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Certain Properties for Spiral-Like Functions Associated with Ruscheweyh-Type q-Difference Operator

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Abstract

In this paper, making use of the Ruscheweyh- type q-difference operator $\mathscr{D}_q(\mathscr{R}_q^{\alpha} f(z))$ we introduce a new subclass of spiral-like functions and discuss some subordination results and Fekete-Szegö problem for this generalized function class. Further, some known and new results which follow as special cases of our results are also mentioned.

Keywords: *Convex functions, Fekete-Szegö problem, Hadamard product, Ruschewyh- type q-difference operator, spiral-like functions, starlike functions, subordinating factor sequence, univalent functions.*
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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

which are analytic and univalent in the open disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of functions that are univalent in $\mathbb{U}$.
A function $f\in\mathscr{A}$ is said to be in the class of γ-spiral-like functions of order λ in $\mathbb{U}$, denoted by $\mathscr{S}^*(\gamma,\lambda)$ if

$$\Re\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)>\lambda\cos\gamma,\ \ z\in\mathbb{U} \tag{1.2}$$

for $0\le\lambda<1$ and some real γ with $|\gamma|<\frac{\pi}{2}$. The class $\mathscr{S}^*(\gamma,\lambda)$ was studied by Libera [6] and Keogh and Merkes [5]. Note that $\mathscr{S}^*(\gamma,0)$ is the class of spiral-like functions introduced by Špaček [15], $\mathscr{S}^*(0,\lambda)=\mathscr{S}^*(\lambda)$ is the class of starlike functions of order λ and $\mathscr{S}^*(0,0)=\mathscr{S}^*$ is the familiar class of starlike functions.
Let $\mathscr{B}$ be the class of all analytic functions w in $\mathbb{U}$ that satisfy the conditions $w(0)=0$ and $|w(z)|<1, z\in\mathbb{U}$.
For functions $f\in\mathscr{A}$ given by (1.1) and $g\in\mathscr{A}$ given by $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\ z\in\mathbb{U}. \tag{1.3}$$

We briefly recall here the notion of *q-operators* i.e. *q-difference operator* that play a vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [3] (also see [2, 11]). For the applications of q-calculus in geometric function theory, one may see the papers of Mohamad and Darus [7], Purohit and Raina [11], Mohamad and Sokól, [8].
Consider $0<q<1$ and a non-negative integer n. The q-integer number or basic number n is defined by

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\ldots+q^{n-1},\ [0]_q=0.$$

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For a non-integer number t we will denote $[t]_q = \dfrac{1-q^t}{1-q}$.
The q-shifted factorial is defined as follow

$[0]! = 1,\ [n]! = [1]_q[2]_q \dots [n]_q.$

Note that $\lim_{q\to 1^-}[n]_q = n$ and $\lim_{q\to 1^-}[n]! = n!$.
The Jackson's *q- derivative operator* or *q-difference operator* for a function $f \in \mathscr{A}$ is defined by

$$\mathscr{D}_q f(z) = \begin{cases} \dfrac{f(qz)-f(z)}{z(q-1)} & , z \neq 0 \\ f'(0) & , z = 0. \end{cases} \tag{1.4}$$

Note that for $n \in \mathbb{N} = \{1,2,\dots\}$ and $z \in \mathbb{U}$

$$\mathscr{D}_q z^n = [n]_q z^{n-1}. \tag{1.5}$$

For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, the *q-generalized Pochhammer symbol* is defined by

$[t]_n = [t]_q[t+1]_q[t+2]_q \dots [t+n-1]_q.$

Moreover, for $t > 0$ the *q-Gamma function* is given by

$\Gamma_q(t+1) = [t]_q\Gamma_q(t)$ and $\Gamma_q(1) = 1.$

For details on *q-calculus* one can refer to [1, 3] and also the reference cited therein.
Using the definition of Ruscheweyh differential operator [12], in [4] Kanas and Răducanu introduced the *Ruscheweyh q-differential operator* defined by

$$\mathscr{R}_q^\alpha f(z) = f(z) * F_{q,\alpha+1}(z) \quad z \in \mathbb{U}, \alpha > -1 \tag{1.6}$$

where $f \in \mathscr{A}$ and

$$F_{q,\alpha+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\alpha)}{[n-1]!\Gamma_q(1+\alpha)} z^n = z + \sum_{n=2}^{\infty} \frac{[\alpha+1]_{n-1}}{[n-1]!} z^n, \quad z \in \mathbb{U}. \tag{1.7}$$

From (1.6) we have

$$\mathscr{R}_q^0 f(z) = f(z), \qquad \mathscr{R}_q^1 f(z) = z\mathscr{D}_q f(z)$$

and

$$\mathscr{R}_q^m f(z) = \frac{z\mathscr{D}_q^m \left(z^{m-1} f(z)\right)}{[m]!} \quad m \in \mathbb{N}.$$

For $f \in \mathscr{A}$ given by (1.1), in view of (1.6) and (1.7), we obtain

$$\mathscr{R}_q^\alpha f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\alpha)}{[n-1]!\Gamma_q(1+\alpha)} a_n z^n = z + \sum_{n=2}^{\infty} \frac{[\alpha+1]_{n-1}}{[n-1]!} a_n z^n \quad z \in \mathbb{U}. \tag{1.8}$$

It is easy to check that

$$\lim_{q\to 1^-} F_{q,\alpha+1}(z) = \frac{z}{(1-z)^{\alpha+1}}$$

and

$$\lim_{q\to 1^-} \mathscr{R}_q^\alpha f(z) = f(z) * \frac{z}{(1-z)^{\alpha+1}}.$$

From (1.8) we get

$$\mathscr{D}_q(\mathscr{R}_q^\alpha f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q \Psi_q(n,\alpha) a_n z^{n-1} \tag{1.9}$$

where

$$\Psi_q(n,\alpha) = \frac{\Gamma_q(n+\alpha)}{[n-1]!\Gamma_q(1+\alpha)} = \frac{[\alpha+1]_{n-1}}{[n-1]!}. \tag{1.10}$$

Making use of the generalized Ruscheweyh q-differential operator $\mathscr{R}_q^\alpha f(z)$, we introduce a new subclass of spiral-like functions.
For $0 \le \lambda \le 1$, $0 \le \gamma < 1$ and $\frac{-\pi}{2} < \eta < \frac{\pi}{2}$, we let $\mathscr{G}_q^\alpha(\eta,\gamma,\lambda)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic condition:

$$\Re\left(e^{i\eta} \frac{z\mathscr{D}_q(\mathscr{R}_q^\alpha f(z))}{(1-\lambda)z + \lambda\mathscr{R}_q^\alpha f(z)} \right) > \gamma\cos\eta, \quad z \in \mathbb{U}, \tag{1.11}$$

where $\mathscr{D}_q(R_q^\alpha f(z))$ is given by (1.9).

Example 1.1. *For $\lambda=1$, $0\leq\gamma<1$ and $\frac{-\pi}{2}<\eta<\frac{\pi}{2}$, we let $\mathscr{G}_q^{\alpha}(\eta,\gamma,1)\equiv\mathscr{S}_q^{\alpha}(\eta,\gamma)$ be the subclass of $\mathscr{A}$ consisting of functions of the form (1.1) and satisfying the analytic condition:*

$$\Re\left(e^{i\eta}\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{\mathscr{R}_q^{\alpha}f(z)}\right)>\gamma\cos\eta,\quad z\in\mathbb{U}, \tag{1.12}$$

where $\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))$ is given by (1.9).

Example 1.2. *For $\lambda=0$, $0\leq\gamma<1$ and $\frac{-\pi}{2}<\eta<\frac{\pi}{2}$, we let $\mathscr{G}_q^{\alpha}(\eta,\gamma,0)\equiv\mathscr{R}_q^{\alpha}(\eta,\gamma)$ be the subclass of $\mathscr{A}$ consisting of functions of the form (1.1) and satisfying the analytic condition:*

$$\Re\left(e^{i\eta}\ \mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))\right)>\gamma\cos\eta,\ z\in\mathbb{U}, \tag{1.13}$$

where $\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))$ is given by (1.9).

The object of the present paper is to investigate the coefficient estimates and subordination properties for the class of functions $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$. Some interesting consequences of the results are also pointed out.

2. Membership characterizations

In this section we obtain several sufficient conditions for a function $f\in\mathscr{A}$ to be in the class $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$.

Theorem 2.1. *Let $f\in\mathscr{A}$ and let δ be a real number with $0\leq\delta<1$. If*

$$\left|\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}-1\right|\leq1-\delta,\ z\in\mathbb{U} \tag{2.1}$$

then $f\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ provided that

$$|\gamma|\leq\cos^{-1}\left(\frac{1-\delta}{1-\lambda}\right).$$

Proof. From (2.1) it follows that

$$\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}=1+(1-\delta)w(z),$$

where $w(z)\in\mathscr{B}$. We have

$$\begin{aligned}\Re\left(e^{i\eta}\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}\right)&=\Re[e^{i\eta}(1+(1-\delta)w(z))]\\&=\cos\eta+(1-\delta)\Re(e^{i\eta}w(z))\\&\geq\cos\eta-(1-\delta)|e^{i\eta}w(z)|\\&>\cos\eta-(1-\delta)\\&\geq\gamma\cos\eta,\end{aligned}$$

provided that $|\eta|\leq\cos^{-1}\left(\frac{1-\delta}{1-\gamma}\right)$. Thus, the proof is completed. □

If in Theorem 2.1 we take $\delta=1-(1-\gamma)\cos\eta$ we obtain the following result.

Corollary 2.2. *Let $f\in\mathscr{A}$. If*

$$\left|\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}-1\right|\leq(1-\gamma)\cos\eta,\ z\in\mathbb{U} \tag{2.2}$$

then $f\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$.

In the following theorem, we obtain a sufficient condition for f to be in $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$.

Theorem 2.3. *A function $f(z)$ of the form (1.1) is in $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ if*

$$\sum_{n=2}^{\infty}[([n]_q-\lambda)\sec\eta+(1-\gamma)\lambda]\Psi_q(n,\alpha)|a_n|\leq1-\gamma, \tag{2.3}$$

where $|\eta|<\frac{\pi}{2}$, $0\leq\lambda\leq1$, $0\leq\gamma<1$ and $\Psi_q(n,\alpha)$ is given by (1.10).

Proof. In virtue of Corollary 2.2, it suffices to show that the condition (2.3) is satisfied. We have

$$\left|\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}-1\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}([n]_q-\lambda)\Psi_q(n,\alpha)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\lambda\Psi_q(n,\alpha)a_nz^{n-1}}\right|$$

$$<\frac{\sum_{n=2}^{\infty}([n]_q-\lambda)\Psi_q(n,\alpha)|a_n|}{1-\sum_{n=2}^{\infty}\lambda\Psi_q(n,\alpha)|a_n|}.$$

The last expression is bounded above by $(1-\gamma)\cos\eta$, if

$$\sum_{n=2}^{\infty}([n]_q-\lambda)\Psi_q(n,\alpha)|a_n|\leq(1-\gamma)\cos\eta\left(1-\sum_{n=2}^{\infty}\lambda\Psi_q(n,\alpha)|a_n|\right)$$

which is equivalent to

$$\sum_{n=2}^{\infty}[([n]_q-\lambda)\sec\eta+(1-\gamma)\lambda]\Psi_q(n,\alpha)|a_n|\leq 1-\gamma.$$

This completes the proof of the Theorem 2.3. □

In view of Examples 1.1 and 1.2, we state the following corollaries.

Corollary 2.4. *A function $f(z)$ of the form (1.1) is in $\mathscr{S}_q^{\alpha}(\eta,\gamma)$ if*

$$\sum_{n=2}^{\infty}(([n]_q-1)\sec\eta+(1-\gamma))\Psi_q(n,\alpha)|a_n|\leq 1-\gamma,$$

where $|\eta|<\frac{\pi}{2}$, and $0\leq\gamma<1$.

Corollary 2.5. *A function $f(z)$ of the form (1.1) is in $\mathscr{R}_q^{\alpha}(\eta,\gamma)$ if*

$$\sum_{n=2}^{\infty}([n]_q\sec\eta)\Psi_q(n,\alpha)|a_n|\leq 1-\gamma,$$

where $|\eta|<\frac{\pi}{2}$ and $0\leq\gamma<1$.

Remark 2.6. *We observe that Corollary 2.4, yields the result of Silverman [13] for special values of η and γ.*

3. Subordination result

Before stating and proving our subordination result for the class $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$, we need the following definitions and a lemma due to Wilf [17].

Definition 3.1. *Let $g,h\in\mathscr{A}$. The function g is said to be subordinate to the function h, denoted by $g\prec h$, if there exists a function $w\in\mathscr{B}$ such that $g(z)=h(w(z))$, for all $z\in\mathbb{U}$.*

Definition 3.2. *[17]. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)=\sum_{n=1}^{\infty}a_nz^n, a_1=1$ is regular, univalent and convex in $\mathbb{U}$, we have*

$$\sum_{n=1}^{\infty}b_na_nz^n\prec f(z),\ z\in\mathbb{U}. \tag{3.1}$$

Lemma 3.3. *[17] The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\Re\left\{1+2\sum_{n=1}^{\infty}b_nz^n\right\}>0,\ z\in\mathbb{U}. \tag{3.2}$$

Theorem 3.4. *Let $f\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ and $g(z)$ be any function in the usual class of convex functions C, then*

$$\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)]}(f*g)(z)\prec g(z) \tag{3.3}$$

where $|\eta|<\frac{\pi}{2}, 0\leq\gamma<1, 0\leq\lambda<1$, with

$$\Psi_q(2,\alpha)=\frac{\Gamma_q(2+\alpha)}{\Gamma_q(1+\alpha)} \tag{3.4}$$

and

$$\Re\{f(z)\} > -\frac{1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)},\ z\in\mathbb{U}. \tag{3.5}$$

The constant factor $\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)]}$ *in (3.3) cannot be replaced by a larger number.*

Proof. Let $f\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ satisfy the coefficient inequality (2.3) and suppose that $g(z)=z+\sum_{n=2}^{\infty}c_nz^n\in C$. Then, by Definition 3.2, the subordination (3.3) of our theorem will hold true if the sequence

$$\left\{\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 3.3, it is equivalent to the inequality

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}a_nz^n\right\}>0,\ z\in\mathbb{U}. \tag{3.6}$$

By noting the fact that $\frac{[([n]_q-\lambda)\sec\eta+(1-\gamma)\lambda]\Psi_q(n,\alpha)}{(1-\gamma)}$ is an increasing function for $n\geq 2$ and in view of (2.3), when $|z|=r<1$, we obtain

$$\begin{aligned}
&\Re\left\{1+\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}\sum_{n=1}^{\infty}a_nz^n\right\}\\
&\geq 1-\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}r\\
&-\frac{1-\gamma}{1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}r>0,\ |z|=r<1.
\end{aligned}$$

This evidently proves the inequality (3.6) and hence also the subordination result (3.3) asserted by Theorem 3.4. The inequality (3.5) follows from (3.3) by taking $g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in C$.

The sharpness of the multiplying factor in (3.3) can be established by considering a function $F(z)=z-\frac{1-\gamma}{1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}z^2$, where $|\eta|<\frac{\pi}{2}, 0\leq\gamma<1, 0\leq\lambda\leq 1$ and $\Psi_q(2,\alpha)$ is given by (3.4). Clearly $F\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$. Using (3.3) we infer that

$$\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)]}F(z)\prec\frac{z}{1-z},$$

and it follows that

$$\min\left\{\Re\left(\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)]}F(z)\right)\right\}=-\frac{1}{2},\ z\in\mathbb{U}.$$

This shows that the constant $\frac{(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(([2]_q-\lambda)\sec\eta+\lambda(1-\gamma))\Psi_q(2,\alpha)]}$ cannot be replaced by any larger one. □

For $\lambda=1$, we state the following corollary.

Corollary 3.5. *If $f\in\mathscr{S}_q^{\alpha}(\eta,\gamma)$, then*

$$\frac{(q\sec\eta+(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(q\sec\eta+(1-\gamma))\Psi_q(2,\alpha)]}(f*g)(z)\prec g(z) \tag{3.7}$$

where $|\eta|<\frac{\pi}{2}, 0\leq\gamma<1, g\in C$ and

$$\Re\{f(z)\}>-\frac{1-\gamma+(q\sec\eta+(1-\gamma))\Psi_q(2,\alpha)}{(q\sec\eta+(1-\gamma))\Psi_q(2,\alpha)},\ z\in\mathbb{U}.$$

The constant factor $\frac{(q\sec\eta+(1-\gamma))\Psi_q(2,\alpha)}{2[1-\gamma+(q\sec\eta+(1-\gamma))\Psi_q(2,\alpha)]}$ *in (3.7) cannot be replaced by a larger one.*

By taking $\lambda=0$, we state the next corollary.

Corollary 3.6. *If $f\in\mathscr{R}_q^{\alpha}(\eta,\gamma,\lambda)$, then*

$$\frac{(1+q)\sec\eta\Psi_q(2,\alpha)}{2[1-\gamma+(1+q)\sec\eta\Psi_q(2,\alpha)]}(f*g)(z)\prec g(z) \tag{3.8}$$

where $|\eta|<\frac{\pi}{2}, 0\leq\gamma<1, g\in C$ and

$$\Re\{f(z)\}>-\frac{1-\gamma+(1+q)\sec\eta\Psi_q(2,\alpha)}{(1+q)\sec\eta\Psi_q(2,\alpha)},\ z\in\mathbb{U}.$$

The constant factor $\frac{(1+q)\sec\eta\Psi_q(2,\alpha)}{2[1-\gamma+(1+q)\sec\eta\Psi_q(2,\alpha)]}$ *in (3.8) cannot be replaced by a larger one.*

4. The Fekete-Szegö problem

The Fekete-Szegö problem consists in finding sharp upper bounds for the functional $|a_3-\mu a_2^2|$ for various subclasses of $\mathscr{A}$ (see [10], [16]). In order to obtain sharp upper-bounds for $|a_3-\mu a_2^2|$ for the class $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ the following lemma is required (see, e.g., [9], p.108).

Lemma 4.1. *Let the function $w\in\mathscr{B}$ be given by*

$$w(z)=\sum_{n=1}^{\infty}w_nz^n,\ \ z\in\mathbb{U}.$$

Then

$$|w_1|\leq 1\ \ and\ \ |w_2|\leq 1-|w_1|^2 \tag{4.1}$$

and

$$|w_2-sw_1^2|\leq \max\{1,|s|\}\ \ for\ any\ complex\ number\ s. \tag{4.2}$$

The functions $w(z)=z$ and $w(z)=z^2$ or one of their rotations show that both inequalities (4.1) and (4.2) are sharp.

For the constants γ,η with $0\leq\gamma<1$ and $|\eta|<\dfrac{\pi}{2}$ denote

$$p_{\gamma,\eta}(z)=\frac{1+e^{-i\eta}(e^{-i\eta}-2\gamma\cos\eta)z}{1-z},\ \ z\in\mathbb{U}. \tag{4.3}$$

The function $p_{\gamma,\eta}(z)$ maps the open unit disk onto the half-plane

$$H_{\gamma,\eta}=\left\{z\in\mathbb{C}:\Re(e^{i\eta}z)>\gamma\cos\eta\right\}.$$

If

$$p_{\gamma,\eta}(z)=1+\sum_{n=1}^{\infty}p_nz^n$$

then it is easy to check that

$$p_n=2e^{-i\eta}(1-\gamma)\cos\eta,\ \ \text{for all}\ \ n\geq 1. \tag{4.4}$$

First we obtain sharp upper-bounds for the Fekete-Szegö functional $|a_3-\mu a_2^2|$ with μ real parameter .

Theorem 4.2. *Let $f\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ be given by (1.1) and let μ be a real number. Then*

$$|a_3-\mu a_2^2|\leq$$

$$\begin{cases}\dfrac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}\left[1+\dfrac{2(1-\gamma)\lambda}{1+q-\lambda}-\mu\dfrac{2(1-\gamma)(1+q+q^2-\lambda)}{(1+q-\lambda)^2}\cdot\dfrac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}\right], & if\ \mu\leq\sigma_1\\ \dfrac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}, & if\ \sigma_1\leq\mu\leq\sigma_2\\ \dfrac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}\left[\mu\dfrac{2(1-\gamma)(1+q+q^2-\lambda)}{(1+q-\lambda)^2}\cdot\dfrac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}+\dfrac{2(1-\gamma)\lambda}{1+q-\lambda}-1\right], & if\ \mu\geq\sigma_2\end{cases} \tag{4.5}$$

where

$$\sigma_1=\frac{\lambda(1+q-\lambda)}{1+q+q^2-\lambda}\cdot\frac{\Psi_q^2(2,\alpha)}{\Psi_q(3,\alpha)} \tag{4.6}$$

$$\sigma_2=\frac{(1+q-\lambda)(1+q-\lambda\gamma)}{(1-\gamma)(1+q+q^2-\lambda)}\cdot\frac{\Psi_q^2(2,\alpha)}{\Psi_q(3,\alpha)} \tag{4.7}$$

and $\Psi_q(2,\alpha),\Psi_q(3,\alpha)$ are defined by (1.10) with $n=2$ and $n=3$ respectively. All estimates are sharp.

Proof. Suppose that $f\in\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ is given by (1.1). Then, from the definition of the class $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$, there exists $w\in\mathscr{B}$, $w(z)=w_1z+w_2z^2+w_3z^3+\ldots$ such that

$$\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}=p_{\gamma,\eta}(w(z)),\ \ z\in\mathbb{U}. \tag{4.8}$$

We have

$$\frac{z\mathscr{D}_q(\mathscr{R}_q^{\alpha}f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^{\alpha}f(z)}$$

$$=1+([2]_q-\lambda)\Psi_q(2,\alpha)a_2z+[(\lambda^2-[2]_q\lambda)\Psi_q^2(2,\alpha)a_2^2+([3]_q-\lambda)\Psi_q(3,\alpha)a_3]z^2+\ldots$$

or

$$\frac{z\mathscr{D}_q(\mathscr{R}_q^\alpha f(z))}{(1-\lambda)z+\lambda\mathscr{R}_q^\alpha f(z)}$$

$$=1+(1+q-\lambda)\Psi_q(2,\alpha)a_2z+[(\lambda^2-q\lambda-\lambda)\Psi_q^2(2,\alpha)a_2^2+(1+q+q^2-\lambda)\Psi_q(3,\alpha)a_3]z^2+\ldots. \tag{4.9}$$

Set $p_{\gamma,\eta}(z)=1+p_1z+p_2z^2+p_3z^3+\ldots$. From (4.4) we have

$$p_1=p_2=2e^{-i\eta}(1-\gamma)\cos\eta.$$

Equating the coefficients of z and z^2 on both sides of (4.8) and taking into account (4.9), we obtain

$$a_2=\frac{p_1w_1}{(1+q-\lambda)\Psi_q(2,\alpha)} \text{ and } a_3=\frac{1}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}\left[p_1w_2+\left(p_2+\frac{\lambda}{1+q-\lambda}p_1^2\right)w_1^2\right].$$

and thus we obtain

$$a_2=\frac{2e^{-i\eta}(1-\gamma)\cos\eta}{(1+q-\lambda)\Psi_q(2,\alpha)}w_1 \tag{4.10}$$

and

$$a_3=\frac{2e^{-i\eta}(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}\left[w_2+\left(1+\frac{2\lambda e^{-i\eta}(1-\gamma)\cos\eta}{1+q-\lambda}\right)w_1^2\right]. \tag{4.11}$$

It follows

$$|a_3-\mu a_2^2|\le\frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}$$

$$\times\left\{|w_2|+\left|1+\frac{2e^{-i\eta}(1-\gamma)\cos\eta}{1+q-\lambda}\left(\lambda-\mu\frac{1+q+q^2-\lambda}{1+q-\lambda}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}\right)\right||w_1|^2\right\}.$$

Making use of Lemma 4.1 (4.1) we have

$$|a_3-\mu a_2^2|\le\frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}$$

$$\times\left\{1+\left[\left|1+\frac{2e^{-i\eta}(1-\gamma)\cos\eta}{1+q-\lambda}\left(\lambda-\mu\frac{1+q+q^2-\lambda}{1+q-\lambda}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}\right)\right|-1\right]|w_1|^2\right\}.$$

or

$$\left|a_3-\mu a_2^2\right|\le\frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\eta)\Psi_q(3,\alpha)}\left[1+\left(\sqrt{1+M(2+M)\cos^2\eta}-1\right)|w_1|^2\right], \tag{4.12}$$

where

$$M=\frac{2(1-\gamma)}{1+q-\lambda}\left(\lambda-\mu\frac{1+q+q^2-\lambda}{1+q-\lambda}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}\right). \tag{4.13}$$

Denote by

$$F(x,y)=1+\left(\sqrt{1+M(2+M)x^2}-1\right)y^2 \text{ where } x=\cos\eta,\ y=|w_1| \text{ and } (x,y)\in[0,1]\times[0,1].$$

Simple calculation shows that the function $F(x,y)$ does not have a local maximum at any interior point of the open rectangle $(0,1)\times(0,1)$. Thus, the maximum must be attained at a boundary point. Since $F(x,0)=1, F(0,y)=1$ and $F(1,1)=|1+M|$, it follows that the maximal value of $F(x,y)$ may be $F(0,0)=1$ or $F(1,1)=|1+M|$.
Therefore, from (4.12) we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\eta)\Psi_q(3,\alpha)}\max\{1,|1+M|\}. \tag{4.14}$$

where M is given by (4.13).
Consider first the case $|1+M|\ge1$. If $\mu\le\sigma_1$, where σ_1 is given by (4.6), then $M\ge0$ and from (4.14) we obtain

$$|a_3-\mu a_2^2|\le\frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}\left[1+\frac{2(1-\gamma)\lambda}{1+q-\lambda}-\mu\frac{2(1-\gamma)(1+q+q^2-\lambda)}{(1+q-\lambda)^2}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}\right]$$

which is the first part of the inequality (4.5). If $\mu\ge\sigma_2$, where σ_2 is given by (4.7), then $M\le-2$ and it follows from (4.14) that

$$|a_3-\mu a_2^2|\le\frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}\left[\mu\frac{2(1-\gamma)(1+q+q^2-\lambda)}{(1+q-\lambda)^2}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}+\frac{2(1-\gamma)\lambda}{1+q-\lambda}-1\right]$$

and this is the third part of (4.5).
Next, suppose $\sigma_1 \le \mu \le \sigma_2$. Then, $|1+M| \le 1$ and thus, from (4.14) we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\lambda)\Psi_q(3,\alpha)}$$

which is the second part of the inequality (4.5).
In view of Lemma 4.1, the results are sharp for $w(z) = z$ and $w(z) = z^2$ or one of their rotations. □

Next, we consider the Fekete-Szegö problem for the class $\mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ with μ complex parameter.

Theorem 4.3. *Let $f \in \mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$ be given by (1.1) and let μ be a complex number. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\eta)\Psi_q(3,\alpha)}$$

$$\times \max\left\{1, \left|\frac{2(1-\gamma)\cos\eta}{1+q-\lambda}\left(\mu\frac{1+q+q^2-\lambda}{1+q-\lambda}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)} - \lambda\right) - e^{i\eta}\right|\right\} \tag{4.15}$$

The result is sharp.

Proof. Assume that $f \in \mathscr{G}_q^{\alpha}(\eta,\gamma,\lambda)$. Making use of (4.10) and (4.11) we obtain

$$\left|a_3 - \mu a_2^2\right| \le \frac{2(1-\gamma)\cos\eta}{(1+q+q^2-\eta)\Psi_q(3,\alpha)}$$

$$\times\left|w_2 - \left[-\frac{2e^{-i\eta}(1-\gamma)\cos\eta}{1+q-\lambda}\left(\lambda - \mu\frac{1+q+q^2-\lambda}{1+q-\lambda}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)}\right) - 1\right] w_1^2\right|$$

The inequality (4.15) follows as an application of Lemma 4.1(4.2) with

$$s = \frac{2e^{-i\eta}(1-\gamma)\cos\eta}{1+q-\lambda}\left(\mu\frac{1+q+q^2-\lambda}{1+q-\lambda}\cdot\frac{\Psi_q(3,\alpha)}{\Psi_q^2(2,\alpha)} - \lambda\right) - 1.$$

□

Remark 4.4. *By specializing the parameters $\lambda = 0$ and $\lambda = 1$ one can state the above discussed results for function f in the subclasses defined in Example 1.1 and 1.2 respectively.*

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