

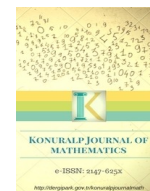
PAPER DETAILS

TITLE: Close-to-Convex Functions By Means Of Bounded Boundary Rotation

AUTHORS: Yasar POLATOGLU,Asena ÇETINKAYA,Oya MERT

PAGES: 136-139

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/698046>



Close-to-Convex Functions By Means Of Bounded Boundary Rotation

Yaşar Polatoğlu¹, Asena Çetinkaya¹ and Oya Mert^{2*}

¹Department of Mathematics and Computer Sciences, İstanbul Kültür University, İstanbul, Turkey

²Department of Basic Sciences, Altınbaş University, İstanbul, Turkey

*Corresponding author E-mail: oya.mert@altinbas.edu.tr

Abstract

In the present paper, we study on the class $\mathcal{CC}_k(A, B)$ of the close-to-convex functions with bounded boundary rotation. We investigate distortion theorem, growth theorem and coefficient inequality for the class $\mathcal{CC}_k(A, B)$.

Keywords: Bounded boundary rotation, Coefficient inequality, Distortion theorem, Growth theorem

2010 Mathematics Subject Classification: 30C45.

1. Introduction

Let \mathcal{A} be the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and let g be an element of \mathcal{A} . If g satisfies the condition

$$\operatorname{Re} \left(1 + z \frac{g''(z)}{g'(z)} \right) > 0,$$

then g is called convex function. The class of such functions is denoted by \mathcal{C} . For more details of convex functions one may refer to [1].

Let Ω be the family of functions ϕ regular in \mathbb{D} and satisfying the condition $\phi(0) = 0$ and $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. For arbitrary fixed number A, B , $-1 \leq B < A \leq 1$ denote by $\mathcal{P}(A, B)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 \dots$, analytic in \mathbb{D} such that p in $\mathcal{P}(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}$$

for some function $\phi \in \Omega$ and every $z \in \mathbb{D}$. A function p of the form $p(z) = 1 + p_1 z + p_2 z^2 \dots$, analytic in \mathbb{D} with $p(0) = 1$ is said to be in the class of $\mathcal{P}_k(A, B)$, $k \geq 2$, $-1 \leq B < A \leq 1$ if and only if there exists $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}(A, B)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2} \right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2} \right) p_2^{(2)}(z).$$

We now give the following definition.

Definition 1.1. Let f of the form (1.1) be an element of \mathcal{A} . Then $f \in \mathcal{CC}_k(A, B)$, if there exists a function $g \in \mathcal{C}$ such that

$$\frac{f'(z)}{g'(z)} = p(z), \quad p \in \mathcal{P}_k(A, B), \quad (1.2)$$

with $k \geq 2$, $-1 \leq B < A \leq 1$, then f is called close-to-convex function with bounded boundary rotation denoted by $\mathcal{CC}_k(A, B)$.

We investigate distortion theorem, growth theorem and coefficient inequality for the class $\mathcal{CC}_k(A, B)$. Details of bounded boundary rotation can be found in [3],[4].

2. Main Results

Let p be an element of $\mathcal{P}(A, B)$ and $|z| = r < 1$, then we have

$$\begin{aligned} \frac{1-Ar}{1-Br} \leq \operatorname{Rep}(z) \leq |p(z)| \leq \frac{1+Ar}{1+Br}, \quad B \neq 0, \\ 1-Ar \leq \operatorname{Rep}(z) \leq |p(z)| \leq 1+Ar, \quad B = 0. \end{aligned} \quad (2.1)$$

After simple calculations in (2.1), we get

$$\begin{aligned} \frac{1 - \frac{k}{2}(A-B)r - AB r^2}{1 - B^2 r^2} \leq \operatorname{Rep}(z) \leq \frac{1 + \frac{k}{2}(A-B)r - AB r^2}{1 - B^2 r^2}, \quad B \neq 0, \\ 1 - \frac{k}{2}Ar \leq \operatorname{Rep}(z) \leq 1 + \frac{k}{2}Ar, \quad B = 0. \end{aligned} \quad (2.2)$$

(2.2) shows that the set of variability of $p \in \mathcal{P}_k(A, B)$ is the closed disc

$$\begin{aligned} \left| p(z) - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{\frac{k}{2}(A-B)r}{1 - B^2 r^2}, \quad B \neq 0, \\ |p(z) - 1| \leq \frac{k}{2}Ar, \quad B = 0. \end{aligned} \quad (2.3)$$

On the other hand from definition of $\mathcal{CC}_k(A, B)$, we can write

$$\begin{aligned} \left| \frac{f'(z)}{g'(z)} - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{\frac{k}{2}(A-B)r}{1 - B^2 r^2}, \quad B \neq 0, \\ \left| \frac{f'(z)}{g'(z)} - 1 \right| \leq \frac{k}{2}Ar, \quad B = 0. \end{aligned} \quad (2.4)$$

Since g is convex, using distortion bound of convex functions and above inequalities, we obtain

$$\begin{aligned} \frac{1 - \frac{k}{2}(A-B)r - AB r^2}{(1+r)^2(1-B^2 r^2)} \leq |f'(z)| \leq \frac{1 + \frac{k}{2}(A-B)r - AB r^2}{(1-r)^2(1-B^2 r^2)}, \quad B \neq 0, \\ \frac{1 - \frac{k}{2}Ar}{(1+r)^2} \leq |f'(z)| \leq \frac{1 + \frac{k}{2}Ar}{(1-r)^2}, \quad B = 0. \end{aligned} \quad (2.5)$$

Sharpness follows from the function f and g (convex) where

$$\begin{aligned} \frac{f'(z)}{g'(z)} = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1-Az}{1-Bz} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1+Az}{1+Bz}, \quad B \neq 0, \\ \frac{f'(z)}{g'(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)(1-Az) - \left(\frac{k}{4} - \frac{1}{2}\right)(1+Az), \quad B = 0. \end{aligned} \quad (2.6)$$

Since $p(z) = \frac{f'(z)}{g'(z)}, g \in \mathcal{C}$,

$$\left| p(z) - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{\frac{k}{2}(A-B)r}{1 - B^2 r^2}$$

then the set of variability of $\frac{f'}{g'}$ is the closed disc

$$\left| \frac{f'(z)}{g'(z)} - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{\frac{k}{2}(A-B)r}{1 - B^2 r^2}.$$

Therefore this set could be written in the following form

$$w(\mathbb{D}_r) = \left\{ \frac{f'(z)}{g'(z)} : \left| \frac{f'(z)}{g'(z)} - \frac{1 - AB r^2}{1 - B^2 r^2} \right| \leq \frac{\frac{k}{2}(A-B)r}{1 - B^2 r^2}, \quad 0 < r < 1 \right\}. \quad (2.7)$$

On the other hand, we define the function ϕ by

$$\begin{aligned} \frac{f(z)}{g(z)} = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1 - A\phi(z)}{1 - B\phi(z)} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1 + A\phi(z)}{1 + B\phi(z)} = \frac{1 - \frac{k}{2}(A-B)\phi(z) - AB(\phi(z))^2}{1 - B^2(\phi(z))^2}, \quad B \neq 0, \\ \frac{f(z)}{g(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)(1 - A\phi(z)) - \left(\frac{k}{4} - \frac{1}{2}\right)(1 + A\phi(z)) = 1 - \frac{k}{2}A\phi(z), \quad B = 0. \end{aligned} \quad (2.8)$$

Note that ϕ is well-defined analytic function and $\phi(0) = 0$. Now, we want to show that $|\phi(z)| < 1$ for every $z \in \mathbb{D}$. Taking derivative and simplifying both sides of (2.8), we get

$$\frac{f'(z)}{g'(z)} = \frac{1 - \frac{k}{2}(A-B)\phi(z) - AB(\phi(z))^2}{1 - B^2(\phi(z))^2} + \frac{z\phi'(z)\left(-\frac{k}{2}(A-B)(1+B^2(\phi(z))^2) - 2B(A-B)\phi(z)\right)}{(1 - B^2(\phi(z))^2)^2} \frac{g(z)}{zg'(z)}, \quad B \neq 0,$$

$$\frac{f'(z)}{g'(z)} = 1 - \frac{k}{2}A\phi(z) - \frac{k}{2}Az\phi'(z) \frac{g(z)}{zg'(z)}, \quad B = 0.$$

Suppose that there exists a $z_0 \in \mathbb{D}_r$ such that $|\phi(z_0)| = 1$, then by Jack's Lemma [2], we obtain

$$\frac{f'(z_0)}{g'(z_0)} = \frac{1 - \frac{k}{2}(A-B)\phi(z_0) - AB(\phi(z_0))^2}{1 - B^2(\phi(z_0))^2} + \frac{m\phi(z_0)\left(-\frac{k}{2}(A-B)(1+B^2(\phi(z_0))^2) - 2B(A-B)\phi(z_0)\right)}{(1 - B^2(\phi(z_0))^2)^2} \frac{g(z_0)}{z_0g'(z_0)} \notin w(\mathbb{D}_r), B \neq 0,$$

$$\frac{f'(z_0)}{g'(z_0)} = 1 - \frac{k}{2}A\phi(z_0) - \frac{k}{2}Am\phi(z_0) \frac{g(z_0)}{z_0g'(z_0)} \notin w(\mathbb{D}_r), \quad B = 0,$$

where $m \geq 1$ is a real number and $\frac{g(z_0)}{z_0g'(z_0)} = (1 + \phi(z_0))$. But this contradicts with (2.7), therefore we have $|\phi(z)| < 1$ for all $z \in \mathbb{D}$, and obtain the following subordination

$$\frac{f(z)}{g(z)} \prec \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1-Az}{1-Bz} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1+Az}{1+Bz}, \quad B \neq 0,$$

$$\frac{f(z)}{g(z)} \prec \left(\frac{k}{4} + \frac{1}{2}\right)(1-Az) - \left(\frac{k}{4} - \frac{1}{2}\right)(1+Az), \quad B = 0.$$

Using the subordination principle, we obtain

$$|g(z)|^{\frac{1-\frac{k}{2}(A-B)r-ABr^2}{1-B^2r^2}} \leq |f(z)| \leq |g(z)|^{\frac{1+\frac{k}{2}(A-B)r-ABr^2}{1-B^2r^2}}, \quad B \neq 0, \quad (2.9)$$

$$|g(z)|(1 - \frac{k}{2}Ar) \leq |f(z)| \leq |g(z)|(1 + \frac{k}{2}Ar), \quad B = 0.$$

Using the growth theorem for convex functions, the inequalities in (2.9) can be written in the following form

$$rF(A, B, k, -r) \leq |f(z)| \leq rF(A, B, k, r),$$

$$rF(A, k, -r) \leq |f(z)| \leq rF(A, k, r), \quad (2.10)$$

where

$$F(A, B, k, r) = \frac{1 + \frac{k}{2}(A-B)r - ABr^2}{(1-r)(1-B^2r^2)},$$

$$F(A, k, r) = \frac{1 + \frac{k}{2}Ar}{(1-r)}.$$

These inequalities are sharp because the extremal function is given in (2.6).

The following lemma helps us to prove the next theorem.

Lemma 2.1. Let $p(z) = 1 + p_1z + p_2z^2 + \dots$ be an element of $\mathcal{P}_k(A, B)$, then

$$|p_n| \leq \frac{k}{2}(A-B).$$

for all $n \geq 1, k \geq 2, -1 \leq B < A \leq 1$. This result is sharp for the functions

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(1)}(z),$$

where $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}(A, B)$.

Proof. Let $p_1^{(1)} = 1 + a_1z + a_2z^2 + \dots$ and $p_2^{(2)} = 1 + b_1z + b_2z^2 + \dots$. Since $p \in \mathcal{P}_k(A, B)$, then we have

$$\begin{aligned} p(z) &= \left(\frac{k}{4} + \frac{1}{2}\right) p_1^{(1)}(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2^{(2)}(z) \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) (1 + a_1z + a_2z^2 + \dots) - \left(\frac{k}{4} - \frac{1}{2}\right) (1 + b_1z + b_2z^2 + \dots). \end{aligned}$$

Then, for n th term, we have

$$p_n = \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n.$$

Since $p_1^{(1)}, p_2^{(2)} \in \mathcal{P}(A, B)$, then $|a_n| \leq (A - B), |b_n| \leq (A - B)$ for all $n \geq 1$, and

$$\begin{aligned} |p_n| &= \left| \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n \right| \\ &\leq \left(\frac{k}{4} + \frac{1}{2}\right) |a_n| + \left(\frac{k}{4} - \frac{1}{2}\right) |b_n| \\ &\leq \left(\frac{k}{4} + \frac{1}{2}\right) (A - B) + \left(\frac{k}{4} - \frac{1}{2}\right) (A - B). \end{aligned}$$

This shows that, $|p_n| \leq \frac{k}{2}(A - B)$. □

Theorem 2.2. If f of the form (1.1) is an element of $\mathcal{CC}_k(A, B)$, then

$$|a_n| \leq 1 + \frac{k(A - B)(n - 1)}{4}. \quad (2.11)$$

This result is sharp for each $n \geq 2$.

Proof. Using Definition 1.1 and subordination principle, we write

$$\frac{f'(z)}{g'(z)} = p(z)$$

for some $g \in \mathcal{C}$, where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$. Since $p(0) = 1$, it shows that $p \in \mathcal{P}_k(A, B)$, where $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Therefore, above equality is equivalent to

$$\left(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right) = \left(1 + \sum_{n=2}^{\infty} n b_n z^{n-1}\right) \left(1 + \sum_{n=1}^{\infty} p_n z^n\right).$$

This equation yields,

$$1 + 2a_2z + 3a_3z^2 + \dots = 1 + (2b_2 + p_1)z + (3b_3 + 2b_2p_1 + p_2)z^2 + \dots \quad (2.12)$$

Comparing the coefficients of z^{n-1} on both sides, we obtain

$$n a_n = n b_n + (n - 1) b_{n-1} p_1 + (n - 2) b_{n-2} p_2 + \dots + 2 b_2 p_{n-2} + p_{n-1}.$$

Using Lemma 2.1, we get

$$n |a_n| \leq n |b_n| + \frac{k}{2} (A - B) \left[(n - 1) |b_{n-1}| + \dots + 2 |b_2| + 1 \right].$$

Since g is convex, then $|b_n| \leq 1$, it follows from that

$$|a_n| \leq 1 + \frac{k(A - B)(n - 1)}{4}.$$

This completes our proof. □

Remark 2.3. Letting $k = 2, A = 1, B = -1$, Theorem 2.2 gives the well-known coefficient inequality for close-to-convex functions.

We can conclude that an analytic functions $p \in \mathcal{P}_k(A, B)$, $k \geq 2$ if and only if there exists $p_1, p_2 \in \mathcal{P}(A, B)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

then new results are obtained for the class $\mathcal{CC}_k(A, B)$ by means of classes of bounded boundary rotation and bounded radius rotation. Giving the special values to A, B and k we obtain the growth theorem, distortion theorem and coefficient inequality for the other subclasses.

References

- [1] A. W. Goodman, *Univalent functions vol. I and II*, Polygonal Pub. House, 1983.
- [2] I. S. Jack, *Functions starlike and convex of order alpha*, J. London Math. Soc. (2)(3)(1971), 469-474.
- [3] K. I. Noor, B. Malik, S. Mustafa, *A Survey on Functions Bounded Boundary and Bounded Radius Rotation*, Appl. Math. E-Notes. 12 (2012), 136-152.
- [4] B. Pinchuk, *Functions of bounded boundary rotation*, Isr. J. Math. 10 (1971), 7-19.