({}^V(\nabla_Z A), {}^V B) - {}^{CG}g({}^V A, {}^V(\nabla_Z B)) \\
&= {}^V(\nabla_Z \frac{1}{\alpha})^V (G(A,B) + G(A,t)G(B,t)) \\
&\quad + \frac{1}{\alpha}^V (\nabla_Z(G(A,B) + G(A,t)G(B,t))) \\
&\quad - \frac{1}{\alpha}^V (G((\nabla_Z A), B) + G((\nabla_Z A), t)G(B,t)) \\
&\quad - \frac{1}{\alpha}^V (G(A, (\nabla_Z B)) + G(A,t)G((\nabla_Z B), t)) \\
&= {}^V(\nabla_Z \frac{1}{\alpha})^V (G(A,B) + G(A,t)G(B,t)) \\
&\quad + \frac{1}{\alpha}^V ((\nabla_Z G)(A,B)) + \frac{1}{\alpha}^V (\nabla_Z(G(A,t)G(B,t))) \\
&\quad - \frac{1}{\alpha}^V (G((\nabla_Z A), t)G(B,t)) - \frac{1}{\alpha}^V (G(A,t)G((\nabla_Z B), t))
\end{aligned}$$

□

Definition 2.3. Let M be an $n-$dimensional differentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im_0^1(M)$ if

$$\begin{aligned}
L_X f &= Xf, \forall f \in \Im_0^0(M), \\
L_X Y &= [X,Y], \forall X, Y \in \Im_0^1(M).
\end{aligned} \tag{2.3}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [5, 6, 12, 18]

$$(L_X F)Y = [X, FY] - F[X,Y]. \tag{2.4}$$

Definition 2.4. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{cases}
[{}^V A, {}^V B] = 0\ , \\
[{}^H X, {}^V A] = {}^V(\nabla_X A), \\
[{}^H X, {}^H Y] = {}^H[X,Y] + (\tilde{\gamma} - \gamma)R(X,Y),
\end{cases} \tag{2.5}$$

where R denotes the curvature tensor field of the connection ∇, and $\tilde{\gamma} - \gamma : \varphi \to \Im_0^1(T_1^1(M))$ is the operator defined by

$$(\tilde{\gamma} - \gamma)\varphi = \begin{pmatrix} 0 \\ t_m^i \varphi_j^m - t_j^m \varphi_m^i \end{pmatrix}$$

for any $\varphi \in \Im_1^1(M)$ [17].

Theorem 2.2. *Let* ^{CG}g *be the Cheeger-Gromoll type metric* ^{CG}g *defined by (1.5),(1.6),(1.7) and* L_X *the operator Lie derivation with respect to* X*. From Definintion 2.3 and Definintion 2.4, we get the following results*

$$
\begin{aligned}
i)\ (L_{^VC}{}^{CG}g)(^VA,{}^VB) &= 0\\
ii)\ (L_{^VC}{}^{CG}g)(^HX,{}^HY) &= 0\\
iii)\ (L_{^HZ}{}^{CG}g)(^VA,{}^HY) &= -{}^{CG}g(^VA,(\tilde{\gamma}-\gamma)R(Z,Y))\\
iv)\ (L_{^HZ}{}^{CG}g)(^HX,{}^VB) &= -{}^{CG}g((\tilde{\gamma}-\gamma)R(Z,X),{}^VB)\\
v)\ (L_{^VC}{}^{CG}g)(^VA,{}^HY) &= \frac{1}{\alpha}^V(G(A,(\nabla_YC))+G(A,t)G((\nabla_YC),t))\\
vi)\ (L_{^VC}{}^{CG}g)(^HX,{}^VB) &= \frac{1}{\alpha}^V(G((\nabla_XC),B)+G((\nabla_XC),t)G(B,t))\\
vii)\ (L_{^HZ}\,{}^{CG}g)(^HX,{}^HY) &= {}^V((L_Zg)(X,Y)) -{}^{CG}g((\tilde{\gamma}-\gamma)R(Z,X),{}^HY)\\
&\quad -{}^{CG}g(^HX,(\tilde{\gamma}-\gamma)R(Z,Y))\\
viii)\ (L_{^HZ}\,{}^{CG}g)(^VA,{}^VB) &= {}^V(\nabla_Z\frac{1}{\alpha})^V(G(A,B)+G(A,t)G(B,t))\\
&\quad +\frac{1}{\alpha}^V((\nabla_ZG)(A,B))+\frac{1}{\alpha}^V(\nabla_Z(G(A,t)G(B,t)))\\
&\quad -\frac{1}{\alpha}^V(G(A,t)G((\nabla_ZB),t))\\
&\quad -\frac{1}{\alpha}^V(G((\nabla_ZA),t)G(B,t))
\end{aligned}
$$

where the vertical lift $^VA \in \Im^1_0(T^1_1M)$ of $A \in \Im^1_1(M)$ and the horizontal lifts $^HX \in \Im^1_0(T^1_1M)$ of $X \in \Im^1_0(M)$ defined by (1.1) and (1.2), respectively.

Proof. *i)*

$$
\begin{aligned}
(L_{^VC}{}^{CG}g)(^VA,{}^VB) &= L_{^VC}\,{}^{CG}g(^VA,{}^VB) -{}^{CG}g(L_{^VC}{}^VA,{}^VB) -{}^{CG}g(^VA,L_{^VC}{}^VB)\\
&= 0
\end{aligned}
$$

ii)

$$
\begin{aligned}
(L_{^VC}{}^{CG}g)(^HX,{}^HY) &= L_{^VC}{}^{CG}g(^HX,{}^HY) -{}^{CG}g(L_{^VC}{}^HX,{}^HY) -{}^{CG}g(^HX,L_{^VC}{}^HY)\\
&= L_{^VC}{}^V(g(X,Y)) +{}^{CG}g(^V(\nabla_XC),{}^HY) +{}^{CG}g(^HX,{}^V(\nabla_YC))\\
&= {}^VC^V(g(X,Y))\\
&= 0
\end{aligned}
$$

iii)

$$
\begin{aligned}
(L_{^HZ}{}^{CG}g)(^VA,{}^HY) &= L_{^HZ}\,{}^{CG}g(^VA,{}^HY) -{}^{CG}g(L_{^HZ}{}^VA,{}^HY) -{}^{CG}g(^VA,L_{^HZ}{}^HY)\\
&= -{}^{CG}g(^V(\nabla_ZA),{}^HY) -{}^{CG}g(^VA,{}^H[Z,Y]+(\tilde{\gamma}-\gamma)R(Z,Y))\\
&= -{}^{CG}g(^VA,{}^H(L_ZY)) -{}^{CG}g(^VA,(\tilde{\gamma}-\gamma)R(Z,Y))\\
&= -{}^{CG}g(^VA,(\tilde{\gamma}-\gamma)R(Z,Y))
\end{aligned}
$$

iv)

$$
\begin{aligned}
(L_{^HZ}{}^{CG}g)(^HX,{}^VB) &= L_{^HZ}\,{}^{CG}g(^HX,{}^VB) -{}^{CG}g(L_{^HZ}{}^HX,{}^VB) -{}^{CG}g(^HX,L_{^HZ}{}^VB)\\
&= -{}^{CG}g(^H[Z,X]+(\tilde{\gamma}-\gamma)R(Z,X),{}^VB) -{}^{CG}g(^HX,{}^V(\nabla_ZB))\\
&= -{}^{CG}g((\tilde{\gamma}-\gamma)R(Z,X),{}^VB)
\end{aligned}
$$

v)

$$\begin{aligned}
(L_{^VC}{}^{CG}g)(^VA,^HY) &= L_{^VC}{}^{CG}g(^VA,^HY) -^{CG} g(L_{^VC}{}^VA,^HY) -^{CG} g(^VA, L_{^VC}{}^HY) \\
&= {}^{CG}g(^VA,^V(\nabla_Y C)) \\
&= \frac{1}{\alpha}^V(G(A,(\nabla_Y C)) + G(A,t)G((\nabla_Y C),t)))
\end{aligned}$$

vi)

$$\begin{aligned}
(L_{^VC}{}^{CG}g)(^HX,^VB) &= L_{^VC}{}^{CG}g(^HX,^VB) -^{CG} g(L_{^VC}{}^HX,^VB) -^{CG} g(^HX, L_{^VC}{}^VB) \\
&= +^{CG}g(^V(\nabla_X C),^VB) \\
&= \frac{1}{\alpha}^V(G((\nabla_X C),B) + G((\nabla_X C),t)G(B,t))
\end{aligned}$$

vii)

$$\begin{aligned}
(L_{^HZ}{}^{CG}g)(^HX,^HY) &= L_{^HZ}{}^{CG}g(^HX,^HY) -^{CG} g(L_{^HZ}{}^HX,^HY) -^{CG} g(^HX, L_{^HZ}{}^HY) \\
&= {}^HZ^V(g(X,Y)) -^{CG} g(^H[Z,X] + (\tilde{\gamma}-\gamma)R(Z,X),^HY) \\
&\quad -^{CG}g(^HX,^H[Z,Y] + (\tilde{\gamma}-\gamma)R(Z,Y)) \\
&= {}^V(L_Z g(X,Y)) -^V(g((L_Z X),Y)) -^V(g(X,(L_Z Y))) \\
&\quad -^{CG}g((\tilde{\gamma}-\gamma)R(Z,X),^HY) -^{CG} g(^HX,(\tilde{\gamma}-\gamma)R(Z,Y)) \\
&= {}^V((L_Z g)(X,Y)) -^{CG} g((\tilde{\gamma}-\gamma)R(Z,X),^HY) \\
&\quad -^{CG}g(^HX,(\tilde{\gamma}-\gamma)R(Z,Y))
\end{aligned}$$

viii)

$$\begin{aligned}
(L_{^HZ}{}^{CG}g)(^VA,^VB) &= L_{^HZ}{}^{CG}g(^VA,^VB) -^{CG} g(L_{^HZ}{}^VA,^VB) -^{CG} g(^VA, L_{^HZ}{}^VB) \\
&= {}^HZ(\frac{1}{\alpha}^V(G(A,B) + G(A,t)G(B,t)))^{CG}g(^V(\nabla_Z A),^VB) \\
&\quad -^{CG}g(^VA,^V(\nabla_Z B) \\
&= {}^V(\nabla_Z \frac{1}{\alpha})^V(G(A,B) + G(A,t)G(B,t)) \\
&\quad + \frac{1}{\alpha}^V(\nabla_Z(G(A,B) + G(A,t)G(B,t))) \\
&\quad - \frac{1}{\alpha}^V(G((\nabla_Z A),B) + G((\nabla_Z A),t)G(B,t)) \\
&\quad - \frac{1}{\alpha}^V(G(A,(\nabla_Z B)) + G(A,t)G((\nabla_Z B),t)) \\
&= {}^V(\nabla_Z \frac{1}{\alpha})^V(G(A,B) + G(A,t)G(B,t)) + \frac{1}{\alpha}^V((\nabla_Z G)(A,B)) \\
&\quad + \frac{1}{\alpha}^V(\nabla_Z(G(A,t)G(B,t))) - \frac{1}{\alpha}^V(G((\nabla_Z A),t)G(B,t)) \\
&\quad - \frac{1}{\alpha}^V(G(A,t)G((\nabla_Z B),t))
\end{aligned}$$

□

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