Konuralp Journal of Mathematics, 10 (1) (2022) 30-33

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

On Constructing Complete $A-$Metric Spaces

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Abstract

The $A-$metric space has been developed on the idea of measuring simultaneously the distance between n points is the latest generalization of the usual metric spaces in the literature. Our main motivation is to answer the question "how to get another a new complete $A-$metric space from a complete $A-$metric space ?"

Keywords: *A−metric space; Complete metric spaces; Generalized metric spaces.*
2010 Mathematics Subject Classification: *54E50, 54C05, 47H05, 47H09, 47H10.*

1. Introduction

We know that the traditional definition of distance in space is based on the idea of "how far apart" two points in space are. But, measuring the distance between more than two points or elements is somewhat tedious and time-consuming. This measurement is usually done by combining the binary distance values for all pairs of items in a collection into an aggregated measure. However, the generalizations of traditional metric spaces in recent years has given rise to the ability to simultaneously measure the distance between more than two elements. In this sense, $2-$metric, $D-$metric and $G-$metric spaces, which are the generalizations of the usual metric spaces, and in which the distance between three points can be measured at the same time, were introduced. Today, serious studies are still being carried out on these spaces. The last and most important of these generalizations is $A-$metric space, because this space has been developed on the idea of measuring simultaneously the distance between n points. The definition of $A-$metric space, introduced by Mujahid Abbas et al. [1] in 2015, is as follows:

Definition 1.1. *Let $X \neq \emptyset$ and let the function $A : \overbrace{X \times X \times \cdots \times X}^{n-times} \longrightarrow [0,\infty)$ $(n \geq 3)$ be satisfying the following axioms:*
$(A1)$ $A(x_1,x_2,\ldots,x_n) = 0$ *iff* $x_1 = x_2 = \ldots = x_n$,
$(A2)$ $A(x_1,x_2,\ldots,x_n) \leq \sum\limits_{i=1}^{n} A(x_i,x_i,\ldots,x_i,w)$, x_i *and* $w \in X$.
Then the function A is called the $A-$metric and the pair (X,A) is called the $A-$metric space.

The concepts we need while constructing a new complete $A-$metric space from a given complete $A-$metric space will be given in the following section.

2. Preliminaries

Definition 2.1. *[1], [8] Let (X,A) be an $A-$metric space;*
i. *A sequence $\{x_n\}$ is said to $A-$converge to a point x if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $n > N$ implies $A(x_n,\ldots,x_n,x) < \varepsilon$. Briefly, $\lim\limits_{k\to\infty} A(x_k,x_k,\ldots,x_k,x) = 0 \Longleftrightarrow \lim\limits_{k\to\infty} x_k = x$.*

ii. *A sequence $\{x_n\}$ is called an $A-$Cauchy sequence if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that for all $r,s > N$ implies $A(x_r,\ldots,x_r,x_s) < \varepsilon$.*

iii. *An $A-$metric space is called the complete $A-$metric space in which every $A-$Cauchy sequence is $A-$convergent.*

iv. *Let (Y,A^*) be an A-metric space. A function $f : X \to Y$ is called A-continuous at $a \in X$ if and only if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that if $A(x,\ldots,x,a) < \delta$, then $A(f(x),\ldots,f(x),f(a)) < \varepsilon$.*

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Lemma 2.2. *[1] Let $\{x_n\}$ be a sequence in the $A-$metric space (X,A). Then, for every $k,m\in\mathbb{N}$*

$$A(x_k,\dots,x_k,x_m)\le(n-1)\left[\sum_{i=k}^{m-2}A(x_i,\dots,x_i,x_{i+1})\right]+A(x_{m-1},\dots,x_{m-1},x_m).$$

3. The Main Results

If a continuous map f defined on a complete metric space is not a contraction but f^k is a contraction for some positive integer k, then f has a unique fixed point. This is a well-know theorem due to [2], [4], [6]. However, in 1968, V. B. Bryant [3] showed that the assumption of continuity used in this theorem is not necessary. In 2018, Y. U. O. Gaba [5] has managed to generalize this theorem to G-metric spaces [7] with order three. In this section, using the methodology introduced by Gaba, it will be proved in Theorem 3.4 that from an $A-$metric space where F^n is contraction map while F is not a contraction map, a new $A-$metric space can be constructed where F is a contraction map. After the proof of Theorem 3.4, a new complete $A-$metric space will be constructed from a given complete $A-$metric space.

Definition 3.1. *A map $T:X\to X$, where (X,A) is a $A-$metric space, is a contraction if there exist the constant $\lambda\in[0,1)$ such that $A(T(x_1),\dots,T(x_n))\le\lambda A(x_1,\dots,x_n)$ for all $x_1,\dots,x_n\in X$.*

Theorem 3.2. *Let (X,A) be a complete $A-$metric space, then every contraction has a unique fixed point.*

Proof. To prove the existence and uniqueness of the contraction mapping's fixed point, we will respectively benefit from induction and contradiction.

By condition of contraction, let x_0 be an arbitrary point in X. Because the map $F:X\to X$ is function, each element of the domain X is mapped to by at least one element of the codomain X. So, we can write $x_{k+1}=F(x_k)$ for all $k\in\mathbb{N}$. There are two cases according to $x_{k+1}=F(x_k)$:

Case 1: If $x_{k+1}=x_k$, then, x_k is a fixed point of F.
Case 2: If $x_{k+1}\neq x_k$, then, by the condition of contraction, it follows that $A(x_{k+1},x_{k+2},\dots,x_{k+n+1})\le\lambda A(x_k,x_{k+1},\dots,x_{k+n})$. So, by induction we have

$$\begin{aligned}A(x_{k+1},x_{k+2},\dots,x_{k+n+1}) &\le \lambda A(x_k,x_{k+1},\dots,x_{k+n})\\ &\vdots\\ &\le \lambda^{k+1}A(x_0,x_1,\dots,x_n),\end{aligned}$$

which implies that $A(x_{k+1},x_{k+2},\dots,x_{k+n+1})\to 0$ as $k\to\infty$. Therefore, the sequence (x_n) is an $A-$Cauchy sequence in X. Let $\lim_{k\to\infty}x_k=L$. It follows that $A(x_{k+1},F(L),\dots,F(L))\le\lambda A(x_k,L,\dots,L)$. As $k\to\infty$, $A(L,F(L),\dots,F(L))\le A(L,L,\dots,L)=0$. Thus $F(L)=L$ is obtained by the condition $(A1)$. Assume that $F(L_1)=L_1$ and $F(L_2)=L_2$ for L_1, $L_2\in X$ such that $L_1\neq L_1$. For $\lambda\in[0,1)$

$$A(L_1,\dots,L_1,L_2)=A(F(L_1),\dots,F(L_1),F(L_2))\le\lambda A(L_1,\dots,L_1,L_2).$$

Therefore, $A(L_1,\dots,L_1,L_2)=0$ is obtained which implies that $L_1=L_2$. But this is a contradiction. □

Corollary 3.3. *Let (X,A) be a complete $A-$metric space and $T:X\to X$ be a mapping such that there exist a constant λ with $\lambda\in[0,1)$ satisfying $A(T^n(x_1),\dots,T^n(x_n))\le\lambda A(x_1,\dots,x_n)$ for all $x_1,\dots,x_n\in X$. Then T has an unique fixed point.*

Proof. We define $T^n(x)$ inductively by $T^0(x)=x$ and $T^{n+1}(x)=T(T^n(x))$ for any $x\in X$ and $n\in\{0,1,\dots\}$. Also, T^n is a contraction mapping because of the above inequality. So, $T^n(x)=x$ can be written. Also, from the equations $T^{n+1}(x)=T(T^n(x))=T(x)=T^n(T(x))$, $T(x)$ is obtained as the fixed point of T^n. Since the fixed point of T^n must be unique, x is unique which satisfies $T(x)=x$. □

Theorem 3.4. *Let (X,d_1) be an $A-$metric space and $F:(X,d_1)\to(X,d_1)$ be a map such that F^n is a contraction mapping with constant μ in (X,d_1). Let d_2 be defined just as in [5]. More precisely,*

$$\begin{aligned}d_2\ :\ \overbrace{X\times X\times\cdots\times X}^{n-times} &\to [0,\infty)\\ (x_1,\dots,x_n) &\mapsto d_2(x_1,\dots,x_n):=\sum_{i=0}^{n-1}\lambda^i d_1\left(F^ix_1,\dots,F^ix_n\right)\end{aligned}$$

where $\lambda\in[0,\infty)$ such that $\mu^{\frac{1}{n}}<\frac{1}{\lambda}<1$. Then,
i) *(X,d_2) is an $A-$metric space,*
ii) *F is a contraction mapping with constant $\frac{1}{\lambda}$ in (X,d_2).*

Proof. Firstly, it is proved that d_2 is an $A-$metric. So the axioms A1 and A2 are valid for all $x_1,\cdots,x_n,w\in X$ and for $I=\{0,1,\dots n-1\}$. Actually,
A1) Since d_1 is an $A-$metric on X, we have that ;

$$\begin{aligned}x_1=\cdots=x_n=0 \iff&\ \forall i\in I,\ d_1\left(F^ix_1,\dots,F^ix_n\right)=0\\ \iff&\ \sum_{i=0}^{n-1}\lambda^i d_1\left(F^ix_1,\dots,F^ix_n\right)=0\\ \iff&\ d_2(x_1,\dots,x_n)=0.\end{aligned}$$

A2) Using the definition of d_2, we get following equations

$$d_2(x_1,\ldots,x_1,w)=\sum_{i=0}^{n-1}\lambda^i d_1\left(F^i x_1,\ldots,F^i x_1,F^i w\right),$$

$$d_2(x_2,\ldots,x_2,w)=\sum_{i=0}^{n-1}\lambda^i d_1\left(F^i x_2,\ldots,F^i x_2,F^i w\right),$$

$$d_2(x_n,\ldots,x_n,w)=\sum_{i=0}^{n-1}\lambda^i d_1\left(F^i x_n,\ldots,F^i x_n,F^i w\right).$$

For the sake of shortness, let $d_2(x_1,\ldots,x_1,w)+d_2(x_2,\ldots,x_2,w)+\cdots+d_2(x_n,\ldots,x_n,w)$ be denoted by K, then

$$K=\sum_{i=0}^{n-1}\lambda^i\left[d_1\left(F^i x_1,\ldots,F^i x_1,F^i w\right)+\cdots+d_1\left(F^i x_n,\ldots,F^i x_n,F^i w\right)\right].$$

Since d_1 is an $A-$metric on X, it follows that

$$\begin{aligned}
K &= \sum_{i=0}^{n-1}\lambda^i\left[d_1\left(F^i x_1,\ldots,F^i x_1,F^i w\right)+\cdots+d_1\left(F^i x_n,\ldots,F^i x_n,F^i w\right)\right]\\
&\geq \sum_{i=0}^{n-1}\lambda^i d_1\left(F^i x_1,\ldots,F^i x_n\right)\\
&= d_2(x_1,\ldots,x_n).
\end{aligned}$$

Consequently, d_2 is an $A-$metric on X. Therefore, the pair (X,d_2) is an $A-$metric space. Now, we prove that F is a contaction mapping with constant $\frac{1}{\lambda}$ in (X,d_2) such that $\mu^{\frac{1}{n}}<\frac{1}{\lambda}<1$. Using the definition of d_2, we get following equations

$$d_2(Fx_1,Fx_2,\ldots,Fx_n)=\sum_{i=0}^{n-1}\lambda^i d_1\left(F^i Fx_1,\ldots,F^i Fx_n\right),$$

$$d_2(x_1,\ldots,x_n)=\sum_{i=0}^{n-1}\lambda^i d_1\left(F^i x_1,\ldots,F^i x_n\right).$$

$\lambda d_2(Fx_1,\ldots,Fx_n)-d_2(x_1,\ldots,x_n)=-d_1(x_1,\ldots,x_n)+\lambda^n d_1(Fx_1,\ldots,Fx_n)$ is obtained by a simple comptutation. Since F^n is a contraction mapping with constant μ in (X,d_1), and $\mu^{\frac{1}{n}}<\frac{1}{\lambda}<1$, we get that

$$\begin{aligned}
d_2(Fx_1,Fx_2,\ldots,Fx_n) &= \frac{1}{\lambda}\left[d_2(x_1,\ldots,x_n)-d_1(x_1,\ldots,x_n)\right]+\lambda^{n-1}d_1(Fx_1,Fx_2,\ldots,Fx_n)\\
&\leq \frac{1}{\lambda}\left[d_2(x_1,\ldots,x_n)-d_1(x_1,\ldots,x_n)\right]+\mu\lambda^{n-1}d_1(x_1,\ldots,x_n)\\
&= \frac{1}{\lambda}d_2(x_1,\ldots,x_n)+\left[\mu-\frac{1}{\lambda^n}\right]\lambda^{n-1}d_1(x_1,\ldots,x_n)\\
&\leq \frac{1}{\lambda}d_2(x_1,\ldots,x_n).
\end{aligned}$$

Therefore, we have that F is a contraction mapping with constant $\frac{1}{\lambda}$ in (X,d_2). □

Now, we can give our main results in the following theorem:

Theorem 3.5. *$F:(X,d_1)\to(X,d_1)$ be a uniformly continuous map, where (X,d_1) is a complete $A-$metric space. Then, (X,d_2) is given Theorem 3.4 is a complete $A-$metric space.*

Proof. Let (x_n) be any $A-$Cauchy sequence in (X,d_2). Then, (x_n) is a $A-$Cauchy sequence in (X,d_1) because of $d_2(x_1,\ldots,x_n)\leq d_1(x_1,\ldots,x_n)$. So, $\exists\, L\in X\ni(x_n)\overset{(X,d_1)}{\longrightarrow}L$ $[\because (X,d_1)$ is the complete $A-$metric space$]$. Let $\beta=\max\left\{\lambda^i:i=1,\ldots,n-1\right\}$, then $\beta\geq\lambda>1$. Also, $F(x_n)\overset{(X,d_1)}{\longrightarrow}F(L)$ $[\because F$ is a uniformly continuous map in $(X,d_1)]$. Since all the power of F are also uniformly continuous in (X,d_1), for any $\varepsilon>0$, there is a $\delta>0$ such that for all $x_1,\ldots,x_n\in X$ and $i=1,\ldots,n-1$. $d_1(x_1,\ldots,x_n)<\delta\Longrightarrow d_1\left(F^i x_1,\ldots,F^i x_n\right)<\varepsilon/\beta n$. Also, $(x_n)\overset{(X,d_1)}{\longrightarrow}L\Longrightarrow\forall\delta>0,\ \exists n_0\in\mathbb{N}\ \ni\ \forall k\geq n_0,\ d_1(x_k,\ldots,x_k,L)<\delta$. Thus, for $i=1,\ldots,n-1$, $k\geq n_0\Longrightarrow d_1\left(F^i x_k,\ldots,F^i x_k,F^i L\right)<\varepsilon/\beta n$. Therefore, $d_2(x_k,\ldots,x_k,L)<\frac{\varepsilon}{n}\left[\frac{1}{\beta}+\frac{\lambda}{\beta}+\cdots+\frac{\lambda^{n-1}}{\beta}\right]<\varepsilon$. Consequently, $A-$Cauchy sequence (x_n) is $A-$convergent to L. So, (X,d_2) is a complete $A-$metric space. □

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