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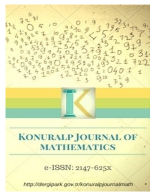

I-statistical Convergence of Double Sequences Defined by Weight Functions in a Locally Solid Riesz Space

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Abstract

In this work, we introduce the concepts of $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence of double sequences defined by weight functions in a locally solid Riesz space based on the notion of the ideal of subsets of $\mathbb{N}\times\mathbb{N}$. We also examine some inclusion relations of these concepts.

Keywords: *locally solid Riesz space; $\mathscr{I}$-statistical convergence; ideal convergence; double sequences; Pringsheim convergence.*
2010 Mathematics Subject Classification: *40A35; 40G15; 46A40; 46A45*

1. Introduction

The notion of $\mathscr{I}$-convergence was studied at initial stage by Kostyrko et al. [22] as a generalization of statistical convergence which had formally been introduced by Fast [14], Steinhaus [37], Schoenberg [36] and has still been discussed and investigated in the theory of Fourier analysis, ergodic theory, number theory under different names and varied points of view in many fields of mathematics. For more details, see also [7, 8, 9, 10, 11, 19, 20, 21, 23, 24, 25, 26, 27, 30, 31, 39, 40, 41].

In 1928, the concept of Riesz space was first introduced by Riesz [33], at the International Mathematical Congress in Bologna, Italy. A Riesz space is an ordered vector space which is lattice at the same time. A locally solid Riesz space is a Riesz space equipped with a linear topology that has a base consisting of solid sets. Soon after, in the mid-thirties, Freudental [15] and Kantrovich [17] independently set up the axiomatic foundation and derived a number of properties dealing with the lattice structure of Riesz space. Riesz space have many applications in measure theory, operator theory and optimization. They have also some applications in economics [3]. For further results on this topic, we may refer to [2, 12, 13, 18, 28, 38, 42].

Let w_2 be the set of all real or complex double sequences. By the convergence of a double sequence we mean the convergence in the Pringsheim's sense, that is; the double sequence $x=(x_{k,l})$ has a Pringsheim limit L denoted by P-$\lim x = L$ provided that, given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $|x_{k,l}-L|<\varepsilon$ holds whenever $k,l\geq N$, we will describe such an x more briefly as "P-convergent" (see [32]).

Recently, Balcerzak et al. [6] show that one can further extend the concept of natural or asymptotic density (as well as natural density of order α) by considering natural density of weight g where $g:\mathbb{N}\to[0,\infty)$ is a function with $\lim_{n\to\infty} g(n)=\infty$ and $n/g(n)\nrightarrow 0$ as $n\to\infty$.

In this work, our aim is to introduce the concepts of $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence of double sequences with respect to weight functions in locally solid Riesz space. We also investigate some basic properties and examine some inclusion relations of these concepts.

2. Definitions and Preliminaries

Let X be a real vector space and "$\leq$" be a partial order on this space. Then, X is said to be an ordered vector space if it satisfies the following properties:

1. $\forall x,y\in X$ and $y\leq x$ imply $y+z\leq x+z$ for each $z\in X$,
2. $\forall x,y\in X$ and $y\leq x$ imply $\alpha y\leq \alpha x$ for each $\alpha\geq 0$.

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In addition, if X is a lattice with respect to the partial order, then X is said to be a Riesz space (or a vector lattice) [42]. For an element x of a Riesz space X the positive part of x is defined by $x^{+}=x\vee\theta=\sup\{x,\theta\}$, the negative part of x by $x^{-}=(-x)\vee\theta=\sup\{-x,\theta\}$ and the absolute value of x by $|x|=x\vee(-x)=\sup\{x,-x\}$, where θ is the element zero of X. If for each pair elements $x,y\in X$ has the supremum and infimum of the set x,y both exists in X. A subset S of a Riesz space X is said to be solid if $y\in S$ and $|x|\leq|y|$ implies $x\in S$. Some examples of Riesz spaces can be given as follows:

Example 2.1. *[1] Let $X=\mathbb{R}^n$, $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$, $y=(y_1,y_2,...,y_n)\in\mathbb{R}^n$ and $x\preceq y\Longleftrightarrow x_i\leq y_i$ $(1\leq i\leq n)$. Then Euclidean space $\mathbb{R}^n$ is Riesz space with the partial order " $\preceq$ ".*

A topological vector space (X,τ) is a vector space which has a linear topology τ such that the algebraic operations of addition and scalar multiplication in X are continuous.
Every linear topology τ on a vector space X has a base $\mathscr{N}_{sol}$ for the neighbourhoods of zero satisfying the following properties:

1. Each $Y\in\mathscr{N}_{sol}$ is a balanced set, that is, $\alpha x\in Y$ holds for all $x\in Y$ and every $\alpha\in\mathbb{R}$ with $|\alpha|\leq 1$.
2. Each $Y\in\mathscr{N}_{sol}$ is an absorbing set, that is, for every $x\in X$ there exists $\alpha>0$ such that $\alpha x\in Y$.
3. For each $Y\in\mathscr{N}_{sol}$, there exists some $W\in Y$ with $W+W\subseteq Y$.

A linear topology τ on a Riesz space X is said to be locally solid Riesz space (shortly, LSRS) if τ has a base at zero consisting of solid sets. A locally solid Riesz space (X,τ) is a Riesz space equipped with a locally solid topology τ [34].

Definition 2.2. *[16]* Let $E\subseteq\mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and it is defined by $\delta(E):=\lim_{n\to\infty}\frac{1}{n}|\{k\in E:k\leq n\}|$, if the limit exists. Here the vertical bars denote the cardinality of the respective set.

The sequence $x=(x_k)$ is said to be statistically convergent to x_0 if for every $\varepsilon>0$, the set $E_\varepsilon:=\{k\in E:|x_k-x_0|\geq\varepsilon\}$ has natural density zero, i.e., $\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:|x_k-x_0|\geq\varepsilon\}|=0$. In this case, we write $x_0=st\text{-}\lim x$. Note that every convergence sequence is statistically convergent, but not conversely, [14].

Let $g:\mathbb{N}\to[0,\infty)$ be a function satisfying $\lim_{n\to\infty}g(n)=\infty$. The upper and the lower density of weight g were defined respectively by the formula $\bar{d}_g(A)=\limsup\limits_{n\to\infty}\frac{A(1,n)}{g(n)}$ and $\underline{d}_g(A)=\liminf\limits_{n\to\infty}\frac{A(1,n)}{g(n)}$ for $A\subseteq\mathbb{N}$ where $A(1,n)$ denotes the cardinality of the set $A\cap[1,n]$. If the $\lim_{n\to\infty}A(1,n)/g(n)$ exists, then we say that the density of weight g of the set A exists and we denote it by $d_g(A)$ [6].

Definition 2.3. [29] Let (X,τ) be a LSRS. Then a double sequence $x=(x_{k,l})$ in X is said to be statistically τ-convergent to the number x_0 in X if for every τ-neighbourhood V of zero, $P-\lim\limits_{m,n\to\infty}\frac{1}{mn}\left|\left\{(k,l),k\leq m \text{ and } l\leq n:x_{k,l}-x_0\notin V\right\}\right|=0$. In this case, we write $S^\tau-\lim x=x_0$ or $x_{k,l}\stackrel{S^\tau}{\to}x_0$.

A family of sets $\mathscr{I}\subset P(\mathbb{N})$ (power sets of $\mathbb{N}$) is said to be an ideal if;

1. $\emptyset\in\mathscr{I}$,
2. $A\cup B\in\mathscr{I}$, for each $A,B\in\mathscr{I}$,
3. for each $A\in\mathscr{I}$ and for each $B\subseteq A$ imply $B\in\mathscr{I}$.

We can give such a simple example: Let $X=\{a,b\}$ and then $P(X)=\{\emptyset,\{a\},\{b\},X\}$. If we take $\mathscr{I}=\{\emptyset,\{a\},\{a,b\}\}$ so $\mathscr{I}\subset P(X)$. Let $A=\{a,b\}\in\mathscr{I}$ and $B=\{b\}$ then it is clear that $B\subset A$. Although $B\subset A$ and $A\in\mathscr{I}$, $B\notin\mathscr{I}$. Thus, $\mathscr{I}$ cannot be an ideal because the third condition of definition of ideal cannot be satisfied.

A non-empty family of sets $\mathscr{F}\subset P(\mathbb{N})$ is said to be a filter on $\mathbb{N}$ if:

1. $\emptyset\notin\mathscr{F}$,
2. $A\cap B\in\mathscr{F}$, for each $A,B\in\mathscr{F}$,
3. for each $A\in\mathscr{F}$ and $B\supset A$ imply $B\in\mathscr{F}$.

$\mathscr{I}$ is called non-trivial if $\mathscr{I}\neq\emptyset$ and $\mathbb{N}\notin\mathscr{I}$. A non-trivial ideal $\mathscr{I}$ is called admissible ideal if it contains all singletons of $\mathbb{N}$, i.e., $\{\{x\}:x\in\mathbb{N}\}\subseteq\mathscr{I}$.

Definition 2.4. [22] Let $\mathscr{I}$ be a non trivial ideal in $\mathbb{N}$. A sequence $x=(x_n)$ of real numbers is said to be $\mathscr{I}$-convergent to $x_0\in\mathbb{R}$ if for every $\varepsilon>0$ the set $A(\varepsilon)=\{n:|x_n-x_0|\geq\varepsilon\}$ belongs to $\mathscr{I}$.

If we take $\mathscr{I}=\mathscr{I}_f=\{A\subseteq\mathbb{N}:\text{A is finite subset}\}$, then $\mathscr{I}_f$ is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence.

We consider again density of weight g of the set A. The family $\mathscr{I}_g=\left\{A\subset\mathbb{N}:\bar{d}_g(A)=0\right\}$ forms an ideal. It has been observed in [6] that $\mathbb{N}\in\mathscr{I}_g$ if and only if $\frac{n}{g(n)}\to 0$ as $n\to\infty$. So, we additionally assume that $\frac{n}{g(n)}\nrightarrow 0$ as $n\to\infty$ so that $\mathbb{N}\notin\mathscr{I}_g$ and $\mathscr{I}_g$ is a proper admissible ideal of $\mathbb{N}$. The collection of all such weight functions g satisfying the above properties will be denoted by G.

Definition 2.5. *[4]* Let (X,τ) be a LSRS. A double sequence $x=(x_{k,l})$ of points in X is said to be $\mathscr{I}(\tau)$-convergent to an element x_0 in X if for each τ-neighbourhood V of zero $\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\notin V\right\}\in\mathscr{I}$, that is; $\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:x_{k,l}-x_0\in V\right\}\in\mathscr{F}(\mathscr{I})$. In this case, we write $\mathscr{I}(\tau)\text{-}\lim x_{k,l}=x_0$ or $x_{k,l}\stackrel{\mathscr{I}(\tau)}{\to}x_0$.

By a double lacunary sequence $\theta_{r,s}=\{(k_r,l_s)\}$ where $k_0=0$ and $l_0=0$, we shall mean two increasing sequences of non-negative integers with $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$ and $\bar{h}_s=l_s-l_{s-1}\to\infty$ as $s\to\infty$. Let us denote $k_{rs}=k_rl_s$, $h_{rs}=h_r\bar{h}_s$ and the intervals determined by θ_{rs} will be denoted by $I_{rs}=\{(k,l):k_{r-1}<k\leq k_r \text{ and } l_{s-1}<l\leq l_s\}$, $q_{rs}=q_r\bar{q}_s$ where $q_r=\frac{k_r}{k_{r-1}}$ and $\bar{q}_s=\frac{l_s}{l_{s-1}}$.

Throughout the paper, the symbol $\mathscr{N}_{sol}$ will denote any base at zero consisting of solid sets and satisfying the above conditions (1), (2), (3) in a locally solid Riesz topology τ. For abbreviation, here and in where follows, we shall write a word "LSRS" instead of a locally solid Riesz space and we mean $\lim_{k,l\to\infty}x_{k,l}$ by $\lim x$. We also assume that $\mathscr{I}$ is an admissible ideal of $\mathbb{N}\times\mathbb{N}$.

3. Main Results

Definition 3.1. Let $(m,n) \in K \subseteq \mathbb{N}\times\mathbb{N}$ and $g \in G$. By $K_{m,n}$, we denote the cardinality of the set $\{(k,l) \in K : 1 \leq k \leq m \text{ and } 1 \leq l \leq n\}$. Now, we give the definition of the lower and upper double density of weight g of the set K, respectively;
$\underline{d}_{g,g}(K) = P - \liminf_{m,n} \frac{K(m,n)}{g(m)g(n)}$ and $\bar{d}_{g,g}(K) = P - \limsup_{m,n} \frac{K(m,n)}{g(m)g(n)}$.
If the limit $P - \lim_{m,n} \frac{K(m,n)}{g(mn)}$ exists in Pringsheim's sense, then we say that the double density of weight g of the set K exists and we shall denote it by

$$d_{g,g}(K) = P - \lim_{m,n} \frac{K(m,n)}{g(m)g(n)}. \tag{3.1}$$

Definition 3.2. Let (X,τ) be a LSRS. A double sequence $x = (x_{k,l})$ in X is said to be $S_{g,g}^{(\tau)}(\mathscr{I})$- convergent to $x_0 \in X$ if for every neighbourhood V of zero and $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - x_0 \notin V\right\}\right| \geq \delta\right\} \in \mathscr{I},$$

or equivalently

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - x_0 \in V\right\}\right| < \delta\right\} \in \mathscr{F}(\mathscr{I}).$$

In this case, we write $x_{k,l} \to x_0(S_{g,g}^{(\tau)}(\mathscr{I}))$ or $x_{k,l} \overset{S_{g,g}^{(\tau)}(\mathscr{I})}{\to} x_0$. The class of $\mathscr{I}$-statistical convergence of weight g of all double sequences will be denoted simply by $S_{g,g}^{(\tau)}(\mathscr{I})$.

Remark 3.3. If $\mathscr{I}_f = \{A \subset \mathbb{N}\times\mathbb{N}, A \textit{ is a finite set}\}$ and $g(m)$=m, $g(n)$=n, then $S_{g,g}^{(\tau)}(\mathscr{I})$-convergence reduces to the statistically τ-convergent in LSRS, in [29].

Definition 3.4. Let (X,τ) be a LSRS. A double sequence $x = (x_{k,l})$ in X is said to be $S_{g,g}^{(\tau)}(\mathscr{I})$-bounded if for every τ-neighbourhood V of zero and for any $\delta > 0$, there exists $\lambda > 0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : \lambda x_{k,l} \notin V\right\}\right| \geq \delta\right\} \in \mathscr{I}.$$

Definition 3.5. Let (X,τ) be a LSRS. We say that a double sequence $x = (x_{k,l})$ in X is said to be $S_{g,g}^{(\tau)}(\mathscr{I})$-Cauchy if for each τ-neighbourhood V of zero and for any $\delta > 0$, there exists $n_0, m_0 \in \mathbb{N}$ such that for all $k, p \geq n_0$, $l, q \geq m_0$

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - x_{p,q} \notin V\right\}\right| \geq \delta\right\} \in \mathscr{I}.$$

Theorem 3.6. Let (X,τ) be a Hausdorff LSRS and $x = (x_{k,l})$, $y = (y_{k,l})$ be two sequences in X. Then, the followings hold:

1. If $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = x_0$ and $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = y_0$, then $x_0 = y_0$.
2. If $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = x_0$, then $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim \alpha x_{k,l} = \alpha x_0$, for every $\alpha \in \mathbb{R}$,
3. If $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = x_0$ and $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim y_{k,l} = y_0$, then $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim(x_{k,l} + y_{k,l}) = x_0 + y_0$.

Proof. 1. Suppose that $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = x_0$ and $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = y_0$. Let V be any τ-neighbourhood of zero. Then, there exists a $Y \in \mathscr{N}_{sol}$ such that $Y \subseteq V$. Choose $W \in \mathscr{N}_{sol}$ such that $W + W \subseteq Y$. Since $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = x_0$ and $S_{g,g}^{(\tau)}(\mathscr{I}) - \lim x_{k,l} = y_0$, then we define the following sets

$$A_1 = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} \frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - x_0 \notin W\right\}\right| < \delta\right\} \in \mathscr{F}(\mathscr{I})$$

and

$$A_2 = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} \frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - y_0 \notin W\right\}\right| < \delta\right\} \in \mathscr{F}(\mathscr{I}).$$

Now let $A = A_1 \cup A_2 \in \mathscr{I}$. For any $0 < \delta < 1$,

$$\frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - x_0 \in W\right\}\right| > 1 - \delta$$

similarly,

$$\frac{1}{g(m)g(n)} \left|\left\{(k,l), k \leq m \text{ and } l \leq n : x_{k,l} - y_0 \in W\right\}\right| > 1 - \delta.$$

Then, we have $x_0 - y_0 = x_0 - x_{k,l} + x_{k,l} - y_0 \in W + W \subseteq Y \subseteq V$. Hence, for every τ-neighbourhood V of zero, we have $(x_0 - y_0) \in V$. Since (X,τ) is Hausdorff, the intersection of all τ-neighbourhood V of zero is the singleton set $\{\theta\}$. Thus, we get $x_0 - y_0 = \theta$, i.e., $x_0 = y_0$.

2. Let V be an arbitrary τ-neighbourhood of zero and $S_{g,g}^{(\tau)}(\mathscr{I})-\lim x_{k,l}=x_0$. Then, there exists a $Y\in\mathscr{N}_{sol}$ such that $Y\subseteq V$. Let $0<\delta<1$. We have

$$B=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\notin Y\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}),$$

i.e., for every $m,n\in B$ and any $0<\delta<1$,

$$\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\in V\right\}\right|>1-\delta.$$

Since Y is balanced, $x_{k,l}-x_0\in Y$ implies that $\alpha(x_{k,l}-x_0)\in Y$ for every $\alpha\in\mathbb{R}$ with $|\alpha|\leq 1$. Hence,

$$\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\in Y\right\}\subseteq\left\{(k,l),k\leq m\text{ and }l\leq n:\alpha\left(x_{k,l}-x_0\right)\in Y\right\}\subseteq\left\{(k,l),k\leq m\text{ and }l\leq n:\alpha\left(x_{k,l}-x_0\right)\in V\right\}$$

for $m,n\in B$. Thus, we obtain

$$\tfrac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:\alpha\left(x_{k,l}-x_0\right)\in V\right\}\right|\geq\tfrac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\in V\right\}\right|>1-\delta$$

which implies that

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:\alpha(x_{k,l}-x_0)\notin V\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}).$$

Now, let $|\alpha|>1$ and $[\![\alpha]\!]$ be the smallest integer greater than or equal to $|\alpha|$. Then, there exists $W\in N_{sol}$ such that $[\![\alpha]\!]W\subseteq Y$. Since $S_{g,g}^{(\tau)}(\mathscr{I})-\lim x_{k,l}=x_0$, then again we take for any $\delta>0$,

$$B=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\notin W\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}).$$

Therefore, we have $\left|\alpha\left(x_{k,l}-x_0\right)\right|=|\alpha|\left|x_{k,l}-x_0\right|\leq[\![\alpha]\!]\left|x_{k,l}-x_0\right|\in[\![\alpha]\!]W\subseteq Y\subseteq V$. Since Y is solid, we have $\alpha\left(x_{k,l}-x_0\right)\in Y$ and this implies that $\alpha\left(x_{k,l}-x_0\right)\in V$. Consequently,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:\alpha x_{k,l}-\alpha x_0\notin W\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}).$$

This proves that $S_{g,g}^{(\tau)}(\mathscr{I})-\lim\alpha x_{k,l}=\alpha x_0$.

3. Let V be an arbitrary $\tau-$neighbourhood of zero. Then, there exists $Y\in\mathscr{N}_{sol}$ such that $Y\subseteq V$. We choose $W\in\mathscr{N}_{sol}$ such that $W+W\subseteq Y$. Since $S_{g,g}^{(\tau)}(\mathscr{I})-\lim x_{k,l}=x_0$ and $S_{g,g}^{(\tau)}(\mathscr{I})-\lim y_{k,l}=y_0$, then we have

$$B_1=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\notin W\right\}\right|<\frac{\delta}{2}\right\}\in\mathscr{F}(\mathscr{I}),$$

and

$$B_2=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:y_{k,l}-y_0\notin W\right\}\right|<\frac{\delta}{2}\right\}\in\mathscr{F}(\mathscr{I}).$$

Let $B_3:=B_1\cap B_2$. Hence, we have

$$\left(\left(x_{k,l}+y_{k,l}\right)-(x_0+y_0)\right)=\left(x_{k,l}-x_0\right)+\left(y_{k,l}-y_0\right)\in W+W\subseteq Y\subseteq V,$$

for all $(k,l)\in B_3$. Therefore, we get

$$\tfrac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:\left(x_{k,l}+y_{k,l}\right)-(x_0+y_0)\notin W\right\}\right|$$

$$\leq\tfrac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:x_{k,l}-x_0\notin W\right\}\right|+\tfrac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:y_{k,l}-y_0\notin W\right\}\right|<\tfrac{\delta}{2}+\tfrac{\delta}{2}=\delta.$$

Consequently,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\leq m\text{ and }l\leq n:\left(x_{k,l}+y_{k,l}\right)-(x_0+y_0)\notin W\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}).$$

Since V is arbitrary, we have $S_{g,g}^{(\tau)}(\mathscr{I})-\lim x_{k,l}+y_{k,l}=x_0+y_0$. □

Theorem 3.7. Let (X,τ) be a LSRS. If a sequence $x=(x_{k,l})$ is $S_{g,g}^{(\tau)}(\mathscr{I})$-convergent, then $x=(x_{k,l})$ is $S_{g,g}^{(\tau)}(\mathscr{I})$-bounded.

Proof. Suppose that $x=(x_{k,l})$ is $S_{g,g}^{(\tau)}(\mathscr{I})-\lim x=x_0$. Let V be an arbitrary τ-neighbourhood of zero. Then, there exists $Y\in\mathscr{N}_{sol}$ such that $Y\subseteq V$. We choose another element $W\in\mathscr{N}_{sol}$ such that $W+W\subseteq Y$. Since $x=(x_{k,l})$ is $S_{g,g}^{(\tau)}(\mathscr{I})-\lim x=x_0$, then we have following fact;

$$D=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin W\right\}\right|\ge\delta\right\}\in\mathscr{I}.$$

Since W is absorbing, there exists $a>0$ such that $ax_0\in W$. Let b be such that $0<b\le 1$ and $b\le a$. Since W is solid and $|bx_o|\le|ax_0|$, we have $bx_0\in W$. Since W is balanced, $x_{k,l}-x_0\in W$ implies that $b(x_{k,l}-x_0)\in W$. Then, we have $ax_{k,l}=a(x_{k,l}-x_o)+ax_0\in W+W\subseteq Y\subseteq V$ for $(k,l)\in(\mathbb{N}\times\mathbb{N})\backslash D$. Thus,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}\notin V\right\}\right|\ge\delta\right\}\in\mathscr{I}.$$

This shows that $S_{g,g}^{(\tau)}(\mathscr{I})$-convergent sequence (x_{kl}) is $S_{g,g}^{(\tau)}(\mathscr{I})$-bounded. □

Theorem 3.8. Let (X,τ) be a LSRS. If a double sequence $x=(x_{k,l})$ is $S_{g,g}^{(\tau)}(\mathscr{I})$-convergent, then it is $S_{g,g}^{(\tau)}(\mathscr{I})$-Cauchy.

Proof. Let $x=(x_{k,l})$ be $S_{g,g}^{(\tau)}(\mathscr{I})$-convergent to x_0 in X. Let V be an arbitrary τ-neighbourhood of zero. Choose $V,W\in\mathscr{N}_{sol}$ such that $W+W\subseteq Y\subseteq V$. Let $0<\delta<1$. Then

$$E=\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin W\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}).$$

For all $m,n\in E$, $\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin W\right\}\right|<\delta$, i.e., $\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\in W\right\}\right|>1-\delta$. For $m,n\in E$ choose $p,q\in\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\in W\right\}$. Then, $x_{p,q}-x_0\in W$. So $x_{k,l}-x_{p,q}=x_{k,l}-x_0+x_0-x_{p,q}\in W+W\subseteq Y\subseteq V$. For any $\delta>0$

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(m)g(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_{p,q}\notin W\right\}\right|<\delta\right\}\in\mathscr{F}(\mathscr{I}).$$

Hence, $x=(x_{k,l})$ is $S_{g,g}^{(\tau)}(\mathscr{I})$-Cauchy. □

Theorem 3.9. Let (X,τ) be a LSRS and $g_1,g_2\in G$ be such that there exists $M>0$ and $(m_0,n_0)\in\mathbb{N}\times\mathbb{N}$ such that $\frac{g_1(m)g_1(n)}{g_2(m)g_2(n)}\le M$ for all $m\ge m_0$, $n\ge n_0$. Then, $S_{g_1,g_1}^{(\tau)}(\mathscr{I})\subset S_{g_2,g_2}^{(\tau)}(\mathscr{I})$.

Proof. Let (X,τ) be a LSRS and $x=(x_{k,l})$ belongs to $S_{g_1,g_1}^{(\tau)}(\mathscr{I})$. For any τ-neighbourhood V of zero there exists $Y\subseteq V$. Then for $m\ge m_0$, $n\ge n_0$

$$\begin{aligned}\frac{1}{g_2(m)g_2(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin V\right\}\right| &= \frac{g_1(m)g_1(n)}{g_2(m)g_2(n)}\frac{1}{g_1(m)g_1(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin V\right\}\right|\\ &\le M\frac{1}{g_1(m)g_1(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin V\right\}\right|.\end{aligned}$$

For any $\delta>0$

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g_1(m)g_1(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin V\right\}\right|\ge\delta\right\}$$

$$\subset\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g_2(m)g_2(n)}\left|\left\{(k,l),k\le m \text{ and } l\le n: x_{k,l}-x_0\notin V\right\}\right|\ge\delta\right\}.$$

Since $x_{k,l}\in S_{g_1,g_1}^{(\tau)}(\mathscr{I})$, then the set on the right hand side belongs to $\mathscr{I}$ and hence the set on the left hand side belongs to $\mathscr{I}$. This shows that $S_{g_1,g_1}^{(\tau)}(\mathscr{I})\subset S_{g_2,g_2}^{(\tau)}(\mathscr{I})$. □

Now, we will introduce the definition of $\mathscr{I}$-lacunary statistical convergence of weight g of double sequences and investigate inclusion relation between $S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)$.

Definition 3.10. Let (X,τ) be a LSRS and $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. We say that a double sequence $x=(x_{k,l})$ is said to be $S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)$-convergent to x_0 in X if for every τ-neighbourhood V of zero and $\delta>0$,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(h_r)g(\bar{h}_s)}\left|\left\{(k,l)\in I_{rs}: x_{k,l}-x_0\notin V\right\}\right|\ge\delta\right\}\in\mathscr{I},$$

where $g(h_{rs})=g(h_r\cdot\bar{h}_s)=g(h_r)g(\bar{h}_s)$. In this case, we write $x_{k,l}\to x_0\left(S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)\right)$ or $S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)-\lim x_{k,l}=x_0$.

Definition 3.11. Let (X,τ) be a LSRS and $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. We say that a double sequence $x=(x_{k,l})$ is said to be $S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)$-bounded if for every τ-neighbourhood V of zero and $\delta>0$, there exists $\alpha>0$ such that

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{g(h_r)g(\bar{h}_s)}\left|\left\{(k,l)\in I_{rs}: \alpha x_{k,l}\notin V\right\}\right|\ge\delta\right\}\in\mathscr{I}.$$

Theorem 3.12. Let (X,τ) be a LSRS and $\theta_{r,s}=\{(k_r,l_s)\}$ be a double lacunary sequence. If $\liminf_{r,s}\frac{g(h_r)g(\bar{h}_s)}{g(k_r)g(k_s)}>1$, then $S_{g,g}^{(\tau)}(\mathscr{I})\subset S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)$.

Proof. Since $\liminf_{r,s}\frac{g(h_r)g(\bar{h}_s)}{g(k_r)g(k_s)}>1$, then we can find a $H>0$ such that $\liminf_{r,s}\frac{g(h_r)g(\bar{h}_s)}{g(k_r)g(l_s)}\geq 1+H$ for sufficiently large values of r,s. Assume that $x_{k,l}\overset{S_{g,g}^{(\tau)}(\mathscr{I})}{\rightarrow}x_0$. Hence, for every V neighbourhood of zero and for sufficiently large values of r,s we have

$$\begin{aligned}\frac{1}{g(k_r)g(l_s)}\left|\left\{(k,l),k\leq k_r \text{ and } l\leq l_s : x_{k,l}-x_0\notin V\right\}\right| &\geq \frac{1}{g(k_r)g(l_s)}\left|\left\{(k,l)\in I_{rs}: x_{k,l}-x_0\notin V\right\}\right| \\ &= \frac{g(h_r)g(\bar{h}_s)}{g(k_r)g(l_s)}\frac{1}{g(h_r)g(\bar{h}_s)}\left|\left\{(k,l)\in I_{rs}: x_{k,l}-x_0\notin V\right\}\right| \\ &\geq (1+H)\frac{1}{g(h_r)g(\bar{h}_s)}\left|\left\{(k,l)\in I_{rs}: x_{k,l}-x_0\notin V\right\}\right|.\end{aligned}$$

For any $\delta>0$, we get $\left\{\frac{1}{g(h_r)g(\bar{h}_s)}\left|\left\{(k,l)\in I_{rs}: x_{k,l}-x_0\notin V\right\}\right|\geq\frac{1}{(1+H)}\delta\right\}\subset\left\{\frac{1}{g(k_r)g(l_s)}\left|\left\{k\leq k_r \text{ and } l\leq l_s : x_{k,l}-x_0\notin V\right\}\right|\geq\delta\right\}$. Since $x=(x_{k,l})\in S_{g,g}^{(\tau)}(\mathscr{I})$, then the set on the right hand side belongs to $\mathscr{I}$. This shows that $S_{g,g}^{(\tau)}(\mathscr{I})\subset S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)$. □

3.1. Generalized Kinds of Density of Weight g

An arithmetical function f which is not identically zero is called multiplicative if

(1) $f(mn)=f(m).f(n)$ whenever $(m,n)=1$, that is, when m and n are mutually prime, and it is called completely multiplicative if

(2) $f(mn)=f(m).f(n)$ for all m and n [5].

Following the idea in Definition 3.1, we may redefine the density of $K\subseteq\mathbb{N}\times\mathbb{N}$ as follows: where $K_{m,n}=\{(k,l)\in K:\ 1\leq k\leq m \text{ and } 1\leq l\leq n\}$ and g,h are weight functions such that $g,h:\mathbb{N}\to[0,\infty)$ and $\lim_{n\to\infty}g(n)=\lim_{n\to\infty}h(n)=\infty$. Then

$$d_{g,h}(K)=P-\lim_{m,n}\frac{K(m,n)}{g(m)h(n)}. \tag{3.2}$$

If we take $g=h$ in the equation (3.2), then it reduces to (3.1). If g is completely multiplicative function, then the equation 3.1 can be rewritten as

$$d_g(K)=P-\lim_{m,n\to\infty}\frac{K(m,n)}{g(mn)} \tag{3.3}$$

and then (3.3) reduces to the definition given in [35] for $X=\mathbb{R}$.

In equation (3.1):

1. For $g(m)=m$, $h(n)=n$, we obtain the classical definition of statistical convergence of double sequences in LSRS (see, [29])

$$d(K)=P-\lim_{m,n\to\infty}\frac{K(m,n)}{mn}.$$

2. If $h(n)=n$, then it reduces to

$$d_{g,n}(K)=P-\lim_{m,n\to\infty}\frac{K(m,n)}{g(m)n}. \tag{3.4}$$

3. For $g(m)=m$, then

$$d_{m,h}(K)=P-\lim_{m,n\to\infty}\frac{K(m,n)}{h(n)m}. \tag{3.5}$$

As it can be seen in the equations (3.4) and (3.5), we obtain different kinds of densities with respect to the weight functions g and h. The different classes of densities that may emerge here are comparable. Also, following the idea in Definition 3.1, $S_{g,h}{}^{(\tau)}(\mathscr{I})$-convergence can be defined for any $g,h\in G$ and all results given here can be generalized.

4. Conclusion

In this paper, we introduce the concepts of $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence of weight g in locally solid Riesz space for double sequence and give some topological properties of these concepts. We still do not know under what condition, $S_{g,g}^{(\tau)}\left(\mathscr{I}_{\theta_{r,s}}\right)$-convergence implies $S_{g,g}(\mathscr{I})(\tau)$-convergence. So, we leave it as an open problem for readers.

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