Konuralp Journal of Mathematics, 7 (1) (2019) 168-174

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Some Additive Inequalities for Heinz Operator Mean

Silvestru Sever Dragomir[1*]

[1]*Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia*

[*]*Corresponding author E-mail: sever.dragomir@vu.edu.au*

Abstract

In this paper we obtain some new additive inequalities for Heinz operator mean, namely the operator $H_\nu(A,B) := \frac{1}{2}(A\sharp_\nu B + A\sharp_{1-\nu}B)$ where $A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}$ is the weighted geometric mean for the positive invertible operators A and B, and $\nu \in [0,1]$.

Keywords: *Young's Inequality, Real functions, Arithmetic mean-Geometric mean inequality, Heinz means*

2010 Mathematics Subject Classification: *47A63, 47A30, 15A60, 26D15, 26D10.*

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We use the following notations for operators and $\nu \in [0,1]$

$$A\nabla_\nu B := (1-\nu)A + \nu B,$$

the *weighted operator arithmetic mean,* and

$$A\sharp_\nu B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2},$$

the *weighted operator geometric mean* [14]. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.
Define the *Heinz operator mean* by

$$H_\nu(A,B) := \frac{1}{2}(A\sharp_\nu B + A\sharp_{1-\nu}B).$$

The following interpolatory inequality is obvious

$$A\sharp B \le H_\nu(A,B) \le A\nabla B \tag{1.1}$$

for any $\nu \in [0,1]$.
We recall that *Specht's ratio* is defined by [16]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.
The following result provides an upper and lower bound for the Heinz mean in terms of the operator geometric mean $A\sharp B$:

Theorem 1.1 (Dragomir, 2015 [6]). *Assume that A and B are positive invertible operators and the constants $M > m > 0$ are such that*

$$mA \le B \le MA. \tag{1.3}$$

Then we have

$$\omega_\nu(m,M)A\sharp B \le H_\nu(A,B) \le \Omega_\nu(m,M)A\sharp B, \tag{1.4}$$

Email addresses: **sever.dragomir@vu.edu.au (Silvestru Sever Dragomir)**

where

$$\Omega_{\nu}(m,M) := \begin{cases} S\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \max\left\{S\left(m^{|2\nu-1|}\right), S\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \le 1 \le M, \\ S\left(M^{|2\nu-1|}\right) \text{ if } 1<m \end{cases} \tag{1.5}$$

and

$$\omega_{\nu}(m,M) := \begin{cases} S\left(M^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } M<1, \\ 1 \text{ if } m \le 1 \le M, \\ S\left(m^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } 1<m, \end{cases} \tag{1.6}$$

where $\nu \in [0,1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h>0. \tag{1.7}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h>0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h>0$.
We have:

Theorem 1.2 (Dragomir, 2015 [7])**.** *Assume that A and B are positive invertible operators and the constants $M>m>0$ are such that the condition (1.3) is valid. Then for any $\nu \in [0,1]$ we have*

$$(A\sharp B \le) H_{\nu}(A,B) \le \exp[\Theta_{\nu}(m,M) - 1] A\sharp B \tag{1.8}$$

where

$$\Theta_{\nu}(m,M) := \begin{cases} K\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \max\left\{K\left(m^{|2\nu-1|}\right), K\left(M^{|2\nu-1|}\right)\right\} \text{ if } m \le 1 \le M, \\ K\left(M^{|2\nu-1|}\right) \text{ if } 1<m \end{cases} \tag{1.9}$$

and

$$(0 \le) H_{\nu}(A,B) - A\sharp B \le \frac{1}{4m^{1-\nu}} \max_{x\in[m,M]} D\left(x^{2\nu-1}\right) A, \tag{1.10}$$

where the function $D:(0,\infty) \to [0,\infty)$ *is defined by* $D(x) = (x-1)\ln x$.

The following bounds for the Heinz mean $H_{\nu}(A,B)$ in terms of $A\nabla B$ are also valid:

Theorem 1.3 (Dragomir, 2015 [7])**.** *With the assumptions of Theorem 1.2 we have*

$$(0 \le) A\nabla B - H_{\nu}(A,B) \le \nu(1-\nu)\Upsilon(m,M) A, \tag{1.11}$$

where

$$\Upsilon(m,M) := \begin{cases} (m-1)\ln m \text{ if } M<1, \\ \max\{(m-1)\ln m, (M-1)\ln M\} \text{ if } m \le 1 \le M, \\ (M-1)\ln M \text{ if } 1<m \end{cases} \tag{1.12}$$

and

$$A\nabla B \exp[-4\nu(1-\nu)(\digamma(m,M)-1)] \le H_{\nu}(A,B) (\le A\nabla B) \tag{1.13}$$

where

$$\digamma(m,M) := \begin{cases} K(m) \text{ if } M<1, \\ \max\{K(m), K(M)\} \text{ if } m \le 1 \le M, \\ K(M) \text{ if } 1<m. \end{cases} \tag{1.14}$$

For other recent results on operator geometric mean inequalities, see [1]-[13], [15] and [17]-[18].
Motivated by the above results, we establish in this paper some inequalities for the quantities

$$H_{\nu}(A,B) - A\sharp B \text{ and } A\nabla B - H_{\nu}(A,B)$$

under various assumptions for positive invertible operators A and B.

2. Bounds for $H_\nu(A,B) - A\sharp B$

We first notice the following simple result:

Theorem 2.1. *Assume that A and B are positive invertible operators and the constants $M > m > 0$ are such that the condition (1.3) holds. If we consider the function $f_\nu : [0,\infty) \to \mathbb{R}$ for $\nu \in [0,1]$ defined by*

$$f_\nu(x) = \frac{1}{2}\left(x^\nu + x^{1-\nu}\right),$$

then we have

$$f_\nu(m)A \le H_\nu(A,B) \le f_\nu(M)A. \tag{2.1}$$

Proof. We observe that

$$f_\nu'(x) = \frac{1}{2}\left(\nu x^{\nu-1} + (1-\nu)x^{-\nu}\right),$$

which is positive for $x \in (0,\infty)$.
Therefore f_ν is increasing on $(0,\infty)$ and

$$f_\nu(m) = \min_{x\in[m,M]} f_\nu(x) \le f_\nu(x) \le \max_{x\in[m,M]} f_\nu(x) = f_\nu(M)$$

for any $x \in [m,M]$.
Using the continuous functional calculus, we have for any operator X with $mI \le X \le MI$ that

$$f_\nu(m)I \le \frac{1}{2}\left(X^\nu + X^{1-\nu}\right) \le f_\nu(M)I. \tag{2.2}$$

From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI \le A^{-1/2}BA^{-1/2} \le MI.$$

Now, writing the inequality (2.2) for $X = A^{-1/2}BA^{-1/2}$, we get

$$f_\nu(m)I \le \frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^\nu + \left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}\right] \le f_\nu(M)I. \tag{2.3}$$

Finally, if we multiply both sides of (2.3) by $A^{1/2}$ we get the desired result (2.1). □

Corollary 2.2. *Let A and B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*
(i) If $0 < mI \le A \le m'I < M'I \le B \le MI$, then

$$f_\nu(h')A \le H_\nu(A,B) \le f_\nu(h)A. \tag{2.4}$$

(ii) If $0 < mI \le B \le m'I < M'I \le A \le MI$, then

$$\frac{f_\nu(h)}{h}A \le H_\nu(A,B) \le \frac{f_\nu(h')}{h'}A. \tag{2.5}$$

Proof. If the condition (i) is valid, then we have for $X = A^{-1/2}BA^{-1/2}$

$$I < \frac{M'}{m'}I = h'I \le X \le hI = \frac{M}{m}I,$$

which, by (2.2) gives the desired result (2.4).
If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \le X \le \frac{1}{h'}I < I,$$

which, by (2.2) gives

$$f_\nu\left(\frac{1}{h}\right)A \le H_\nu(A,B) \le f_\nu\left(\frac{1}{h'}\right)A$$

that is equivalent to (2.5), since

$$f_\nu\left(\frac{1}{h}\right) = \frac{f_\nu(h)}{h}.$$

□

We need the following lemma in order to prove our first main result:

Lemma 2.3. *Consider the function $g_\nu : [0,\infty) \to \mathbb{R}$ for $\nu \in (0,1)$ defined by*

$$g_\nu(x) = \frac{1}{2}\left(x^\nu + x^{1-\nu}\right) - \sqrt{x} \geq 0. \tag{2.6}$$

Then $g_\nu(0) = g_\nu(1) = 0$, g_ν is increasing on $(0, x_\nu)$ with a local maximum in

$$x_\nu := \left(\frac{\nu}{1-\nu}\right)^{\frac{2}{1-2\nu}} \in (0,1), \tag{2.7}$$

is decreasing on $(x_\nu, 1)$ with a local minimum in $x = 1$ and increasing on $(1,\infty)$ with $\lim_{x\to\infty} g_\nu(x) = \infty$.

Proof. (i). If $\nu \in \left(0, \frac{1}{2}\right)$, then

$$g_\nu'(x) = \frac{1}{2}\left(\frac{\nu}{x^{1-\nu}} + \frac{1-\nu}{x^\nu} - \frac{1}{x^{1/2}}\right)$$
$$= \frac{1}{2}\frac{\nu + (1-\nu)x^{1-2\nu} - x^{\frac{1-2\nu}{2}}}{x^{1-\nu}}.$$

If we denote $u = x^{\frac{1-2\nu}{2}}$, then we have

$$\nu + (1-\nu)x^{1-2\nu} - x^{\frac{1-2\nu}{2}} = (1-\nu)u^2 - u + \nu.$$
$$= (1-\nu)\left(u - \frac{\nu}{1-\nu}\right)(u-1)$$
$$= (1-\nu)\left(x^{\frac{1-2\nu}{2}} - \frac{\nu}{1-\nu}\right)\left(x^{\frac{1-2\nu}{2}} - 1\right).$$

We observe that $g_\nu'(x) = 0$ only for $x = 1$ and $x_\nu = \left(\frac{\nu}{1-\nu}\right)^{\frac{2}{1-2\nu}} \in (0,1)$. Also $g_\nu'(x) > 0$ for $x \in (0,x_\nu) \cup (1,\infty)$ and $g_\nu'(x) < 0$ for $x \in (x_\nu, 1)$. These imply the desired conclusion.
(ii) If $\nu \in \left(\frac{1}{2}, 1\right)$, then

$$g_\nu'(x) = \frac{1}{2}\frac{1-\nu + \nu x^{2\nu-1} - x^{\frac{2\nu-1}{2}}}{x^\nu}.$$

If we denote $z = x^{\frac{2\nu-1}{2}}$, then we have

$$1-\nu + \nu x^{2\nu-1} - x^{\frac{2\nu-1}{2}} = \nu z^2 - z + 1 - \nu$$
$$= \nu\left(z - \frac{1-\nu}{\nu}\right)(z-1)$$
$$= \nu\left(x^{\frac{2\nu-1}{2}} - \frac{1-\nu}{\nu}\right)\left(x^{\frac{2\nu-1}{2}} - 1\right).$$

We observe that $g_\nu'(x) = 0$ only for $x = 1$ and $x_\nu = \left(\frac{1-\nu}{\nu}\right)^{\frac{2}{2\nu-1}} = \left(\frac{\nu}{1-\nu}\right)^{\frac{2}{1-2\nu}} \in (0,1)$. Also $g_\nu'(x) > 0$ for $x \in (0,x_\nu) \cup (1,\infty)$ and $g_\nu'(x) < 0$ for $x \in (x_\nu, 1)$. These imply the desired conclusion. □

The above lemma allows us to obtain various bounds for the nonnegative quantity

$$H_\nu(A,B) - A\sharp B$$

when some conditions for the involved operators A and B are known.

Theorem 2.4. *Assume that A and B are positive invertible operators with $B \leq A$. Then for $\nu \in (0,1)$ we have*

$$(0 \leq) H_\nu(A,B) - A\sharp B \leq g_\nu(x_\nu)A, \tag{2.8}$$

where g_ν is defined by (2.6) and x_ν by (2.7).

Proof. From Lemma 2.3 we have for $\nu \in (0,1)$ that

$$0 \leq \frac{1}{2}\left(x^\nu + x^{1-\nu}\right) - \sqrt{x} \leq g_\nu(x_\nu)$$

for any $x \in [0,1]$.
Using the continuous functional calculus, we have for any operator X with $0 \leq X \leq I$ that

$$0 \leq \frac{1}{2}\left(X^\nu + X^{1-\nu}\right) - X^{1/2} \leq g_\nu(x_\nu) \tag{2.9}$$

for $\nu \in (0,1)$.
By multiplying both sides of the inequality $0 \leq B \leq A$ with $A^{-1/2}$ we get

$$0 \leq A^{-1/2}BA^{-1/2} \leq I.$$

If we use the inequality (2.9) for $X=A^{-1/2}BA^{-1/2}$, then we get

$$0\leq\frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{\nu}+\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}\right]-\left(A^{-1/2}BA^{-1/2}\right)^{1/2} \leq g_{\nu}\left(x_{\nu}\right)I \tag{2.10}$$

for $\nu\in(0,1)$.
Finally, if we multiply both sides of (2.10) with $A^{1/2}$, then we get the desired result (2.8). □

Theorem 2.5. *Assume that A and B are positive invertible operators and the constants $M>m\geq 0$ are such that the condition (1.3) holds. Let $\nu\in(0,1)$.*
(i) If $0\leq m<M\leq 1$, then

$$\gamma_{\nu}(m,M)A\leq H_{\nu}(A,B)-A\sharp B\leq\Gamma_{\nu}(m,M)A, \tag{2.11}$$

where

$$\gamma_{\nu}(m,M):=\begin{cases} g_{\nu}(m) \text{ if } 0\leq m<M\leq x_{\nu},\\ \min\{g_{\nu}(m),g_{\nu}(M)\} \text{ if } 0\leq m\leq x_{\nu}\leq M\leq 1,\\ g_{\nu}(M) \text{ if } x_{\nu}\leq m<M\end{cases} \tag{2.12}$$

and

$$\Gamma_{\nu}(m,M):=\begin{cases} g_{\nu}(M) \text{ if } 0\leq m<M\leq x_{\nu},\\ g_{\nu}(x_{\nu}) \text{ if } 0\leq m\leq x_{\nu}\leq M\leq 1,\\ g_{\nu}(m) \text{ if } x_{\nu}\leq m\leq M\leq 1,\end{cases} \tag{2.13}$$

where g_{ν} is defined by (2.6) and x_{ν} by (2.7).
(ii) If $1\leq m<M<\infty$, then

$$g_{\nu}(m)A\leq H_{\nu}(A,B)-A\sharp B\leq g_{\nu}(M)A. \tag{2.14}$$

Proof. (i) If $0\leq m<M\leq 1$ then by Lemma 2.3 we have for $\nu\in(0,1)$ that

$$\begin{cases} g_{\nu}(m) \text{ if } 0\leq m<M\leq x_{\nu}\\ \min\{g_{\nu}(m),g_{\nu}(M)\} \text{ if } 0\leq m\leq x_{\nu}\leq M\leq 1\\ g_{\nu}(M) \text{ if } x_{\nu}\leq m<M\end{cases}$$
$$\leq g_{\nu}(x)$$
$$\leq\begin{cases} g_{\nu}(M) \text{ if } 0\leq m<M\leq x_{\nu}\\ g_{\nu}(x_{\nu}) \text{ if } 0\leq m\leq x_{\nu}\leq M\leq 1\\ g_{\nu}(m) \text{ if } x_{\nu}\leq m<M\leq 1\end{cases}$$

for any $x\in[m,M]$.
Now, on making use of a similar argument to the one in the proof of Theorem 2.4, we obtain the desired result (2.13).
(ii) Obvious by the properties of function g_{ν}. □

The interested reader may obtain similar bounds for other locations of $0\leq m<M<\infty$. The details are omitted.
The following particular case holds:

Corollary 2.6. *Let A and B be two positive operators. For positive real numbers m, m', M, M', put $h:=\frac{M}{m}$, $h':=\frac{M'}{m'}$ and let $\nu\in(0,1)$.*
(i) If $0<mI\leq A\leq m'I<M'I\leq B\leq MI$, then

$$g_{\nu}\left(h'\right)A\leq H_{\nu}(A,B)-A\sharp B\leq g_{\nu}(h)A. \tag{2.15}$$

(ii) If $0<mI\leq B\leq m'I<M'I\leq A\leq MI$, then

$$\tilde{\gamma}_{\nu}\left(h,h'\right)A\leq H_{\nu}(A,B)-A\sharp B\leq\tilde{\Gamma}_{\nu}\left(h,h'\right)A, \tag{2.16}$$

where

$$\tilde{\gamma}_{\nu}\left(h,h'\right):=\begin{cases} \frac{g_{\nu}(h)}{h} \text{ if } 0\leq\frac{1}{h}<\frac{1}{h'}\leq x_{\nu},\\ \min\left\{\frac{g_{\nu}(h)}{h},\frac{g_{\nu}(h')}{h'}\right\} \text{ if } 0\leq\frac{1}{h}\leq x_{\nu}\leq\frac{1}{h'}\leq 1,\\ \frac{g_{\nu}(h')}{h'} \text{ if } x_{\nu}\leq\frac{1}{h}<\frac{1}{h'}\end{cases} \tag{2.17}$$

and

$$\tilde{\Gamma}_\nu\left(h,h'\right) := \begin{cases} \frac{g_\nu(h')}{h'} \text{ if } 0 \le \frac{1}{h} < \frac{1}{h'} \le x_\nu, \\ g_\nu\left(x_\nu\right) \text{ if } 0 \le \frac{1}{h} \le x_\nu \le \frac{1}{h'} \le 1, \\ \frac{g_\nu(h)}{h} \text{ if } x_\nu \le \frac{1}{h} < \frac{1}{h'} \le 1. \end{cases} \tag{2.18}$$

3. Bounds for $A\nabla B - H_\nu\left(A,B\right)$

In order to provide some upper and lower bounds for the quantity

$A\nabla B - H_\nu\left(A,B\right)$

where A and B are positive invertible operators, we need the following lemma.

Lemma 3.1. *Consider the function $h_\nu : [0,\infty) \to \mathbb{R}$ for $\nu \in (0,1)$ defined by*

$$h_\nu\left(x\right) = \frac{x+1}{2} - \frac{1}{2}\left(x^\nu + x^{1-\nu}\right) \ge 0. \tag{3.1}$$

Then h_ν is decreasing on $[0,1)$ and increasing on $(1,\infty)$ with $x=1$ its global minimum. We have $h_\nu\left(0\right) = \frac{1}{2}$, $\lim_{x\to\infty} h_\nu\left(x\right) = \infty$ and h_ν is convex on $(0,\infty)$.

Proof. We have

$$h'_\nu\left(x\right) = \frac{1}{2}\left(1 - \frac{\nu}{x^{1-\nu}} - \frac{1-\nu}{x^\nu}\right)$$

and

$$h''_\nu\left(x\right) = \frac{1}{2}\nu\left(1-\nu\right)\left(x^{\nu-2} + x^{-\nu-1}\right)$$

for any $x \in (0,\infty)$ and $\nu \in (0,1)$.
We observe that $h'_\nu\left(1\right) = 0$ and $h''_\nu\left(x\right) > 0$ for any $x \in (0,\infty)$ and $\nu \in (0,1)$. These imply that the equation $h'_\nu\left(x\right) = 0$ has only one solution on $(0,\infty)$, namely $x=1$. Since $h'_\nu\left(x\right) < 0$ for $x \in (0,1)$ and $h'_\nu\left(x\right) > 0$ for $x \in (1,\infty)$, then we deduce the desired conclusion. □

Theorem 3.2. *Assume that A and B are positive invertible operators, the constants $M > m \ge 0$ are such that the condition (1.3) holds and $\nu \in (0,1)$. Then we have*

$$\delta_\nu\left(m,M\right)A \le A\nabla B - H_\nu\left(A,B\right) \le \Delta_\nu\left(m,M\right)A, \tag{3.2}$$

where

$$\delta_\nu\left(m,M\right) := \begin{cases} h_\nu\left(M\right) \text{ if } M < 1, \\ 0 \text{ if } m \le 1 \le M, \\ h_\nu\left(m\right) \text{ if } 1 < m \end{cases} \tag{3.3}$$

and

$$\Delta_\nu\left(m,M\right) := \begin{cases} h_\nu\left(m\right) \text{ if } M < 1, \\ \max\left\{h_\nu\left(m\right), h_\nu\left(M\right)\right\} \text{ if } m \le 1 \le M, \\ h_\nu\left(M\right) \text{ if } 1 < m, \end{cases} \tag{3.4}$$

where h_ν is defined by (3.1).

Proof. Using Lemma 3.1 we have

$$\begin{cases} h_\nu\left(M\right) \text{ if } M < 1, \\ 0 \text{ if } m \le 1 \le M, \\ h_\nu\left(m\right) \text{ if } 1 < m, \end{cases} \le h_\nu\left(x\right)$$

$$\le \begin{cases} h_\nu\left(m\right) \text{ if } M < 1, \\ \max\left\{h_\nu\left(m\right), h_\nu\left(M\right)\right\} \text{ if } m \le 1 \le M, \\ h_\nu\left(M\right) \text{ if } 1 < m \end{cases}$$

for any $x \in [m,M]$ and $\nu \in (0,1)$.
Using the continuous functional calculus, we have for any operator X with $mI \le X \le MI$ that

$$\delta_\nu (m,M) I \le \frac{X+I}{2} - \frac{1}{2}\left(X^\nu + X^{1-\nu}\right) \le \Delta_\nu (m,M) I. \tag{3.5}$$

From (1.3) we have, by multiplying both sides with $A^{-1/2}$ that

$$mI \le A^{-1/2} B A^{-1/2} \le MI.$$

Now, writing the inequality (3.5) for $X = A^{-1/2} B A^{-1/2}$, we get

$$\begin{aligned} &\delta_\nu (m,M) I \\ &\le \frac{A^{-1/2}BA^{-1/2}+I}{2} - \frac{1}{2}\left(\left(A^{-1/2}BA^{-1/2}\right)^\nu + \left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}\right) \\ &\le \Delta_\nu (m,M) I. \end{aligned} \tag{3.6}$$

Finally, if we multiply both sides of (3.6) by $A^{1/2}$ we get the desired result (3.2). □

Corollary 3.3. *Let A and B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in (0,1)$.*
(i) If $0 < mI \le A \le m'I < M'I \le B \le MI$, then

$$h_\nu \left(h'\right) A \le A\nabla B - H_\nu (A,B) \le h_\nu (h) A. \tag{3.7}$$

(ii) If $0 < mI \le B \le m'I < M'I \le A \le MI$, then

$$\frac{h_\nu (h')}{h'} A \le A\nabla B - H_\nu (A,B) \le \frac{h_\nu (h)}{h} A. \tag{3.8}$$

Acknowledgement. The author would like to thank the anonymous referees for their suggestions that have been implemented in the final version of the paper.

References

[1] S. S. Dragomir, Some new reverses of Young's operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 130. [http://rgmia.org/papers/v18/v18a130.pdf].
[2] S. S. Dragomir, On new refinements and reverses of Young's operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 135. [http://rgmia.org/papers/v18/v18a135.pdf].
[3] S. S. Dragomir, Some inequalities for operator weighted geometric mean, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 139. [http://rgmia.org/papers/v18/v18a139.pdf].
[4] S. S. Dragomir, Refinements and reverses of Hölder-McCarthy operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 143. [http://rgmia.org/papers/v18/v18a143.pdf].
[5] S. S. Dragomir, Some reverses and a refinement of Hölder operator inequality, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 147. [http://rgmia.org/papers/v18/v18a147.pdf].
[6] S. S. Dragomir, Some inequalities for Heinz operator mean, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 163. [Online http://rgmia.org/papers/v18/v18a163.pdf].
[7] S. S. Dragomir, Further inequalities for Heinz operator mean, Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 167. [Online http://rgmia.org/papers/v18/v18a167.pdf].
[8] S. Furuichi, On refined Young inequalities and reverse inequalities, *J. Math. Inequal.* **5** (2011), 21-31.
[9] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.
[10] F. Kittaneh and Y. Manasrah, Improved Young and Heinz inequalities for matrix, *J. Math. Anal. Appl.* **361** (2010), 262-269.
[11] F. Kittaneh and Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Lin. Multilin. Alg.,* **59** (2011), 1031-1037.
[12] F. Kittaneh, M. Krnić, N. Lovričević and J. Pečarić, Improved arithmetic-geometric and Heinz means inequalities for Hilbert space operators, *Publ. Math. Debrecen*, 2012, **80**(3-4), 465–478.
[13] M. Krnić and J. Pečarić, Improved Heinz inequalities via the Jensen functional, *Cent. Eur. J. Math.* **11** (9) 2013,1698-1710.
[14] F. Kubo and T. Ando, Means of positive operators, *Math. Ann.* **264** (1980), 205–224.
[15] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
[16] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.
[17] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.H.
[18] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.,* **5** (2011), 551-556.